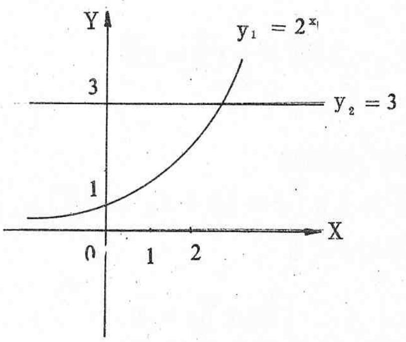
模擬試題解答

何景國 提供

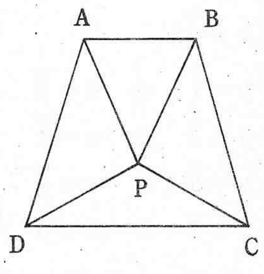
1. (A)(C)(D)

令 $y_1 = 2^x$，$y_2 = m$，得下圖示：



2. (甲)(C) (乙)(D)

(甲)設 G 爲梯形 $ABCD$ 的重心：



得 $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2$

$= 4\overline{PG}^2 + \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2$

← 常數

有最小值之充要條件爲：$P = G$

即點 P 落在兩腰中點連線之中點上。

3. (甲)(D) (乙)(D)(E)

(乙)搬動 64 個銀圈，從一根棒到另一根棒上去須 $2^{64} - 1 \doteqdot 2^{64}$ 秒鐘。令

$$x = 2^{64}$$

得 $\log x = \log 2^{64} = 19.26592$

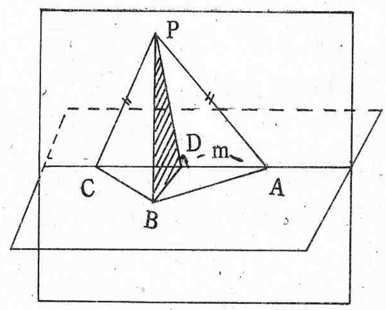
$\Rightarrow x \doteqdot 1.84469 \times 10^{19}$（秒）

$\doteqdot 5.84 \times 10^{11}$ 年

或 $x \doteqdot 5840$ 億年

4. (A)

見圖：



5. (甲)(C) (乙)(E) (丙)(A)

(乙)由(甲)得

$$\tan x = \cot x - 2\cot 2x$$

或

$$\frac{1}{2}\tan x = \frac{1}{2}\cot x - \cot 2x$$

以 $x/2^n$ 代入上式中之 x 得：

$$\frac{1}{2}\tan\frac{x}{2^n} = \frac{1}{2}\cot\frac{x}{2^n} - \cot\frac{2x}{2^n}$$

$$\Rightarrow \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n} - \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}$$

6. (C)(D)

7. (C)

設 x 爲四個數字中最小者，則其餘三個數

分別爲：

$(x+1)$，$(x+7)$ 及 $(x+8)$

8. (A)(C)

設 t 秒後三點共線，則其坐標分別爲

$P(80-5t, 0)$，$R(0, 80-2t)$，

$M(2t, 2t)$

9. (D)

$\because \quad \overrightarrow{wP} = \overrightarrow{wA} + \overrightarrow{wB} + \overrightarrow{wC}$

$\therefore \quad \overrightarrow{AP} = \overrightarrow{wB} + \overrightarrow{wC}$

考慮 $\overrightarrow{AP}\cdot\overrightarrow{BC} = |\overrightarrow{wC}|^2 - |\overrightarrow{wB}|^2 = 0$

故 $\overrightarrow{AP} \perp \overrightarrow{BC}$

同理可得

$\overrightarrow{BP} \perp \overrightarrow{AC}$ 及 $\overrightarrow{CP} \perp \overrightarrow{AB}$

故 P 爲 ΔABC 之垂心。

10. (甲)(C) (乙)(C) (丙)(E) (丁)(B)

(甲) $\because \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BM}$

其中

$$\begin{cases} \overrightarrow{AB} = \overrightarrow{OC} \\ \overrightarrow{BM} = \frac{1}{2}\overrightarrow{BE} = \frac{1}{2}\overrightarrow{OP} \end{cases}$$

$$\therefore \quad \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OC} + \frac{1}{2}\overrightarrow{OP}$$

(乙)設 N 爲 $\overline{BP}$ 之中點，得

$$2\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OP}$$

$$\Rightarrow \overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OP})$$

(丙)令 G 爲 ΔOFQ 之重心，則

$$\overrightarrow{PO} + \overrightarrow{PF} + \overrightarrow{PQ} = 3\overrightarrow{PG}$$

