Bata函數在多重積分上的應用

何志誠

在向量分析中常有些繁雜的計算，尤其是需要用到球面座標轉換，本文將介紹利用*Beta function*來處理這些問題。

I　吾人先觀察下列問題

一個半徑為R，密度為一定值ρ的實心鐵球，若該球繞中心軸旋轉，試求其轉動慣量值。

凡學過轉動力學或普通物理的讀者，都知道轉動慣量值 $I=\frac{2}{5}MR^2$，求I值的方法很多，若以多重積分的方法求之，則為：

$S: x^2+y^2+z^2\leq R^2$，　$I=\int_S r^2\,dm=\rho\iiint_S (x^2+y^2)\,dx\,dy\,dz$

令 $x=r\cos\theta\sin\phi$，$y=r\sin\theta\sin\phi$，$z=r\cos\theta$，$x^2+y^2=r^2\sin^2\theta$

可算得 $Jacobian=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$

$$\iiint_S f(x,y,z)\,dxdydz=\iiint_S f[x(r,\theta,\phi),y(r,\theta,\phi),z(r,\theta,\phi)]\,|Jacobian|\cdot drd\phi d\theta$$

故 $I=\rho\cdot\int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2\sin^2\theta\, r^2\sin\theta\, drd\phi d\theta$

$$=8\cdot\rho\left[\int_0^R r^4dr\right]\cdot\left[\int_0^{\frac{\pi}{2}}d\phi\right]\cdot\left[\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta\right]=\frac{4}{5}R^5\pi\rho\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta$$

其中含有$\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta$的型式，一般而言，球面座標與圓柱座標之三重積分皆常出現$\int_0^{\frac{\pi}{2}}\cos^m\theta\sin^n\theta d\theta$　的型式，若以$x=\sin^2\theta$與$\sin^2\theta+\cos^2\theta=1$之關係式代入，則可化為$\frac{1}{2}\int_0^1 x^{\frac{1}{2}(n-1)}\cdot(1-x)^{\frac{1}{2}(m-1)}dx$之型式，而此即為*Beta function*.

依次定義*Gamma*及*Beta* 函數如下：

(1)　$\Gamma(m+1)=\int_0^{\infty}e^{-x}\cdot x^m dx$　　$(m>0)$

(2)　$B(m,n)=\int_0^1 x^{m-1}\cdot(1-x)^{n-1}dx$

則有下列性質（一般高等微積分書上都有證明）：

(3)　$\Gamma(n+1)=n\cdot\Gamma(n)$

(4)　$\Gamma(\frac{1}{2})=\sqrt{\pi}$

(5) $B(m,n)=\dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

令 $x=\sin^2\theta$，則

(6) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta=\frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

(7) $\Gamma(m)\cdot\Gamma(1-m)=\dfrac{\pi}{\sin m\pi}$ （m不為整數）

故上題中，其轉動慣量值為

$$I=\frac{4}{5}\cdot R^5\pi\rho\cdot\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta=\frac{4}{5}R^5\pi\rho\cdot\frac{2}{3}=\frac{2}{5}MR^2$$

同理，用(6)，(7)式可得

$$\int_0^{\frac{\pi}{2}}\sqrt{\tan x}\,dx=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}x\cdot\cos^{\frac{-1}{2}}x\,dx=\frac{1}{2}B\left(\frac{1+\frac{1}{2}}{2},\frac{1-\frac{1}{2}}{2}\right)=\frac{\pi}{\sqrt{2}}$$

如$Wallis$ 積分公式，由(6)式立即可得

$$W_n=\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta=\int_0^{\frac{\pi}{2}}\cos^n\theta d\theta$$

$$=\begin{cases}\dfrac{2\cdot4\cdot6\cdot\cdots\cdots(n-1)}{3\cdot5\cdot7\cdot\cdots\cdots\cdot n} & n\text{為奇數}\\[2ex]\dfrac{1\cdot3\cdot5\cdot\cdots\cdots(n-1)}{2\cdot4\cdot6\cdot\cdots\cdots\cdot n}\cdot\dfrac{\pi}{2} & n\text{為偶數}\end{cases}$$

II．今考慮 $I=\int_V x_1^{m_1}\cdot x_2^{m_2}\cdots\cdots x_n^{m_n}\,dV$，其中 $m_i>-1$，$i=1$，……，n，

且V：$\left(\dfrac{x_1}{a_1}\right)^{A_1}+\left(\dfrac{x_2}{a_2}\right)^{A_2}+\cdots\cdots+\left(\dfrac{x_n}{a^n}\right)^{A_n}\leq1$，$x_1$，……，$x_n$ 均為正數

