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Eulerian 數的應用

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1. 前言

Eulerian 數早由 Euler [3] 於 1755 年提出。

本文目的是由 Eulerian 數定義出發去討論 Eulerian 數的性質及組合應用。

我們以 D 代表$\frac{d}{dx}$, 將幾何級數 $\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$ $(|x|<1)$ 連續做 xD 的運算得

$$
\begin{aligned}
(xD)^0(\frac{1}{1-x}) &= \sum_{k=0}^{\infty} kx^k = \frac{1}{1-x},\\
(xD)^1(\frac{1}{1-x}) &= \sum_{k=0}^{\infty} k^2x^k = \frac{x}{(1-x)^2}\\
(xD)^2(\frac{1}{1-x}) &= \sum_{k=0}^{\infty} k^3x^k = \frac{x^2+x}{(1-x)^3}\\
(xD)^3(\frac{1}{1-x}) &= \sum_{k=0}^{\infty} k^4x^k = \frac{x^3+4x^2+x}{(1-x)^4}\\
&\vdots
\end{aligned}
$$

於是 Euler 定義 $(xD)^k(\frac{1}{1-x})$ 中分子部分 x^k 的係數爲 Eulerian 數 $\left\langle {n \atop k} \right\rangle$。

$$
\begin{aligned}
&(xD)^0(\frac{1}{1-x})\text{的分子爲}1 = 1 = \left\langle {0 \atop 0} \right\rangle\\
&(xD)^1(\frac{1}{1-x})\text{的分子爲}x = 0+1x = \left\langle {1 \atop 0} \right\rangle + \left\langle {1 \atop 1} \right\rangle x\\
&(xD)^2(\frac{1}{1-x})\text{的分子爲}x^2+x = 0+1x+1x^2 = \left\langle {2 \atop 0} \right\rangle + \left\langle {2 \atop 1} \right\rangle x + \left\langle {2 \atop 2} \right\rangle x^2\\
&(xD)^3(\frac{1}{1-x})\text{的分子爲}x^3+4x^2+x = 0+1x+4x^2+x^3\\
&\qquad\qquad = \left\langle {3 \atop 0} \right\rangle + \left\langle {3 \atop 1} \right\rangle x + \left\langle {3 \atop 2} \right\rangle x^2 + \left\langle {3 \atop 3} \right\rangle x^3\\
&\qquad\vdots
\end{aligned}
$$

$$(xD)^n(\frac{1}{1-x})\text{的分子爲}\left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle x + \left\langle {n \atop 2} \right\rangle x^2 + \cdots + \left\langle {n \atop k} \right\rangle x^k + \cdots + \left\langle {n \atop n} \right\rangle x^n.$$

從而可得到以下 $\left\langle {n \atop k} \right\rangle$ 的表

$n \backslash k$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$\dots$	$k=n$
$n=0$	1	0	0	0	0	$\dots$	0
$n=1$	0	1	0	0	0	$\dots$	0
$n=2$	0	1	1	0	0	$\dots$	0
$n=3$	0	1	4	1	0	$\dots$	0
$n=4$	0	1	11	11	1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$n=n$	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\dots$	$\left\langle {n \atop n} \right\rangle$

由定義我們可以整理出一般式

$$(xD)^n(\frac{1}{1-x}) = \sum_{k=0}^{\infty} k^n x^k = \frac{\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k}{(1-x)^{n+1}}。\tag{1}$$

另外, 也可以用其組合意義來定義 Eulerian 數, $\left\langle {n \atop k} \right\rangle$ 的組合意義爲對稱群 S_n 中恰有 $k-1$ 個下降 (descent) 的 n-置換 (permutation) 的個數, 本文不從組合意義下去探討, 想了解請參閱 Bóna [1]。

在第二節中, 我們討論Eulerian 數與$\sum_{k=0}^{n} k^m$的關聯, 在第三節中我們找出 Eulerian 數與 n 維三角數還有差分之間的關係, 而最後一節我們則討論與 Stirling 數的關聯性。

2. Eulerian 數與 $\sum\limits_{k=0}^{n} k^m$ 的關聯

Eulerian 數具有以下兩個性質, 其證明請參閱 Comtet [2]:

- $\left\langle {n \atop k+1} \right\rangle = \left\langle {n \atop n-k} \right\rangle$,
- $\left\langle {n \atop k} \right\rangle = k\left\langle {n-1 \atop k} \right\rangle + (n-k+1)\left\langle {n-1 \atop k-1} \right\rangle$。

其封閉表達式爲

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n = \sum_{l=0}^{k} \binom{n+1}{k-l} (-1)^{k-l} l^n。$$

Eulerian 數與連續整數冪次和 $\sum_{k=0}^{n} k^m$ 有關, 其結果如下定理。

定理 1. 對任意正整數 n 及 m

$$\sum_{k=0}^{n} k^m = \sum_{i=1}^{m} \left\langle {m \atop i} \right\rangle \binom{n+m+1-i}{n-i}。$$

證明: 由(1) 式知

$$(\frac{1}{1-x})\sum_{k=0}^{\infty} k^m x^k = \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle x^k}{(1-x)^{n+1}},$$

兩邊同乘$\frac{1}{1-x}$可得

$$\begin{aligned}
(\frac{1}{1-x})\sum_{k=0}^{\infty} k^m x^k &= \frac{1}{1-x}\frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle x^k}{(1-x)^{m+1}} \\
&= (\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle x^k)(1-x)^{-(m+2)} \\
&= (\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle x^k)(\sum_{j=0}^{\infty} \binom{j+m+1}{m+1} x^j) \\
&= (\left\langle {m \atop 1} \right\rangle x + \left\langle {m \atop 2} \right\rangle x^2 + \cdots + \left\langle {m \atop m} \right\rangle x^m)(\sum_{j=0}^{\infty} \binom{j+m+1}{m+1} x^j)。
\end{aligned}$$

