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黎曼的級數重排定理的一些省思

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交錯調和級數 $\sum_1^\infty(-1)^{n-1}\frac{1}{n}$ 條件收斂到 $\log 2$。依黎曼的級數重排定理, 固定任意 u, v, $-\infty \leq u \leq v \leq +\infty$, 則有 $\sum_1^\infty(-1)^{n-1}\frac{1}{n}$ 的重排 C, 其部分和的數列以 u 爲下極限, v 爲上極限。黎曼的級數重排定理的證明需要計算部分和, 實際執行有相當程度困難。參考文獻1改用計算項數的方法, 舉出收斂到非原來和的例子。本文延續該方法, 做出重排 C, 其部分和的數列以 u 爲下極限, v 爲上極限。文中出現的重排任意二正項都維持原來的先後次序, 只是間隔可能改變。負項部分也一樣。假設 T 是這樣的重排, 設 T 的第 k 個正項段的最後一項是 $\frac{1}{2p_k-1}$; 第 k 個負項段的最後一項是 $-\frac{1}{2q_k}$, 則 T 由其首項 (1 或 $-\frac{1}{2}$) 及兩個嚴格遞增的正整數序列 $p_1, p_2, p_3, \ldots$ 及 $q_1, q_2, q_3, \ldots$ 完全決定。我們用 $S_n(T)$ 表示 T 的前 n 項和。

假設 T 的首項爲 1, 則當 $p_k + q_k \leq n \leq p_{k+1} + q_{k+1}$ 時,

$$\min\{S_{p_k+q_k}(T), S_{p_{k+1}+q_{k+1}}(T)\} \leq S_n(T) \leq S_{q_k+p_{k+1}}(T)$$

故

$$\liminf S_{p_k+q_k}(T) \leq \liminf S_n(T) \leq \limsup S_n(T) \leq \limsup S_{q_k+p_{k+1}}(T).$$

如果

$$\lim S_{p_k+q_k}(T) = u, \qquad \lim S_{q_k+p_{k+1}}(T) = v,$$

則

$$\liminf S_n(T) = u, \qquad \limsup S_n(T) = v.$$

若是 T 的首項爲 $-\dfrac{1}{2}$, 則當 $p_k + q_k \leq n \leq p_{k+1} + q_{k+1}$ 時,

$$S_{p_k+q_{k+1}}(T) \leq S_n(T) \leq \max\{S_{p_k+q_k}(T), S_{p_{k+1}+q_{k+1}}(T)\}$$

如果

$$\lim S_{p_k+q_k}(T) = v, \qquad \lim S_{p_k+q_{k+1}}(T) = u,$$

則

$$\liminf S_n(T)=u,\qquad \limsup S_n(T)=v.$$

底下分 A: $-\infty<u\le\log 2$, $v<+\infty$; B: $-\infty<u$, $\log 2\le v<+\infty$; C: $-\infty<u<v=+\infty$; D: $-\infty=u<v<+\infty$ 及 E: $-\infty=u=v$ 或 $u=v=+\infty$ 或 $-\infty=u$, $v=+\infty$ 五種情況說明如何確定首項及一組 $p_1,p_2,p_3,\ldots$ 及 $q_1,q_2,q_3,\ldots$, 使對應的重排其部分和的數列以 u 爲下極限, v 爲上極限。對任意 w, $u\le w\le v$, 我們也舉出一個收斂到 w 的 $\{S_n(T)\}$ 的子序列。即此重排的部分和的極限點集爲 $[u,v]$。在此先說明 B, D 二情況我們取首項爲 $-\frac{1}{2}$, 其餘情況首項取爲 1。

A. $u=\log 2-\dfrac{1}{2}\log t$, $v=u+\dfrac{1}{2}\log d$, $1\le t<+\infty$, $1\le d<+\infty$ 。

取 $p_1=1$, $q_1=[t]$; $p_2=[dp_1+1]$, $q_2=[tp_2]$; $\ldots$; $p_{k+1}=[dp_k+1]$, $q_{k+1}=[tp_{k+1}]$。顯然 $p_{k+1}\ge p_k+1$, 於是 $q_{k+1}=[tp_{k+1}]\ge[tp_k+t]\ge[tp_k]+1=q_k+1$。所以 p_1,q_1; p_2,q_2; $\ldots$; p_k,q_k; $\ldots$ 確實定義一個重排 T ($[x]$ 表不超過 x 的最大整數)。