而

$$\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{PO} + \overrightarrow{PQ} + \overrightarrow{PF}$$

$$\Rightarrow \quad 3\overrightarrow{PG} = \overrightarrow{PB}$$

(丁)承(丙)知

$$\overrightarrow{PG} = \frac{1}{3}\overrightarrow{PB}$$

$$\overrightarrow{GN} = \overrightarrow{PN} - \overrightarrow{PG} = \frac{1}{2}\overrightarrow{PB} - \frac{1}{3}\overrightarrow{PB} = \frac{1}{6}\overrightarrow{PB}$$

$$\Rightarrow \quad 2\overrightarrow{GN} = \frac{1}{3}\overrightarrow{PB} = \overrightarrow{PG}$$

11. (甲)(A) (乙)(A)(E)

(甲) $\overline{r} = \{a \mid a = 8k + r,\ k \in \mathbb{Z}\}$，其中 $0 \leqslant r < 8$

$$\begin{cases} \overline{2}x + \overline{6}y = \overline{4} \\ \overline{1}x - \overline{3}y = \overline{0} \end{cases}$$

$$\Rightarrow \begin{cases} \overline{6}x + \overline{6}y = \overline{4} \\ \overline{1}x = \overline{3}y \end{cases}$$

$$\Rightarrow \begin{cases} \overline{4}y = \overline{4} \\ x = \overline{3}y \end{cases}$$

$$\begin{cases} y = \overline{1} \Rightarrow x = \overline{3} \\ y = \overline{3} \Rightarrow x = \overline{1} \\ y = \overline{5} \Rightarrow x = \overline{7} \\ y = \overline{7} \Rightarrow x = \overline{5} \end{cases}$$

解集 $S = \{(\overline{3}, \overline{1}), (\overline{1}, \overline{3}), (\overline{7}, \overline{5}), (\overline{5}, \overline{7})\}$

12. (B)(C)(E)

$$\Delta = -(1+m)^2 + 2(m^2+1)$$
$$= (m-1)^2$$

$$Z_1 = 1 + m\left(\frac{i-1}{i+1}\right) = 1 + mi$$

或

$$Z_2 = m + i$$

$$\because \begin{cases} \overrightarrow{PA_1} = \overrightarrow{OA_1} - \overrightarrow{OP_0} \longleftrightarrow \\ Z_1 - w = (m-1)i \\ \overrightarrow{PA_2} = \overrightarrow{OA_2} - \overrightarrow{OP} \longleftrightarrow \\ Z_2 - w = (m-1)i \end{cases}$$

$\therefore$ $Z_1 - w = i(Z_2 - w)$

或 $Z_2 - w = i(Z_1 - w)$

故 $\overrightarrow{PA_1} \perp \overrightarrow{PA_2}$，且 $|\overrightarrow{PA_1}| = |\overrightarrow{PA_2}|$

即 ΔA_1PA_2 爲一直角等腰三角形。

13. (甲)(A)(C) (乙)(A)(B)(D)(E) (丙)(A)(D)

(甲) $P^2 = P \cdot P = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

其中 $a_2 = a(a-b) + b$；

$b_2 = b(a-b) + b$

即 $P^2 - P = d(P - I_2)$，

其中 $d = a - b$

或 $P^2 = (1+d)P - dI_2$

又

$$P^3 = P^2 \cdot P^1 = (1+d)P^2 - dP$$
$$= \frac{1}{1-d}\left[(1-d^3)P\right.$$
$$\left.-d(1-d^2)I_2\right]$$

令 $P^k = P^{k-1} \cdot P^1$

$$= \frac{1}{1-d}\left[(1-d^{k-1})P^2\right.$$
$$\left.-d(1-d^{k-2})P\right]$$

即

$$P^k = \frac{1}{1-d}\left[(1-d^k)P\right.$$
$$\left.-d(1-d^{k-10})I_2\right]$$

由數學歸納法知：$\forall n \in N$，恆有

$$P^n = \frac{1}{1-d}\left[(1-d^k)P\right.$$
$$\left.-d(1-d^{k-1})I_2\right]$$

(乙)

$$\because \quad P^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$$

其中

$$\begin{cases} a_n = \dfrac{1}{1-d}\left[(1-d^n)a\right. \\ \quad \left.-d(1-d^{n-1})\right] \\ b_n = \dfrac{1}{1-d}\left[(1-d^n)b\right] \\ c_n = \dfrac{1}{1-d}\left[(1-d^n)(1-a)\right] \\ d_n = \dfrac{1}{1-d}\left[(1-d^n)(1-b)\right. \\ \quad \left.-d(1-d^{n-1})\right] \end{cases}$$

$$\therefore \lim_{n\to\infty} a_n = \frac{b}{1-a+b} = \lim_{n\to\infty} b_n$$
$$= \frac{b}{1-a+b}$$
$$\lim_{n\to\infty} c_n = \frac{1-a}{1-a+b} = \lim_{n\to\infty} d_n$$
$$= \frac{1-a}{1-a+b}$$

