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利用遞迴關係式求 r 為自然數 $\sum_{k=1}^{n} k^r$ 的公式解

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研究目的: 本文以遞迴關係式求 $\sum\limits_{k=1}^{n} k^r$ 的公式解, 其中 r 爲自然數, 解題的靈感偶然從腦海中閃過, 我即刻把握當下, 振筆疾書, 記錄這美好的尋求過程, 請看下文的剖析。

研究過程:

1、以 $S_r(n)$ 表示 $\sum\limits_{k=1}^{n} k^r$, 本文探討 r 爲自然數的情形。

2、尋求遞迴關係式:

設 $S_r(n)=\dfrac{1}{r+1}\sum\limits_{i=0}^{r} C_i^{r+1}\cdot B_i\cdot n^{r+1-i}$,

則 $S_r(n-1)=\dfrac{1}{r+1}\sum\limits_{i=0}^{r} C_i^{r+1}\cdot B_i\cdot (n-1)^{r+1-i}$,

$$\Rightarrow n^r=S_r(n)-S_r(n-1)=\frac{1}{r+1}\sum_{i=0}^{r} C_i^{r+1}\cdot B_i\cdot[n^{r+1-i}-(n-1)^{r+1-i}]$$

$$=\frac{1}{r+1}\cdot C_0^{r+1}\cdot B_0\cdot[n^{r+1}-(n-1)^{r+1}]+\frac{1}{r+1}\cdot C_1^{r+1}\cdot B_1\cdot[n^r-(n-1)^r]$$

$$+\frac{1}{r+1}\cdot C_2^{r+1}\cdot B_2\cdot[n^{r-1}-(n-1)^{r-1}]+\cdots+\frac{1}{r+1}\cdot C_r^{r+1}\cdot B_r\cdot[n^1-(n-1)^1]$$

$$=\frac{1}{r+1}\cdot C_0^{r+1}\cdot B_0\cdot[C_1^{r+1}n^r-C_2^{r+1}n^{r-1}+C_3^{r+1}n^{r-2}+\cdots+(-1)^r C_{r+1}^{r+1}]$$

$$+\frac{1}{r+1}\cdot C_1^{r+1}\cdot B_1\cdot[C_1^{r}n^{r-1}-C_2^{r}n^{r-2}+C_3^{r}n^{r-3}+\cdots+(-1)^{r-1} C_{r}^{r}]$$

$$+\frac{1}{r+1}\cdot C_2^{r+1}\cdot B_2\cdot[C_1^{r-1}n^{r-2}-C_2^{r-1}n^{r-3}+C_3^{r-1}n^{r-4}+\cdots+(-1)^{r-2} C_{r-1}^{r-1}]$$

$$
\begin{aligned}
&+\cdots+\frac{1}{r+1}\cdot C_r^{r+1}\cdot B_r\cdot[C_1^1]\\
&=B_0\cdot n^r+\frac{1}{r+1}\cdot[-C_0^{r+1}\cdot B_0\cdot C_2^{r+1}+C_1^{r+1}\cdot B_1\cdot C_1^r]n^{r-1}\\
&+\frac{1}{r+1}\cdot[C_0^{r+1}\cdot B_0\cdot C_3^{r+1}-C_1^{r+1}\cdot B_1\cdot C_2^r+C_2^{r+1}\cdot B_2\cdot C_1^{r-1}]n^{r-2}\\
&+\frac{1}{r+1}\cdot[-C_0^{r+1}\cdot B_0\cdot C_4^{r+1}+C_1^{r+1}\cdot B_1\cdot C_3^r-C_2^{r+1}\cdot B_2\cdot C_2^{r-1}+C_3^{r+1}\cdot B_3\cdot C_1^{r-2}]n^{r-3}\\
&+\cdots+\frac{1}{r+1}\cdot[(-1)^r\cdot C_0^{r+1}\cdot B_0\cdot C_{r+1}^{r+1}+(-1)^{r-1}\cdot C_1^{r+1}\cdot B_1\cdot C_r^r\\
&+(-1)^{r-2}\cdot C_2^{r+1}\cdot B_2\cdot C_{r-1}^{r-1}+\cdots+(-1)^0\cdot C_r^{r+1}\cdot B_r\cdot C_1^1]\\
&=B_0\cdot n^r+\frac{1}{r+1}\cdot[-C_0^2\cdot B_0\cdot C_2^{r+1}+C_1^2\cdot B_1\cdot C_2^{r+1}]n^{r-1}\\
&+\frac{1}{r+1}\cdot[C_0^3\cdot B_0\cdot C_3^{r+1}-C_1^3\cdot B_1\cdot C_3^{r+1}+C_2^3\cdot B_2\cdot C_3^{r+1}]n^{r-2}\\
&+\frac{1}{r+1}\cdot[-C_0^4\cdot B_0\cdot C_4^{r+1}+C_1^4\cdot B_1\cdot C_4^{r+1}-C_2^4\cdot B_2\cdot C_4^{r+1}+C_3^4\cdot B_3\cdot C_4^{r+1}]n^{r-3}+\cdots\\
&+\frac{1}{r+1}\cdot[(-1)^r\cdot C_0^{r+1}\cdot B_0\cdot C_{r+1}^{r+1}+(-1)^{r-1}\cdot C_1^{r+1}\cdot B_1\cdot C_{r+1}^{r+1}\\
&+(-1)^{r-2}\cdot C_2^{r+1}\cdot B_2\cdot C_{r+1}^{r+1}+\cdots+(-1)^0\cdot C_r^{r+1}\cdot B_r\cdot C_{r+1}^{r+1}]\\
&=B_0\cdot n^r+\frac{(-1)^1}{r+1}\cdot C_{1+1}^{r+1}\cdot[C_0^2\cdot B_0-C_1^2\cdot B_1]n^{r-1}\\
&+\frac{(-1)^2}{r+1}\cdot C_{2+1}^{r+1}\cdot[C_0^3\cdot B_0-C_1^3\cdot B_1+C_2^3\cdot B_2]n^{r-2}\\
&+\frac{(-1)^3}{r+1}\cdot C_{3+1}^{r+1}\cdot[C_0^4\cdot B_0-C_1^4\cdot B_1+C_2^4\cdot B_2-C_3^4\cdot B_3]n^{r-3}\\
&+\cdots+\frac{(-1)^r}{r+1}\cdot C_{r+1}^{r+1}\cdot[C_0^{r+1}\cdot B_0-C_1^{r+1}\cdot B_1\\
&+C_2^{r+1}\cdot B_2-C_3^{r+1}\cdot B_3+\cdots+(-1)^r\cdot C_r^{r+1}\cdot B_r]n^{r-r}\\
&=B_0\cdot n^r+\sum_{k=1}^{r}\Big\{\frac{(-1)^k}{r+1}\cdot C_{k+1}^{r+1}\cdot\Big[\sum_{i=0}^{k}(-1)^i\cdot C_i^{k+1}B_i\Big]\Big\}n^{r-k}\\
&\Rightarrow B_0=1 \text{ 且 } \sum_{i=0}^{k}(-1)^i\cdot C_i^{k+1}B_i=0,\ k=1,\ldots,r.
\end{aligned}
$$