吾人先考慮U：$x_1+x_2+\cdots\cdots+x_n\leq1$，$x_1$，$x_2$，$x_3$ 均為正數

令 $1-x_1-x_2-\cdots\cdots-x_k=P_k$，則 $P_{k-1}-x_k$

取 $P_{k-2}\cdot u=x_{k-1}$，故 $P_{k-2}\cdot u=x_{k-1}=P_{k-2}-P_{k-1}$

(8) 故 $P_{k-1}=P_{k-2}\cdot(1-u)$，$dx_{k-1}=P_{k-2}\cdot du$

則 $\iiint\cdots\int_U x_1^{m_1}\cdot x_2^{m_2}\cdots\cdots x_n^{m_n}\,dx_1dx_2\cdots\cdots dx_n$

$$=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\cdots\int_0^{1-x_1\cdots x_n}x_1^{m_1}\cdot x_2^{m_2}\cdot\cdots x_n^{m_n}\,dx_1dx_2\cdots dx_n$$

$$=\int_0^1\int_0^{P_1}\cdots\cdots\int_0^{P_{n-1}}x_1^{m_1}\cdot x_2^{m_2}\cdot\cdots\cdots x_n^{m_n}\,dx_1dx_2\cdots\cdots dx_n$$

$$=\int_0^1\int_0^{P_1}\cdots\cdots\int_0^{P_{n-2}}x_1^{m_1}\cdot x_2^{m_2}\cdots\cdots x_{n-1}^{m_{n-1}}\frac{P_{n-1}^{1+m_n}}{1+m_n}dx_1dx_2\cdots\cdots dx_{n-1}$$

$$=\int_0^1\int_0^{P_1}\cdots\int_0^{P_{n-3}}x_1^{m_1}\bullet x_2^{m_2}\cdots\cdots\ x_{n-2}^{m_{n-2}}\,u^{m_{n-1}}\frac{P_{n-2}^{1+m_n}}{1+m_n}$$

$$\cdot P_{n-2}^{m_{n-1}}(1-u)^{1+m_n}\cdot dx_1\cdot dx_2\cdot dx_3\cdots\cdots$$

$dx_{n-2}\, P_{n-2}\, du$ (用(8))

$$= \frac{1}{1+m_n}\int_0^1 u^{m_{n-1}}(1-u)^{1+m_n}du \cdot \int_0^1\int_0^{P_1}\cdots\int_0^{P_{n-3}} x_1^{m_1}\cdot x_2^{m_2}\cdots x_{n-2}^{m_{n-2}}\cdot P_{n-2}^{2+m_n+m_{n-1}}$$

$dx_1\, dx_2 \cdots\cdots dx_{n-2}$

$$= \frac{B(1+m_{n-1}, 2+m_n)}{(1+m_n)}\cdot\int_0^1\int_0^{P_1}\cdots\int_0^{P_{n-3}} x_1^{m_1}\cdot x_2^{m_2}\cdot\cdots x_{n-3}^{m_{n-3}}\cdot(P_{n-3}^{m_{n-2}}\cdot u^{m_{n-2}})\cdot$$

$[P_{n-3}\cdot(1-u)]^{2+m_n+m_{n-1}}\cdot dx_1 dx_2\cdots dx_{n-2}\, P_{n-3}\, du$ (用(8))

$$= \frac{1}{1+m_n} B(1+m_{n-1}, 2+m_n)\cdot B(1+m_{n-2}, 3+m_n+m_{n-1})$$

$$\cdot\int_0^1\int_0^{P_1}\cdots\int_0^{P_{n-2}} x_1^{m_1} x_2^{m_2}\cdots x_{n-3}^{m_{n-3}}\, dx_1 dx_2\cdots dx_{n-3}$$

$$= \frac{1}{1+m_n} B(1+m_{n-1}, 2+m_n)\cdot B(1+m_{n-2}, 3+m_n+m_{n-1})\cdot\cdots B(1+m_{k-1}$$

$$n-k+2+m_n+m_{n-1}+\cdots+m_{k+1}+m_k)\cdot\int_0^1\int_0^{P_1}\cdots\int_0^{P_{k-3}} x_1^{m_1}\cdot x_2^{m_2}\cdot\cdots x_{k-3}^{m_{k-3}}$$

$$\cdot(P_{k-3}^{m_{k-2}}\cdot u^{m_{k-2}})\cdot[(P_{k-3}\cdot(1-u)]^{2+m_n+\cdots+m_{k-1}}\, dx_1 dx_2\cdots dx_{k-3}\cdot P_{k-3}\cdot du$$