因 $q_k\ge p_k$ 且 $k\to\infty$ 時 $p_k\to\infty$, 於是

$$\begin{aligned}
\lim_{k\to\infty}S_{p_k+q_k}(T)&=\lim_{k\to\infty}\Big\{\Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\Big\}\\
&=\lim_{k\to\infty}\Big\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2p_k-1}-\frac{1}{2p_k}\Big\}\\
&\quad-\lim_{k\to\infty}\Big\{\frac{1}{2p_k+2}+\cdots+\frac{1}{2q_k}\Big\}\\
&=\log 2-\frac{1}{2}\lim_{k\to\infty}\Big\{\frac{1}{p_k+1}+\frac{1}{p_k+2}+\cdots+\frac{1}{[tp_k]}\Big\}\\
&=\log 2-\frac{1}{2}\lim_{k\to\infty}\frac{1}{p_k}\Big\{\frac{1}{1+1/p_k}+\frac{1}{1+2/p_k}+\cdots+\frac{1}{[tp_k]/p_k}\Big\}\\
&=\log 2-\frac{1}{2}\int_1^t\frac{1}{x}dx=\log 2-\frac{1}{2}\log t=u
\end{aligned}$$

又

$$S_{q_k+p_{k+1}}(T)=S_{p_k+q_k}(T)+\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2p_{k+1}-1}\Big\}$$

於是

$$\begin{aligned}
\lim_{k\to\infty}S_{q_k+p_{k+1}}(T)&=\lim_{k\to\infty}S_{p_k+q_k}(T)+\lim_{k\to\infty}\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2p_{k+1}-1}\Big\}\\
&=u+\lim_{k\to\infty}\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2[dp_k+1]-1}\Big\}\\
&=u+\frac{1}{2}\lim_{k\to\infty}\frac{2}{p_k}\Big\{\frac{1}{2+1/p_k}+\frac{1}{2+3/p_k}+\cdots+\frac{1}{(2[dp_k]+1)/p_k}\Big\}
\end{aligned}$$

$$= u + \frac{1}{2}\int_2^{2d} \frac{1}{x}dx = u + \frac{1}{2}\log d = v$$

對任意 λ, $1 \le \lambda \le d$, 令 $\tilde{p}_{k+1} = [\lambda p_k + 1]$, 則

$$\lim_{k\to\infty} S_{q_k+\tilde{p}_{k+1}}(T) = \lim_{k\to\infty} S_{p_k+q_k}(T) + \lim_{k\to\infty}\left\{\frac{1}{2p_k+1} + \frac{1}{2p_k+3} + \cdots + \frac{1}{2\tilde{p}_{k+1}-1}\right\}$$
$$= u + \frac{1}{2}\log\lambda$$

故此重排的部分和的極限點集爲 $[u, v]$。

B. $v = \log 2 + \dfrac{1}{2}\log t$, $u = v - \dfrac{1}{2}\log d$, $1 \le t < +\infty$, $1 \le d < +\infty$.

取 $q_1 = 1$, $p_1 = [t]$; $q_2 = [dq_1 + 1]$, $p_2 = [tq_2]$; $\ldots$; $q_{k+1} = [dq_k + 1]$, $p_{k+1} = [tq_{k+1}]$。即 A 中之 p_k, q_k 互換。設對應之重排爲 R, 而 A 中之重排爲 T, 則

$$S_{p_k+q_k}(T) + S_{p_k+q_k}(R) = \Big(1 + \frac{1}{3} + \cdots + \frac{1}{2p_k-1}\Big) - \Big(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2q_k}\Big)$$
$$+ \Big(1 + \frac{1}{3} + \cdots + \frac{1}{2q_k-1}\Big) - \Big(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p_k}\Big)$$

於是

$$\lim_{k\to\infty}\Big(S_{p_k+q_k}(T) + S_{p_k+q_k}(R)\Big) = 2\log 2$$

得

$$\lim_{k\to\infty} S_{p_k+q_k}(R) = 2\log 2 - \lim_{k\to\infty} S_{p_k+q_k}(T)$$
$$= 2\log 2 - (\log 2 - \frac{1}{2}\log t) = \log 2 + \frac{1}{2}\log t = v$$

同理

$$\lim_{k\to\infty} S_{p_k+q_{k+1}}(R) = 2\log 2 - \lim_{k\to\infty} S_{q_k+p_{k+1}}(T)$$
$$= 2\log 2 - (\log 2 - \frac{1}{2}\log t + \frac{1}{2}\log d) = v - \frac{1}{2}\log d = u$$

對任意 λ, $1 \le \lambda \le d$, 令 $\tilde{q}_{k+1} = [\lambda q_k + 1]$, 則 $\lim_{k\to\infty} S_{p_k+\tilde{q}_{k+1}}(R) = v - \frac{1}{2}\log\lambda$。故此重排的部分和的極限點集爲 $[u, v]$。