到此已尋得遞迴關係式進而可求得 $B_i,\ i=1,\ldots,r$。

3、以 $r=7$ 爲例, 利用上述遞迴關係式可求得 $B_i,\ i=0,1,\ldots,7$:
將遞迴關係式寫成矩陣的形式

$$\begin{bmatrix} C_0^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_0^2 & -C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_0^3 & -C_1^3 & C_2^3 & 0 & 0 & 0 & 0 & 0 \\ C_0^4 & -C_1^4 & C_2^4 & -C_3^4 & 0 & 0 & 0 & 0 \\ C_0^5 & -C_1^5 & C_2^5 & -C_3^5 & C_4^5 & 0 & 0 & 0 \\ C_0^6 & -C_1^6 & C_2^6 & -C_3^6 & C_4^6 & -C_5^6 & 0 & 0 \\ C_0^7 & -C_1^7 & C_2^7 & -C_3^7 & C_4^7 & -C_5^7 & C_6^7 & 0 \\ C_0^8 & -C_1^8 & C_2^8 & -C_3^8 & C_4^8 & -C_5^8 & C_6^8 & -C_7^8 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

解得 $B_0=1,\ B_1=\dfrac{1}{2},\ B_2=\dfrac{1}{6},\ B_3=0,\ B_4=-\dfrac{1}{30},\ B_5=0,\ B_6=\dfrac{1}{42},\ B_7=0$。

4、利用所求得的 $B_i,\ i=0,1,2,\ldots,7$ 與 $S_r(n)=\dfrac{1}{r+1}\sum\limits_{i=0}^{r} C_i^{r+1}\cdot B_i\cdot n^{r+1-i}$, 去求 $S_r(n)$ 的公式解, $r=1,2,\ldots,7$:

$$\begin{aligned}
S_1(n)&=\frac{1}{2}\sum_{i=0}^{1} C_i^2\cdot B_i\cdot n^{2-i}=\frac{1}{2}n^2+\frac{1}{2}n,\\
S_2(n)&=\frac{1}{3}\sum_{i=0}^{2} C_i^3\cdot B_i\cdot n^{3-i}=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n,\\
S_3(n)&=\frac{1}{4}\sum_{i=0}^{3} C_i^4\cdot B_i\cdot n^{4-i}=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2,\\
S_4(n)&=\frac{1}{5}\sum_{i=0}^{4} C_i^5\cdot B_i\cdot n^{5-i}=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n,\\
S_5(n)&=\frac{1}{6}\sum_{i=0}^{5} C_i^6\cdot B_i\cdot n^{6-i}=\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2,\\
S_6(n)&=\frac{1}{7}\sum_{i=0}^{6} C_i^7\cdot B_i\cdot n^{7-i}=\frac{1}{7}n^7+\frac{1}{2}n^6+\frac{1}{2}n^5-\frac{1}{6}n^3+\frac{1}{42}n,\\
S_7(n)&=\frac{1}{8}\sum_{i=0}^{7} C_i^8\cdot B_i\cdot n^{8-i}=\frac{1}{8}n^8+\frac{1}{2}n^7+\frac{7}{12}n^6-\frac{7}{24}n^4+\frac{1}{12}n^2.
\end{aligned}$$

5、結論: 以上, 我們列出了 $S_1(n),\ S_2(n),\ \ldots,\ S_7(n)$ 等的公式解。事實上, 利用遞迴式 $B_0=1$ 且 $\sum\limits_{i=0}^{k}(-1)^i\cdot C_i^{k+1}\cdot B_i=0,\ k=1,\ldots,r$; 並且把此遞迴關係用矩陣的形式表示, 我們可以求得任何 $S_r(n)=\dfrac{1}{r+1}\sum\limits_{i=0}^{r} C_i^{r+1}\cdot B_i\cdot n^{r+1-i}$ 的公式解。

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