但 $0 < b < 1$；$0 < 1-a < 1$

$$\text{故} \quad \lim_{n\to\infty} a_n + \lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} d_n$$
$$= 1$$

(丙)因 P 不可逆 $\Longleftrightarrow$ $\det P = 0$ $\Longleftrightarrow$

$$\begin{vmatrix} a & b \\ 1-a & 1-b \end{vmatrix} = 0$$

故 $a = b$，$d = 0$

$\Rightarrow$ $P^n = P$，$\forall n \in N$

14. (B)(C)(D)(E)

15. (A)(C)(D)(E)

$$\because (0.99)^{10} = 1 - C_1^{10}(0.01) + C_2^{10}(0.01)^2$$
$$- C_3^{10}(0.01)^3 + \cdots$$
$$= 0.90438$$

$\therefore a = 9$，$b = 0$，$c = 4$，$d = 3$

16. (甲)(B) (乙)(C) (丙)(C)

17. (甲)(C) (乙)(A) (丙)(D) (丁)(B)

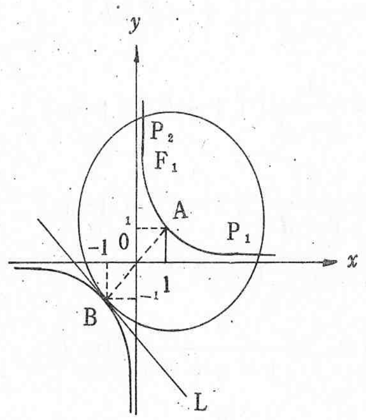
(甲)$F_1: xy = 1$

(乙)$F_2: x^2 + y^2 - 2x - 2y - b = 0$

(丁)F_1, F_2 之公切線方程式為：

$y+1 = m(x+1)$，其中$m = -1$

$\Rightarrow x + y + 2 = 0$



18. (A)(B)(C)(D)

令 $\tan\theta = x$，則

$$f(x) = \frac{x + \tan}{1 - x\tan\alpha} = \tan(\theta + \alpha)$$

$$g(x) = \tan(\theta + \beta)$$

$$\Rightarrow f(g(x)) = \frac{\tan(\theta+\beta) + \tan\alpha}{1 - \tan(\theta+\beta)\tan\alpha} = \tan(\theta + \beta + \alpha)$$

及 $g(x) = \tan(\theta + \alpha + \beta)$

即 $f(g(x)) = g(f(x))$

$\because \tan(\theta + \alpha) = \tan(\theta + \beta)$，

其中 $\tan\theta = x$

$\therefore \alpha = \beta$ （$\because 0 < \alpha, \beta < \pi/2$）

又因 $f_1(x) = f(x) = \tan(\theta + \alpha)$

$$f_2(x) = f(f(x)) = \tan(\theta + 2\alpha)$$

………

$$f_n(x) = f(f_{n-1}(x)) = \tan(\theta + n\alpha) = \frac{\tan\theta + \tan n\alpha}{1 - \tan\theta \cdot \tan n\alpha}$$

即

$$f_n(x) = \frac{x + \tan n\alpha}{1 - x\tan n\alpha}$$

$$\because f(x) = \frac{\sqrt{3}x + 1}{\sqrt{3} - x} = \frac{x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}x}$$

$$\therefore \tan\alpha = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \alpha = \frac{\pi}{6} + k\pi \ (k \in \mathbb{Z})$$

但 $\alpha \in (0, \pi/2)$，故 $\alpha = \pi/6$

又由於 $f_n(x) = \tan(\theta + n\alpha)$

故 $f_{17}(x) = \tan(\theta + 17\alpha)$

$$= \tan\left(\theta + \frac{5\pi}{6}\right) = \frac{x - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}x}$$

19. (A)(B)(E)

令 $341_{10} = 2331_a = 2a^3 + 3a^2 + 3a + 1$

$\Rightarrow (a-5)(2a^2 + 13a + 68) = 0$

$\Rightarrow a = 5$

另一方面知 $a \in N$，且 $a \geqslant 4$

故 $2a^3 < 341$

$\Rightarrow 64 \leqslant a^3 \leqslant 170$

20. (B)(C)

$$(x^2 - x - 11)^{x^2 - x - 6} = 1$$

得 $x^2 - x - 11 = 1$

或 $x^2 - x - 11 \neq 0 \Rightarrow x^2 - x - 6 = 0$

21. (甲)(A) (乙)(B) (丙)(B)

（本解答提供者為延平中學老師）