接著我們取兩邊x^n的係數

$$\begin{aligned}
\sum_{k=0}^{n} k^m &= \left\langle {m \atop 1} \right\rangle \binom{n-1+m+1}{m+1} + \left\langle {m \atop 2} \right\rangle \binom{n-2+m+1}{m+1} + \left\langle {m \atop 3} \right\rangle \binom{n-3+m+1}{m+1} \\
&\quad + \cdots + \left\langle {m \atop m} \right\rangle \binom{n-m+m+1}{m+1} \\
&= \left\langle {m \atop 1} \right\rangle \binom{n+m}{m+1} + \left\langle {m \atop 2} \right\rangle \binom{n+m-1}{m+1} + \cdots + \left\langle {m \atop m} \right\rangle \binom{n+1}{m+1} \\
&= \left\langle {m \atop 1} \right\rangle \binom{n+m}{n-1} + \left\langle {m \atop 2} \right\rangle \binom{n+m-1}{n-2} + \cdots + \left\langle {m \atop m} \right\rangle \binom{n+1}{n-m} \\
&= \sum_{i=0}^{m} \left\langle {m \atop i} \right\rangle \binom{n+m+1-i}{n-i}。
\end{aligned}$$

□

例 1.

$$\begin{aligned}
\sum_{k=0}^{n} k^3 &= \left\langle {3 \atop 1} \right\rangle \binom{n+3}{4} + \left\langle {3 \atop 2} \right\rangle \binom{n+2}{4} + \left\langle {3 \atop 3} \right\rangle \binom{n+1}{4} \\
&= 1\frac{(n+3)(n+2)(n+1)n}{4!} + 4\frac{(n+2)(n+1)n(n-1)}{4!} + 1\frac{(n+1)n(n-1)(n-2)}{4!}
\end{aligned}$$

$$=\frac{n^2(n+1)^2}{4!}。$$

德國數學家 Worpitzky [7] 於 1883 年定義 Eulerian 數爲

$$x^n=\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{n+x-k}{n}=\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k-1}{n}。$$

例 2.

$$3^2=\left\langle{2 \atop 0}\right\rangle\binom{6}{2}+\left\langle{2 \atop 1}\right\rangle\binom{5}{2}+\left\langle{2 \atop 2}\right\rangle\binom{4}{2}=1\times 6+1\times 3,$$

$$3^3=\left\langle{3 \atop 0}\right\rangle\binom{3+3-0}{3}+\left\langle{3 \atop 1}\right\rangle\binom{3+3-1}{3}+\left\langle{3 \atop 2}\right\rangle\binom{3+3-2}{3}+\left\langle{3 \atop 3}\right\rangle\binom{3+3-3}{3}$$

$$=0+1\times 10+4\times 4+1\times 1=27。$$

由此公式也可以得出

$$\begin{aligned}\sum_{k=0}^{n}k^m&=\sum_{k=0}^{n}\sum_{i=0}^{m}\left\langle{m \atop i}\right\rangle\binom{m+k-i}{m}\\&=\sum_{i=0}^{m}\left\langle{m \atop i}\right\rangle\sum_{k=0}^{n}\binom{m+k-i}{m}\\&=\sum_{i=0}^{m}\left\langle{m \atop i}\right\rangle\binom{m+k-i+1}{m+1}\\&=\sum_{i=0}^{m}\left\langle{m \atop i}\right\rangle\binom{m+k-i+1}{k-i}。\end{aligned}$$

與定理 1 相同。

定理 1 是有關 $\sum_{k=0}^{n}k^m$ 與 Eulerian 數的關聯, 自然地我們猜測是否交錯和 Eulerian 數也有類似的關係, 爲了達到此目的我們先證明以下引理。

引理 1. 對任意正整數 m

$$\sum_{k=0}^{\infty}k^m(-x)^k=((-x)D)^m(\frac{1}{1+x})=\frac{\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle(-1)^k x^k}{(1+x)^{m+1}}\ 。$$

證明: 將 x 以 $-x$ 代入(1) 即可得

$$((-x)D)^m(\frac{1}{1+x})=\sum_{k=0}^{\infty}k^m(-x)^k=\frac{\sum_{k=0}^{m}\left\langle{m \atop k}\right\rangle(-1)^k x^k}{(1+x)^{m+1}}$$

接著由引理1來求$\sum_{k=0}^{n}(-1)^k k^m$。

定理 2.

$$\sum_{k=0}^{n}(-1)^k k^m=\sum_{a=0}^{m}\sum_{j=0}^{n-a}(-1)^{n-j}\left\langle {m \atop a}\right\rangle\binom{n-a-j+m}{m}。$$

證明: 由引理1知

$$\sum_{k=0}^{\infty}k^m(-x)^k=((-x)D)^m(\frac{1}{1+x})=\frac{\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle(-1)^k x^k}{(1+x)^{m+1}}。$$

等號兩邊同除$\frac{1}{1-x}$

$$\begin{aligned}(\frac{1}{1-x})\sum_{k=0}^{m}k^m(-x)^k&=\frac{1}{1-x}\frac{\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle(-1)^k(x)^k}{(1+x)^{m+1}}\\ &=(\left\langle {m \atop 1}\right\rangle(-1)^1x+\left\langle {m \atop 2}\right\rangle(-1)^2x^2+\cdots+\left\langle {m \atop m}\right\rangle(-1)^m x^m)(1-x)^{-1}(1+x)^{-(m+1)}\\ &=(\left\langle {m \atop 1}\right\rangle(-)^1x+\left\langle {m \atop 2}\right\rangle(-1)^2x^2+\cdots+\left\langle {m \atop m}\right\rangle(-1)^m x^m)(\sum_{i=0}^{\infty}x^i)(\sum_{j=0}^{\infty}\binom{j+m}{m}(-x)^j)。\end{aligned}$$

我們取等號兩邊x^n的係數

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k^m&=\sum_{a=0}^{m}\left\langle {m \atop a}\right\rangle(-1)^a(\sum_{j=0}^{n-a}\binom{n-a-j+m}{m}(-1)^{n-a-j})\\ &=\sum_{a=0}^{m}\sum_{j=0}^{n-a}(-1)^{n-j}\left\langle {m \atop a}\right\rangle\binom{n-a-j+m}{m}。\end{aligned}$$

□

例 3.

$$\sum_{k=0}^{n}(-1)^k k^3=\sum_{a=0}^{3}\sum_{j=0}^{n-a}(-1)^{n-j}\left\langle {3 \atop a}\right\rangle\binom{n-a-j+3}{3}。$$