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$$= \frac{1}{1+m_n} B(1+m_{n-1}, 2+m_n)\cdot B(1+m_{n-2}, 3+m_n+m_{n-1})\cdot\cdots\cdots$$

$$B(1+m_1, n+m_n+m_{n-1}+\cdots+m_3+m_2)$$

$$= \frac{1}{1+m_n}\frac{\Gamma(1+m_{n-1})\Gamma(2+m_n)}{\Gamma(3+m_n+m_{n-1})}\cdot\frac{\Gamma(1+m_{n-2})\Gamma(3+m_n+m_{n-1})}{\Gamma(4+m_n+m_{n-1}+m_{n-2})}\cdot\cdots\cdots$$

$$\frac{\Gamma(1+m_2)\cdot\Gamma(n-1+m_n+\cdots+m_3)}{\Gamma(n+m_2+\cdots+m_n)}\cdot\frac{\Gamma(1+m_1)\cdot\Gamma(n+m_2+\cdots+m_n)}{\Gamma(n+1+m_1+m_2+\cdots+m_n)}$$

$$= \frac{\Gamma(1+m_{n-1})\Gamma(2+m_n)\cdot\Gamma(1+m_{n-2})\cdot\cdots\cdot\Gamma(1+m_2)\Gamma(1+m_1)}{\Gamma(n+1+m_1+m_2+\cdots\cdots+m_n)\cdot(1+m_n)}$$

$$= \frac{\Gamma(1+m_1)\cdot\Gamma(1+m_2)\cdot\cdots\cdots\cdot\Gamma(1+m_{n-1})\cdot\Gamma(1+m_n)}{\Gamma[(1+m_1)+(1+m_2)+\cdots+(1+m_n)+1]}$$

若U中，以$\left(\frac{x_1}{a_1}\right)^{A_1}$，$\left(\frac{x_2}{a_2}\right)^{A_2}$，…，$\left(\frac{x_n}{a_n}\right)^{A_n}$ 分別依序代入x_1，x_2，x_3，…，x_n則可得下式：

(9) $$I = \int_V x_1^{m_1}\cdot x_2^{m_2}\cdot\cdots\cdot x_n^{m_n}\, dV$$

$$= \frac{a_1^{m_1+1}\cdot a_2^{m_2+1}\cdots a_n^{m_n+1}\cdot\Gamma\left(\frac{m_1+1}{A_1}\right)\cdot\Gamma\left(\frac{m_2+1}{A_2}\right)\cdot\cdots\Gamma\left(\frac{m_n+1}{A_n}\right)}{A_1\cdot A_2\cdot\cdots A_n\cdot\Gamma\left(\frac{m_1+1}{A_1}+\frac{m_2+1}{A_2}+\cdots+\frac{m_n+1}{A_n}+1\right)}$$

其中V：$\left(\frac{x_1}{a_1}\right)^{A_1}+\cdots\cdots+\left(\frac{x_n}{a_n}\right)^{A_n}\leq 1$，$x_1$，$x_2$，…，$x_n$均爲正數

利用(9)式可容易計算出一些多重積分。

以上題求轉動慣量值爲例：

原式可化為 $\left(\frac{x}{R}\right)^2+\left(\frac{y}{R}\right)^2+\left(\frac{z}{R}\right)^2\le 1$

$\therefore\ A_1=A_2=A_3=2\ ,\ a_1=a_2=a_3=R$

$\therefore\ I=\rho\cdot\iiint_{S_1}(x^2+y^2)\,dxdydz\cdot 8$ （因S可分為8個卦限，S_1 為第一卦限）

$=\rho\cdot\iiint_{S_1}x^2\cdot dxdydz\cdot 8\cdot 2$ （因在 S_1 中，x 與 y 為對稱）

$$=\frac{R^{2+1}\cdot R^{0+1}\cdot R^{0+1}}{2\cdot 2\cdot 2}\cdot\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2+1}{2}+\frac{0+1}{2}+\frac{0+1}{2}+1\right)}\cdot 16\cdot\rho$$

$$=2R^5\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)^2}{\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)}\rho=\frac{8}{15}R^5\cdot\rho=\left(\frac{4}{3}\pi R^3\rho\right)\cdot\frac{2}{5}R^2=\frac{2}{5}MR^2$$

例 1：設 $S:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$，試求：

(1) $V_1=\iiint_S dxdydz$ (2) $V_2=\iiint_S x^2y\,dxdydz$ 之值

依(9)式

$$V_1=\frac{a^{0+1}b^{0+1}c^{0+1}}{2\times 2\times 2}\cdot\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{0+1}{2}+\frac{0+1}{2}+\frac{0+1}{2}+1\right)}\cdot 8$$

$$=abc\cdot\frac{\Gamma\left(\frac{1}{2}\right)^3}{\Gamma\left(\frac{5}{2}\right)}=abc\,\frac{\Gamma\left(\frac{1}{2}\right)^3}{\frac{1}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}\right)}=\frac{4}{3}\pi abc$$

$$V_2=\frac{a^{2+1}b^{0+1}c^{0+1}}{2\times 2\times 2}\cdot\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)\cdot 8}{\Gamma\left(\frac{2+1}{2}+\frac{1+1}{2}+\frac{0+1}{2}+1\right)}$$

$$=a^3b^2c\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}=a^3b^2c\frac{\pi}{3!}\cdot\frac{1}{2}=\frac{\pi}{12}a^3b^2c$$

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