C. $v = +\infty$, $u = \log 2 - \dfrac{1}{2}\log s$, $+\infty > s > 0$.

當 $sk < 2$ 時, 取 $p_k = q_k = k$。當 $sk \ge 2$ 時, 取 $p_k = k!$, $q_k = [sp_k]$。先驗證 $\{p_k\}$ 及 $\{q_k\}$ 是嚴格遞增。當 $s(k+1) < 2$ 時, 顯然 $p_{k+1} = q_{k+1} > p_k = q_k$。當 $sk < 2 \le s(k+1)$ 時, $p_{k+1} = (k+1)! > k = p_k$, $q_{k+1} = [s(k+1)!] \ge 2k > k = q_k$。當 $2 \le sk$ 時, $p_{k+1} = (k+1)! > k! = p_k$, $q_{k+1} = [s(k+1)!] > [s \cdot k!] = q_k$ 。

再來計算 $\lim\limits_{k\to\infty} S_{p_k+q_k}(T)$。當 $s\geq 1$ 時, $q_k\geq p_k$ 。又 $k\to\infty$ 時 $p_k\to\infty$。於是

$$\begin{aligned}\lim_{k\to\infty} S_{p_k+q_k}(T) &= \lim_{k\to\infty}\Big\{\Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\Big\}\\ &= \lim_{k\to\infty}\Big\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2p_k-1}-\frac{1}{2p_k}\Big\}\\ &\quad -\lim_{k\to\infty}\Big\{\frac{1}{2p_k+2}+\cdots+\frac{1}{2q_k}\Big\}\\ &= \log 2-\frac{1}{2}\lim_{k\to\infty}\Big\{\frac{1}{p_k+1}+\frac{1}{p_k+2}+\cdots+\frac{1}{[sp_k]}\Big\}\\ &= \log 2-\frac{1}{2}\lim_{k\to\infty}\frac{1}{p_k}\Big\{\frac{1}{1+1/p_k}+\frac{1}{1+2/p_k}+\cdots+\frac{1}{[sp_k]/p_k}\Big\}\\ &= \log 2-\frac{1}{2}\int_1^s\frac{1}{x}dx=\log 2-\frac{1}{2}\log s=u\end{aligned}$$

當 $1>s>0$ 時, $q_k<p_k$。於是

$$\begin{aligned}\lim_{k\to\infty} S_{p_k+q_k}(T) &= \lim_{k\to\infty}\Big\{\Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\Big\}\\ &= \lim_{k\to\infty}\Big\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2p_k-1}-\frac{1}{2p_k}\Big\}\\ &\quad +\lim_{k\to\infty}\Big\{\frac{1}{2q_k+2}+\cdots+\frac{1}{2p_k}\Big\}\\ &= \log 2+\frac{1}{2}\lim_{k\to\infty}\Big\{\frac{1}{[sp_k]+1}+\frac{1}{[sp_k]+2}+\cdots+\frac{1}{p_k}\Big\}+0\\ &= \log 2+\frac{1}{2}\lim_{k\to\infty}\frac{1}{p_k}\Big\{\frac{1}{[sp_k]/p_k+1/p_k}+\frac{1}{[sp_k]/p_k+2/p_k}+\cdots+\frac{1}{1}\Big\}\\ &= \log 2+\frac{1}{2}\int_s^1\frac{1}{x}dx=\log 2-\frac{1}{2}\log s=u\end{aligned}$$

又

$$S_{q_k+p_{k+1}}(T)=S_{p_k+q_k}(T)+\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2p_{k+1}-1}\Big\}$$

對任意正整數 N,

$$\begin{aligned}\liminf_{k\to\infty} S_{q_k+p_{k+1}}(T) &= \lim_{k\to\infty} S_{p_k+q_k}(T)+\liminf_{k\to\infty}\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2(k+1)p_k-1}\Big\}\\ &= u+\liminf_{k\to\infty}\Big\{\frac{1}{2p_k+1}+\frac{1}{2p_k+3}+\cdots+\frac{1}{2(k+1)p_k-1}\Big\}\\ &\geq u+\frac{1}{2}\lim_{k\to\infty}\frac{2}{p_k}\Big\{\frac{1}{2+1/p_k}+\frac{1}{2+3/p_k}+\cdots+\frac{1}{(2Np_k+1)/p_k}\Big\}\\ &= u+\frac{1}{2}\int_2^{2N}\frac{1}{x}dx=u+\frac{1}{2}\log N\end{aligned}$$

因 N 可任意大, 得 $\lim\limits_{k\to\infty} S_{q_k+p_{k+1}}(T)=+\infty$。

對任意 $\lambda\ge 1$, 令 $\tilde{p}_{k+1}=[\lambda p_k]$, 則 $\lim\limits_{k\to\infty} S_{q_k+\tilde{p}_{k+1}}(T)=u+\dfrac{1}{2}\log\lambda$。故此重排的部分和的極限點集爲 $[u,+\infty]$。