令$g=n-a-j$, 因此 g 的範圍是從 0 到 $n-a$,

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k k^3&=\sum_{a=0}^{3}\left\langle {3 \atop a}\right\rangle\sum_{g=0}^{n-a}(-1)^{a+g}\binom{g+3}{3}\\ &=\sum_{a=0}^{3}\left\langle {3 \atop a}\right\rangle(-1)^a\sum_{g=0}^{n-a}(-1)^g\binom{g+3}{3}。\end{aligned}$$

所以當$n=3$時,

$$左式=\sum_{k=0}^{3}(-1)^k k^3=-1+8-27=-20$$

$$
\begin{aligned}
\text{右式} &= \left\langle {3 \atop 1} \right\rangle (-1)^1((-1)^0\binom{3}{3} + (-1)^1\binom{4}{3} + (-1)^2\binom{5}{3}) \\
&\quad + \left\langle {3 \atop 2} \right\rangle (-1)^2((-1)^0\binom{3}{3} + (-1)^1\binom{4}{3}) + \left\langle {3 \atop 3} \right\rangle (-1)^3((-1)^0\binom{3}{3}) \\
&\quad = -1 \times (1-4+10) + 4 \times (1-4) + (-1) \times 1 = -7 - 12 - 1 = -20。
\end{aligned}
$$

接著我們已經知道

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k。$$

因此我們想探討如果只加到有限項 $\sum_{k=0}^{n} x^k$, 則對其做 $(xD)^k$ 會變成怎樣的形式。

定理 3. 令 n 爲正整數, 則

$$
\begin{aligned}
(xD)^k(\frac{1-x^{n+1}}{1-x}) &= \sum_{r=0}^{n} r^k x^r \\
&= -x^{n+1}\sum_{r=0}^{k}\binom{k}{r}(n+1)^{k-r}\sum_{m=0}^{r}\frac{\left\langle {r \atop m} \right\rangle x^m}{(1-x)^{m+1}} + \sum_{r=0}^{k}\frac{\left\langle {k \atop r} \right\rangle x^r}{(1-x)^{k+1}}\ 。
\end{aligned}
$$

證明:

$$
\begin{aligned}
&\sum_{r=0}^{n} r^k x^r = (xD)^k\left(\frac{1-x^{n+1}}{1-x}\right) \\
&= (xD)^k(\frac{-x^{n+1}}{1-x}) + (xD)^k(\frac{1}{1-x}) \\
&= (xD)^k(\frac{-x^{n+1}}{1-x}) + \sum_{r=0}^{k}\frac{\left\langle {k \atop r} \right\rangle x^r}{(1-x)^{k+1}} \\
&= -(xD)^{k-1}(\binom{1}{0}(n+1)x^{n+1}(\frac{1}{1-x}) + \binom{1}{1}x^{n+1}(xD)(\frac{1}{1-x})) + \sum_{r=0}^{k}\frac{\left\langle {k \atop r} \right\rangle x^r}{(1-x)^{k+1}} \\
&= -(xD)^{k-2}(\binom{1}{0}(n+1)^2x^{n+1}(\frac{1}{1-x}) + \binom{1}{1}(n+1)x^{n+1}(xD)(\frac{1}{1-x}) \\
&\quad + \binom{1}{0}(n+1)x^{n+1}(xD)(\frac{1}{1-x}) + \binom{1}{1}x^{n+1}(xD)^2(\frac{1}{1-x})) + \sum_{r=0}^{k}\frac{\left\langle {k \atop r} \right\rangle x^r}{(1-x)^{k+1}} \\
&= -(xD)^{k-2}(\binom{2}{0}(n+1)^2x^{n+1}(\frac{1}{1-x}) + \binom{2}{1}(n+1)x^{n+1}(xD)(\frac{1}{1-x}) \\
&\quad + \binom{2}{2}x^{n+1}(xD)^2(\frac{1}{1-x})) + \sum_{r=0}^{k}\frac{\left\langle {k \atop r} \right\rangle x^r}{(1-x)^{k+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \cdots \\
&= -\sum_{r=0}^{k}\binom{k}{r}(n+1)^{k-r}(xD)^r\left(\frac{1}{1-x}\right)+\sum_{r=0}^{k}\frac{\binom{k}{r}x^r}{(1-x)^{k+1}} \\
&= -x^{n+1}\sum_{r=0}^{k}\binom{k}{r}(n+1)^{k-r}\sum_{m=0}^{r}\frac{\binom{k}{m}x^m}{(1-x)^{m+1}}+\sum_{r=0}^{k}\frac{\binom{k}{r}x^r}{(1-x)^{k+1}}。 \qquad \square
\end{aligned}
$$

註: 在 Hsu 與 Shiue [5] 中有給出類似的證明。

3. $\left\langle {n \atop k} \right\rangle$ 與 n 維三角數的關聯

本節之目的是說明 Eulerian 數與 n 維三角數 [6] 的關聯。首先, 我們先介紹何謂 n 維三角數。像商家平常賣水果一般會先將水果排成一排, 形成一條線的樣子, 這樣就是一維的三角數。之後排不夠我們會排第二、三排等, 形成平面的樣子, 這就是二維的三角數, 然而水果還是很多我們就會將它疊到原本第一及第二排的上面依序排下去, 形成立體的樣子, 這就是三維的三角數, 依此類推可以推到 n 維的三角數。以數學式子來表示則推出以下定理。

定義 1. 令 t_n^k 代表每邊爲 n 的 k 維三角數,

$$t_n^k = t_n^{k-1} + t_{n-1}^{k-1} + \cdots + t_2^{k-1} + t_1^{k-1},$$

其中$t_n^1 = n$, 其堆疊的方式如下圖。

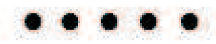

圖 1: $n = 5$ 的一維三角數 t_n^1

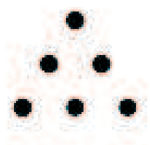

圖 2: $n = 3$ 的二維三角數 t_n^2

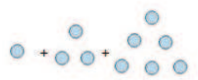

圖 3: $n = 3$ 的三維三角數 t_n^3

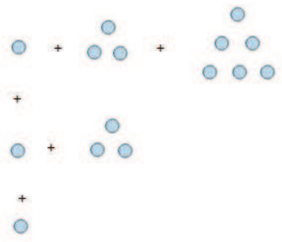

圖 4: $n = 3$ 的四維三角數 t_n^4

觀察 t_n^k $(k=1,2,\ldots)$ 的值:

$$
\begin{aligned}
t_n^1 &= n = \binom{n}{1} \\
t_n^2 &= 1+2+\cdots+n = \sum_{k=0}^{n}\binom{k}{1} = \frac{n(n+1)}{2!} = \binom{n+1}{2} \\
t_n^3 &= (1+2+\cdots+n)+(1+2+\cdots+n-1)+\cdots+(1+2)+(1) \\
&= t_n^2+t_{n-1}^2+\cdots+t_2^2+t_1^2 \\
&= \sum_{k=0}^{n}\binom{k+1}{2} = \frac{n(n+1)(n+2)}{3!} = \binom{n+2}{3} \\
t_n^4 &= t_n^3+t_{n-1}^3+\cdots+t_2^3+t_1^3 \\
&= \sum_{k=0}^{n}\binom{k+2}{3} = \frac{n(n+1)(n+2)(n+3)}{4!} = \binom{n+3}{4} \\
&\vdots \\
t_n^k &= t_n^{k-1}+t_{n-1}^{k-1}+\cdots+t_2^{k-1}+t_1^{k-1} \\
&= \sum_{r=0}^{n}\binom{n+k}{k-1} = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \binom{n+k-1}{k}\text{。}
\end{aligned}
$$

由上面式子我們可以得到以下定理。

定理 4.

$$t_n^k = \binom{n+k-1}{k}\text{。}$$

由定理4我們發現三角數有以下性質。

性質 1.

$$t_n^k = t_{n-1}^k + t_n^{k-1}\text{。}$$

證明:

$$t_n^k = \binom{n+k-1}{k} = \binom{n+k-2}{k} + \binom{n+k-2}{k-1} = t_{n-1}^k + t_n^{k-1}\text{。} \qquad \square$$

性質 2.

$$x^n = \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t_{x+1-i}^n\text{。}$$

證明:

$$x^n = \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle \binom{n+x-i}{n}$$

又

$$t^n_{x+1-i} = \binom{x+1-i+n-1}{n} = \binom{n+x-i}{n}$$

所以可推得

$$x^n = \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t^n_{x+1-i}。 \qquad \square$$

接著我們藉由觀察以下的圖來找出規律。

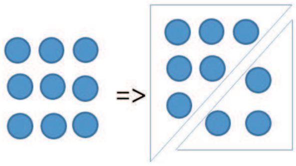

圖 5: 3^2 的圖我們可以將其拆成 $t^2_3 + t^2_2$

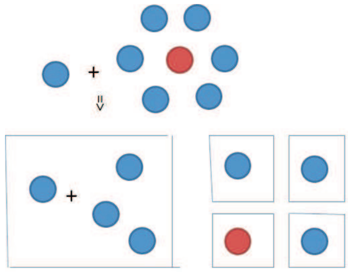

圖 6: 2^3 的圖我們可以將其拆成$t^3_2 + 4t^3_1$

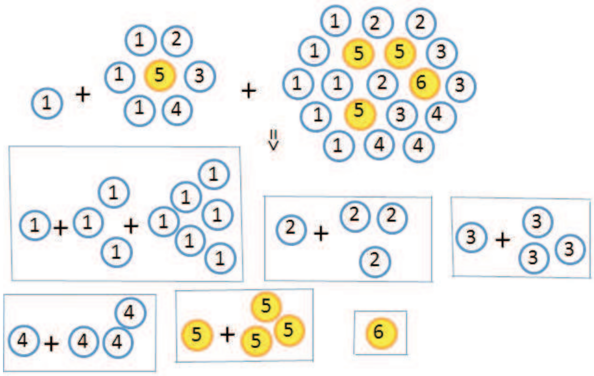

圖 7: 3^3 的圖我們可以將其拆成 $t^3_3 + 4t^3_2 + t^3_1$

圖 6 和圖 7 爲立體圖, 我們將前面的圖疊加在後面的圖後就可以發現我們所說的三角數, 我們會發現當我們從 k^n 到 $(k+1)^n$ 時就相當於在往下多一層而上面不變, 而從圖 5 到圖 6 我們會發現其三角數的維度會增加 1, 且其係數恰爲 Eulerian 數, 確實與性質 2 相同。

註: 另外圖7爲何是這樣分割, 請查閱[6]。

另外我們也發現三角數與差分 $\triangle h$ 也有關係, 差分 $\triangle h$ 如下定義:

定義 2.

$$\triangle f(x) = f(x) - f(x-1)$$

$$\triangle^{k+1} f(x) = \triangle^k f(x) - \triangle^k f(x-1)$$

令 $h_k = k^n$, 則

$$\begin{aligned}
\triangle h_k &= k^n - (k-1)^n \\
\triangle^2 h_k &= \triangle(\triangle h_k) = \triangle h_k - \triangle h_{k-1} = (k^n - (k-1)^n) - ((k-1)^n - (k-2)^n) \\
&= k^n + (k-2)^n \\
&\vdots \\
\triangle^l h_k &= \triangle^{l-1} h_k - \triangle^{l-1} h_{k-1}
\end{aligned}$$

因此由性質1及性質2我們可以得到其差分表。

$$\begin{array}{cccccc}
k^n & \triangle h & \triangle^2 h & \triangle^3 h & \triangle^4 h & \ldots \\
1^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{2-i}^n & & & & & \\
 & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{3-i}^{n-1} & & & & \\
2^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{3-i}^n & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{4-i}^{n-2} & & & \\
 & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{4-i}^{n-1} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{5-i}^{n-3} & & \\
3^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{4-i}^n & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{5-i}^{n-2} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{6-i}^{n-4} & \\
 & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{5-i}^{n-1} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{6-i}^{n-3} & & \\
4^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{5-i}^n & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{6-i}^{n-2} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{7-i}^{n-4} & \\
 & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{6-i}^{n-1} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{7-i}^{n-3} & & \\
5^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{6-i}^n & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{7-i}^{n-2} & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{8-i}^{n-4} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(k-1)^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{k-i}^n & & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{k+1-i}^{n-2} & & & \\
 & \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{k+1-i}^{n-1} & & & & \\
k^n = \sum_{i=0}^n \genfrac{\langle}{\rangle}{0pt}{1}{n}{i} t_{k+1-i}^n & & & & &
\end{array}$$