D. $u=-\infty$, $v=\log 2+\dfrac{1}{2}\log s$, $+\infty>s>0$.

由 B 之討論及 C 之結果, 知只要將 C 中之 p_k, q_k 互換即可。

E. $-\infty=u=v$, 或 $u=v=+\infty$ 或 $-\infty=u$, $v=+\infty$.

E$_1$. 取 $p_k=k$, $q_k=k(k+1)/2$ 則

$$\begin{aligned}S_{q_k+p_{k+1}}(T)&=\Big(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{k(k+1)}\Big)\\&=\Big(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k+1}\Big)\\&\quad-\Big(\frac{1}{2k+2}+\frac{1}{2k+4}+\cdots+\frac{1}{k(k+1)}\Big)\end{aligned}$$

當 $k>2N+1$ 時,

$$-\Big(\frac{1}{2k+2}+\frac{1}{2k+4}+\cdots+\frac{1}{k(k+1)}\Big)<-\frac{1}{2}\Big(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+kN}\Big)$$

於是

$$\begin{aligned}\limsup_{k\to\infty} S_{q_k+p_{k+1}}(T)&\le\log 2-\frac{1}{2}\lim_{k\to\infty}\Big(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+kN}\Big)\\&=\log 2-\frac{1}{2}\log(N+1)\end{aligned}$$

對應之重排 T 滿足 $\lim\limits_{n\to\infty} S_n(T)=-\infty$。

E$_2$. 取 $p_k=k(k+1)/2$, $q_k=k$ 則 $\lim\limits_{n\to\infty} S_n(T)=+\infty$.

E$_3$. 取 $p_k=k\cdot[(k-1)!]^2$, $q_k=(k!)^2$ 則 $p_1=q_1=1$, $p_{k+1}=(k+1)q_k$, $q_k=kp_k$。於是

$$\begin{aligned}S_{q_k+p_{k+1}}(T)&=\Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_{k+1}-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\\&=\Big(1+\frac{1}{3}+\cdots+\frac{1}{2(k+1)q_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\\&=\Big(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2(k+1)q_k-1}-\frac{1}{2(k+1)q_k}\Big)\\&\quad+\Big(\frac{1}{2q_k+2}+\cdots+\frac{1}{2(k+1)q_k}\Big)\end{aligned}$$

$$\lim_{k\to\infty} S_{q_k+p_{k+1}}(T) = \log 2 + \frac{1}{2}\lim_{k\to\infty}\frac{1}{q_k}\Big(\frac{1}{1+1/q_k}+\cdots+\frac{1}{1+kq_k/q_k}\Big) = +\infty$$

對任意 $\lambda \ge 1$, 令 $\tilde{p}_{k+1} = [\lambda q_k]$。當 $k > \lambda$ 時, $q_k \le \tilde{p}_{k+1} < p_{k+1}$。於是

$$\begin{aligned}\lim_{k\to\infty} S_{q_k+\tilde{p}_{k+1}}(T) &= \lim_{k\to\infty}\Big(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2[\lambda q_k]-1}-\frac{1}{2[\lambda q_k]}\Big)\\ &\quad+\lim_{k\to\infty}\Big(\frac{1}{2q_k+2}+\cdots+\frac{1}{2[\lambda q_k]}\Big) = \log 2 + \frac{1}{2}\log\lambda.\end{aligned}$$

而

$$\begin{aligned}S_{p_k+q_k}(T) &= \Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q_k}\Big)\\ &= \Big(1+\frac{1}{3}+\cdots+\frac{1}{2p_k-1}\Big)-\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2kp_k}\Big)\\ &= \Big(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2p_k-1}-\frac{1}{2p_k}\Big)-\Big(\frac{1}{2p_k+2}+\cdots+\frac{1}{2kp_k}\Big)\end{aligned}$$

$$\lim_{k\to\infty} S_{p_k+q_k}(T) = \log 2 - \frac{1}{2}\lim_{k\to\infty}\frac{1}{p_k}\Big(\frac{1}{1+1/p_k}+\cdots+\frac{1}{1+kp_k/p_k}\Big) = -\infty$$

對任意 $\lambda \ge 1$, 令 $\tilde{q}_k = [\lambda p_k]$, 則 $\lim\limits_{k\to\infty} S_{p_k+\tilde{q}_k}(T) = \log 2 - \dfrac{1}{2}\log\lambda$。故此重排的部分和的極限點集爲 $[-\infty, +\infty]$。

參考文獻 1 也討論交錯 p 級數 $\sum_1^\infty (-1)^{n-1}\dfrac{1}{n^p}$, $0 < p < 1$。參考該文及以上作法也可得重排 $\sum_1^\infty (-1)^{n-1}\dfrac{1}{n^p}$ 的結果。

參考文獻

1. 王九逵, 胡門昌。交錯 p 級數的重排。數學傳播季刊, 39(4), 44-48, 2015。

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