從而發現以下定理。

定理 5. 若$h_k = k^n$

$$\triangle^l h_k = \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{k-i+1}^{n-l} = \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle \binom{n-l+k-i}{k-i+1}$$

證明: 我們將利用數學歸納法來證明。

當$l=1$時,

$$\triangle^1 h_x = x^n-(x-1)^n=\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{x+1-i}^{n}-\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{x-i}^{n}=\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{x+1-i}^{n-1} \text{ 成立。}$$

假設$l=a$時,

$$\triangle^a h_k=\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{k-i+1}^{n-a} \text{ 成立。}$$

則當$l=a+1$時,

$$\begin{aligned}\triangle^{a+1}h_k &= \triangle(\triangle^a h_k)\\ &= \sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{k-i+1}^{n-a}-\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{k-i}^{n-a}\\ &= \sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle (t_{k-i+1}^{n-a}-t_{k-i}^{n-a})\\ &= \sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{k-i+1}^{n-(a+1)} \text{ 成立。}\end{aligned}$$

由數學歸納法原理可知

$$\triangle^l h_k=\sum_{i=0}^{n}\left\langle{n \atop i}\right\rangle t_{k-i+1}^{n-l}\text{ 。} \qquad \square$$

4. $\left\langle{n \atop k}\right\rangle$ 與 Stirling 數的關聯

Stirling 數 [2] 有分爲第一類 $s(n,k)$ 跟第二類 $S(n,k)$, 分別由下式定義

$$\text{第一類: } [x]_n=\sum_{k=0}^{n}s(n,k)x^k, \tag{2}$$

$$\text{第二類: } x^n=\sum_{k=0}^{n}S(n,k)[x]_k, \tag{3}$$

其中$[x]_k=x(x-1)(x-2)\cdots(x-k+1)$, $[x]_0=1$。
第二類Stirling 數具有以下性質

$$\sum_{k=0}^{\infty}k^n x^k=\sum_{k=0}^{n}k!S(n,k)\frac{x^k}{(1-x)^{k+1}}, \tag{4}$$

$$S(p,k)=kS(p-1,k)+S(p-1,k-1)\text{。} \tag{5}$$

其證明可分別參閱 Gould[4] 和 Comtet[2]。
此節我們想要尋找 Stirlng 數與 Eulerian 數的關係。

定理 6. 對任意正整數n有

$$\sum_{k=0}^{n} 2^{n-k} \left\langle {n \atop k} \right\rangle = \sum_{k=0}^{n} S(n,k)k!。$$

證明: 由(1) 與 (4) 得

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \frac{x^k}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} k!S(n,k)\frac{x^k}{(1-x)^{k+1}},$$

可推得

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{k=0}^{n} k!S(n,k)x^k(1-x)^{n-k}。$$

當 $x = \frac{1}{2}$ 時

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle (\frac{1}{2})^k = \sum_{k=0}^{n} k!S(n,k)(\frac{1}{2})^k(\frac{1}{2})^{n-k} = \sum_{k=0}^{n} k!S(n,k)(\frac{1}{2})^n,$$

等式兩邊同乘 2^n

$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^{n-k} = \sum_{k=0}^{n} k!S(n,k)。$$

□

例 4. 當 $n = 3$ 時,

$$\begin{aligned}\sum_{k=0}^{3} \left\langle {3 \atop k} \right\rangle 2^{3-k} &= \left\langle {3 \atop 0} \right\rangle 2^3 + \left\langle {3 \atop 1} \right\rangle 2^2 + \left\langle {3 \atop 2} \right\rangle 2^1 + \left\langle {3 \atop 3} \right\rangle 2^0 = 1\times 4 + 4\times 2 + 1\times 1 = 13\\ &= \sum_{k=0}^{3} k!S(3,k) = 0!S(3,0) + 1!S(3,1) + 2!S(3,2) + 3!S(3,3)\\ &= 1 + 2\times 3 + 6 = 13。\end{aligned}$$

由上面定理 6 可反推得以下定理。

定理 7.

$$n! = \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^{n+j} 2^{j-k} \left\langle {j \atop k} \right\rangle s(n,j),$$

其中$s(n,j)$為 *Stirling* 數第一型。

證明: 由定理6可得知,

$$\sum_{k=0}^{n} k!S(n,k) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^{n-k}。$$

其矩陣式爲

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & S(1,1) & 0 & \dots & 0 \\ 0 & S(2,1) & S(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & S(n,1) & S(n,2) & \dots & S(n,n) \end{pmatrix} \begin{pmatrix} 0! \\ 1! \\ 2! \\ \vdots \\ n! \end{pmatrix} = \begin{pmatrix} 1 \\ \sum_{k=0}^{1} \left\langle {1 \atop k} \right\rangle 2^{1-k} \\ \sum_{k=0}^{2} \left\langle {2 \atop k} \right\rangle 2^{2-k} \\ \vdots \\ \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^{n-k} \end{pmatrix}$$

由[2]可知上式中的第二類Stirling 矩陣之反矩陣爲

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & s(1,1) & 0 & \dots & 0 \\ 0 & s(2,1) & s(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s(n,1) & s(n,2) & \dots & s(n,n) \end{pmatrix}$$

故兩邊同乘上反矩陣後可得

$$\begin{pmatrix} 0! \\ 1! \\ 2! \\ \vdots \\ n! \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & s(1,1) & 0 & \dots & 0 \\ 0 & s(2,1) & s(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s(n,1) & s(n,2) & \dots & s(n,n) \end{pmatrix} \begin{pmatrix} 1 \\ \sum_{k=0}^{1} \left\langle {1 \atop k} \right\rangle 2^{1-k} \\ \sum_{k=0}^{2} \left\langle {2 \atop k} \right\rangle 2^{2-k} \\ \vdots \\ \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle 2^{n-k} \end{pmatrix},$$

比較矩陣內的元素可得

$$n! = \sum_{j=1}^{n} s(n,j) \sum_{k=0}^{j} \left\langle {j \atop k} \right\rangle 2^{j-k} = \sum_{j=1}^{n} \sum_{k=0}^{j} 2^{j-k} \left\langle {j \atop k} \right\rangle s(n,j)\ 。$$

又我們已知

$$n! = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle,$$

因此可得

$$n! = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle = \sum_{j=1}^{n} \sum_{k=0}^{j} 2^{j-k} \left\langle {j \atop k} \right\rangle s(n,j)。 \qquad \square$$

例 5. 當 $n=3$ 時,

$$\begin{aligned}
3! &= \sum_{k=0}^{3}\genfrac{\langle}{\rangle}{0pt}{}{3}{k} = \genfrac{\langle}{\rangle}{0pt}{}{3}{0}+\genfrac{\langle}{\rangle}{0pt}{}{3}{1}+\genfrac{\langle}{\rangle}{0pt}{}{3}{2}+\genfrac{\langle}{\rangle}{0pt}{}{3}{3}=6\\
&= \sum_{j=1}^{3}\sum_{k=0}^{j}2^{j-k}\genfrac{\langle}{\rangle}{0pt}{}{j}{k}s(3,j) = \sum k=0^1 2^{1-k}\genfrac{\langle}{\rangle}{0pt}{}{1}{k}s(3,1)\\
&\quad+\sum_{k=0}^{2}2^{2-k}\genfrac{\langle}{\rangle}{0pt}{}{2}{k}s(3,2)+\sum k=0^2 2^{3-k}\genfrac{\langle}{\rangle}{0pt}{}{3}{k}s(3,3)\\
&= 2^{1-0}\genfrac{\langle}{\rangle}{0pt}{}{1}{0}s(3,1)+2^{1-1}\genfrac{\langle}{\rangle}{0pt}{}{1}{1}s(3,1)+2^{2-0}\genfrac{\langle}{\rangle}{0pt}{}{2}{0}s(3,2)+2^{2-1}\genfrac{\langle}{\rangle}{0pt}{}{2}{1}s(3,2)\\
&\quad+2^{2-2}\genfrac{\langle}{\rangle}{0pt}{}{2}{2}s(3,2)+2^{3-0}\genfrac{\langle}{\rangle}{0pt}{}{3}{0}s(3,3)+2^{3-1}\genfrac{\langle}{\rangle}{0pt}{}{3}{1}s(3,3)+2^{3-2}\genfrac{\langle}{\rangle}{0pt}{}{3}{2}s(3,3)\\
&\quad+2^{3-3}\genfrac{\langle}{\rangle}{0pt}{}{3}{3}s(3,3)\\
&= 0+1\times 2+0+2\times(-3)+1\times(-3)+0+4\times 1+2\times 4+1=6。
\end{aligned}$$

引理 2.

$$\sum_{k=0}^{n}k^m=\sum_{r=0}^{n}r!S(m,r)\binom{n+1}{r+1}。$$

證明: 由 (4) 可知

$$\sum_{r=0}^{\infty}r^m x^r=\sum_{r=0}^{n}r!S(m,r)\frac{x^r}{(1-x)^{r+1}},$$

2邊同乘$\frac{1}{1-x}$

$$\begin{aligned}
\frac{1}{1-x}\sum_{r=0}^{\infty}r^m x^r &= \sum_{r=0}^{n}r!S(m,r)\frac{x^r}{(1-x)^{r+2}}\\
&= \sum_{r=0}^{n}r!S(m,r)x^r\sum_{t=0}^{\infty}\binom{t+r+1}{t}x^t。
\end{aligned}$$

我們取兩邊 x^n 的係數

$$\begin{aligned}
\sum_{k=0}^{n}k^m &= \sum_{r=0}^{n}r!S(m,r)\binom{n-r+r+1}{n-r}\\
&= \sum_{r=0}^{n}r!S(m,r)\binom{n+1}{n-r}\\
&= \sum_{r=0}^{n}r!S(m,r)\binom{n+1}{r+1}。
\end{aligned}$$

□

註: 在[2]中也證明了引理2, 但其方法與本文不同。

我們導出兩個第二類 Stirling 數與 Eulerian 數的關係式。定理八將 Stirling 數用 Eulerian 數表示, 而定理九將 Eulerian 數用 Stirling 數表示。

定理 8.

$$S(m,n)=\frac{1}{n!}\sum_{j=1}^{n}\sum_{r=0}^{j}(-1)^{n+j}\binom{n}{j}\left\langle {m \atop r}\right\rangle\binom{n+m+1-r}{n-r}。$$

證明: 由引理2和定理1可推得

$$\sum_{r=0}^{n} r!S(m,r)\binom{n+1}{r+1}=\sum_{r=0}^{n}\left\langle {m \atop r}\right\rangle\binom{n+m+1-r}{n-r}。$$

將其以矩陣形式表示如下

$$\begin{pmatrix} \binom{0}{0} & 0 & 0 & \dots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \dots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \dots & \binom{n+1}{n+1} \end{pmatrix}\begin{pmatrix} 0 \\ 0!S(m,0) \\ 1!S(m,1) \\ \vdots \\ n!S(m,n) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \sum_{r=0}^{1}\left\langle {m \atop r}\right\rangle\binom{1+m+1-r}{1-r} \\ \vdots \\ \sum_{r=0}^{n}\left\langle {m \atop r}\right\rangle\binom{n+m+1-r}{n-r} \end{pmatrix}$$

由二項式定理可知上式中的 $\binom{n}{k}$ 矩陣之反矩陣 [2] 爲

$$\begin{pmatrix} \binom{0}{0} & 0 & \dots & 0 \\ -\binom{1}{0} & \binom{1}{1} & \dots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+0}\binom{n+1}{0} & (-1)^{n+1}\binom{n+1}{1} & \dots & (-1)^{2n+1}\binom{n+1}{n+1} \end{pmatrix}。$$

因此

$$\begin{pmatrix} 0 \\ 0!S(m,0) \\ 1!S(m,1) \\ \vdots \\ n!S(m,n) \end{pmatrix}=\begin{pmatrix} \binom{0}{0} & 0 & 0 & \dots & 0 \\ -\binom{1}{0} & \binom{1}{1} & 0 & \dots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+0}\binom{n+1}{0} & (-1)^{n+1}\binom{n+1}{1} & (-1)^{n+1}\binom{n+1}{2} & \dots & (-1)^{2n+1}\binom{n+1}{n+1} \end{pmatrix}$$

$$\times \begin{pmatrix} 0 \\ 0 \\ \sum_{r=0}^{1} \left\langle {m \atop r} \right\rangle \binom{1+m+1-r}{1-r} \\ \vdots \\ \sum_{r=0}^{n} \left\langle {m \atop r} \right\rangle \binom{n+m+1-r}{n-r} \end{pmatrix}。$$

可推得

$$\begin{aligned} n!S(m,n) &= \sum_{j=1}^{n}(-1)^{n+j}\binom{n}{j}\sum_{r=0}^{j}\left\langle {m \atop r} \right\rangle \binom{n+m+1-r}{n-r} \\ S(m,n) &= \frac{1}{n!}\sum_{j=1}^{n}(-1)^{n+j}\binom{n}{j}\sum_{r=0}^{j}\left\langle {m \atop r} \right\rangle \binom{n+m+1-r}{n-r} \\ &= \frac{1}{n!}\sum_{j=1}^{n}\sum_{r=0}^{j}(-1)^{n+j}\binom{n}{j}\binom{n+m+1-r}{n-r}\left\langle {m \atop r} \right\rangle。 \end{aligned}$$ □

引理 3. 令 $xD = x(\frac{d}{dx})$ 可推得

$$(xD)^m = \sum_{k=0}^{m} S(m,k)x^k D^k$$

證明: 我們利用數學歸納法來證明。

當 $m=1$ 時,

$$xD = \sum_{k=0}^{1} S(1,k)xD = S(1,0)xD + S(1,1)xD = xD\text{成立。}$$

設 $m=n$ 時,

$$(xD)^n = \sum_{k=0}^{n} S(n,k)x^k D^k\text{成立。}$$

則 $m=n+1$ 時,

$$\begin{aligned} (xD)^{n+1} &= (xD)(xD)^n = xD\sum_{k=0}^{n} S(n,k)x^k D^k \\ &= \sum_{k=0}^{n} S(n,k)kx^k D^k + \sum_{k=0}^{n} S(n,k)x^{k+1}D^{k+1} \\ &= \sum_{k=0}^{n+1} S(n,k)kx^k D^k + \sum_{k=1}^{n+1} S(n,k-1)x^k D^k = \sum_{k=0}^{n+1} S(n+1,k)x^k D^k。 \end{aligned}$$ □

引理 4.

$$\sum_{k=0}^{n}\binom{s+k}{k}\binom{n-k}{m}=\binom{s+n+1}{n-m}$$

證明: 當 $n=1$ 時,

$$\text{討論}\binom{s}{0}\binom{1}{m}+\binom{s+1}{1}\binom{0}{m}\text{是否等於}\binom{s+2}{1-m}$$

- 當 $m=0$ 時,
$$\binom{s}{0}\binom{1}{0}+\binom{s+1}{1}\binom{0}{0}=s+2=\binom{s+2}{1}$$
- 當 $m=1$ 時,
$$\binom{s}{0}\binom{1}{1}+\binom{s+1}{1}\binom{0}{1}=1=\binom{s+2}{1-1}$$
- 當 $m\geqslant 2$ 時,
$$\binom{s}{0}\binom{1}{m}+\binom{s+1}{1}\binom{0}{m}=0=\binom{s+2}{1-m}$$

因此可知 $n=1$ 是對的。

設 $n=h$ 時,

$$\sum_{k=0}^{h}\binom{s+k}{k}\binom{h-k}{m}=\binom{s+h+1}{h-m}\text{成立。}$$

則 $n=h+1$時,

$$\begin{aligned}\sum_{k=0}^{h+1}\binom{s+k}{k}\binom{h+1-k}{m}&=\sum_{k=0}^{h}\binom{s+k}{k}(\binom{h-k}{m}+\binom{h-k}{m-1})\\&\quad+\binom{s+h+1}{h+1}\binom{h+1-h-1}{m}\\&=\binom{s+h+1}{h-m}+\binom{s+h+1}{h-m-1}=\binom{s+h+2}{h-m}\ \text{成立。}\end{aligned}$$

由數學歸納法原理得知

$$\sum_{k=0}^{n}\binom{s+k}{k}\binom{n-k}{m}=\binom{s+n+1}{n-m}$$ □

定理 9.

$$\left\langle{n \atop i}\right\rangle=\sum_{k=0}^{n}(-1)^{i-k}\binom{n-k}{i-k}k!S(n,k)\text{。}$$

證明: 由上面引理3可得

$$(xD)^n(\frac{1}{1-x}) = \sum_{k=0}^{n} S(n,k)x^k \sum_{n=0}^{\infty} \frac{(h+k)!}{h!}x^h$$
$$\frac{\sum_{r=0}^{\infty} \left\langle {n \atop r} \right\rangle x^r}{(1-x)^{n+1}} = \sum_{k=0}^{n} S(n,k)x^k \sum_{n=0}^{\infty} \frac{(h+k)!}{h!}x^k。$$

兩邊同乘 $(1-x)^{n+1}$,

$$\begin{aligned}
\sum_{r=0}^{\infty} \left\langle {n \atop r} \right\rangle x^r &= \sum_{k=0}^{n} S(n,k)x^k \sum_{h=0}^{\infty} \frac{(h+k)!}{h!}x^h(1-x)^{n+1} \\
&= \sum_{k=0}^{n} S(n,k)x^k \sum_{h=0}^{\infty} \frac{(h+k)!}{h!}x^h \sum_{i=0}^{n+1} \binom{n+1}{i}(-1)^i x^i \\
&= \sum_{k=0}^{n} \sum_{h=0}^{\infty} \sum_{i=0}^{n+1} S(n,k)\frac{(h+k)!}{h!}\binom{n+1}{i}(-1)^i x^{k+h+i}。
\end{aligned}$$

取兩邊 x^i 項可得,

$$\begin{aligned}
\left\langle {n \atop i} \right\rangle x^i &= \sum_{k=0}^{n} S(n,k)x^k \sum_{h=0}^{\infty} \frac{(h+k)!}{h!}x^h \binom{n+1}{i-k-h}(-1)^{i-k-h}x^{i-k-h} \\
&= \sum_{k=0}^{n} \sum_{h=0}^{\infty} S(n,k)k!\binom{h+k}{h}\binom{n+1}{i-k-h}(-1)^{i-k-h}x^i \\
&= \sum_{k=0}^{n} S(n,k)k! \sum_{h=0}^{\infty} \binom{h+k}{h}\binom{i-h-k-n-2}{-n-2}x^i \\
&= \sum_{k=0}^{n} S(n,k)k! \sum_{h=0}^{i-k-n-2} \binom{h+k}{h}\binom{i-h-k-n-2}{-n-2}x^i
\end{aligned}$$

由引理 4 可知

$$\begin{aligned}
&= \sum_{k=0}^{n} S(n,k)k!\binom{i-n-1}{i-k}x^i \\
&= \sum_{k=0}^{n} S(n,k)k!(-1)^{i-k}\binom{n-k}{i-k}x^i
\end{aligned}$$

因此可得出

$$\left\langle {n \atop i} \right\rangle = \sum_{k=0}^{n} (-1)^{i-k}\binom{n-k}{i-k}k!S(n,k)。$$

例 6. 當$n=3$且$i=2$時,

$$\left\langle {3 \atop 2} \right\rangle = \sum_{k=0}^{3} (-1)^{2-k}\binom{3-k}{2-k}k!S(3,k)$$

$$= (-1)^2\binom{3}{2}0!S(3,0) + (-1)^1\binom{2}{1}1!S(3,1) + (-1)^0\binom{1}{0}2!S(3,2)$$
$$= 0 + (-2) + 6 = 4$$

當$n=4$且$i=2$時,

$$\left\langle {4 \atop 2} \right\rangle = \sum_{k=0}^{4}(-1)^{2-k}\binom{4-k}{2-k}k!S(4,k)$$
$$= (-1)^2\binom{4}{2}0!S(4,0) + (-1)^1\binom{3}{1}1!S(4,1) + (-1)^0\binom{2}{0}2!S(4,2)$$
$$= 0 + (-3) + 14 = 11$$

最後, 在前面的第三節裡我們提到三角數與Eulerian 的關係, 我們將用性質2與 Stirling 數的定義來推得 Stirling 數與三角數的關係

定理 10.

$$S(n,k) = \sum_{j=0}^{x}\sum_{i=0}^{n}(-1)^{x+j}\binom{x}{j}\left\langle {n \atop i} \right\rangle t^n_{j+1-i}$$

證明: 由性質2與 Stirling 數的定義可推得

$$x^n = \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{x+1-i} = \sum_{k=0}^{n}S(n,k)k!\binom{x}{k}。$$

將其以矩陣形式表示如下

$$\begin{pmatrix} \binom{0}{0} & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \ldots & 0 & \ldots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ \binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \ldots & \binom{x}{x} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \ldots & \binom{n}{x} & \ldots & \binom{n}{n} \end{pmatrix} \begin{pmatrix} 0 \\ 0!S(m,0) \\ 1!S(m,1) \\ \vdots \\ x!S(m,x) \\ \vdots \\ n!S(m,n) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{1-i} \\ \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{2-i} \\ \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{3-i} \\ \vdots \\ \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{x+1-i} \\ \vdots \\ \sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t^n_{n+1-i} \end{pmatrix}$$

由二項式定理可知上式中的$\binom{n}{k}$矩陣之反矩陣[2]爲

$$\begin{pmatrix} \binom{0}{0} & 0 & \cdots & 0 \\ -\binom{1}{0} & \binom{1}{1} & \cdots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{x+0}\binom{x}{0} & (-1)^{x+1}\binom{x}{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+0}\binom{n}{0} & (-1)^{n}\binom{n}{1} & \cdots & (-1)^{2n}\binom{n}{n} \end{pmatrix}。$$

因此

$$\begin{pmatrix} 0 \\ 0!S(m,0) \\ 1!S(m,1) \\ \vdots \\ x!S(m,x) \\ \vdots \\ n!S(m,n) \end{pmatrix} = \begin{pmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ -\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{x}\binom{x}{0} & (-1)^{x+1}\binom{x}{1} & (-1)^{x+2}\binom{x}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{n}\binom{n}{0} & (-1)^{n+1}\binom{n}{1} & (-1)^{n+2}\binom{n}{2} & \cdots & (-1)^{2n}\binom{n}{n} \end{pmatrix}$$

$$\times \begin{pmatrix} \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{1-i}^{n} \\ \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{2-i}^{n} \\ \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{3-i}^{n} \\ \vdots \\ \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{x+1-i}^{n} \\ \vdots \\ \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle t_{n+1-i}^{n} \end{pmatrix}。$$

可推得

$$\begin{aligned} x!S(m,x) &= \sum_{j=0}^{x}(-1)^{x+j}\binom{x}{j}\sum_{i=0}^{n}\left\langle {n \atop i} \right\rangle t_{j+1-i}^{i} \\ S(m,x) &= \frac{1}{x!}\sum_{j=0}^{x}(-1)^{x+j}\binom{x}{j}\sum_{r=0}^{n}\left\langle {n \atop i} \right\rangle t_{j+1-i}^{i} \\ &= \frac{1}{x!}\sum_{j=0}^{x}\sum_{r=0}^{n}(-1)^{x+j}\binom{x}{j}\left\langle {n \atop i} \right\rangle t_{j+1-i}^{i}。 \end{aligned}$$

□

例 7. 當$n=4$且$x=2$時,

$$
\begin{aligned}
S(4,2) &= 7 \\
&= \frac{1}{2!}\sum_{j=0}^{2}(-1)^{2+j}\binom{2}{j}\sum_{i=0}^{4}t^4_{j+1-i} \\
&= \frac{1}{2}((-1)^2\binom{2}{0}\sum_{i=0}^{4}t^4_{1-i}+(-1)^3\binom{2}{1}\sum_{i=0}^{4}t^4_{2-i}+(-1)^4\binom{2}{2}\sum_{i=0}^{4}t^4_{3-i}) \\
&= \frac{1}{2}((-2)(\left\langle{4 \atop 1}\right\rangle t^4_1)+(\left\langle{4 \atop 1}\right\rangle t^4_2+\left\langle{4 \atop 2}\right\rangle t^4_1)) \\
&= \frac{1}{2}(-2+5+11)=7
\end{aligned}
$$

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