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利用向量三重積求兩歪斜線的公垂線段的兩端點座標的公式解法

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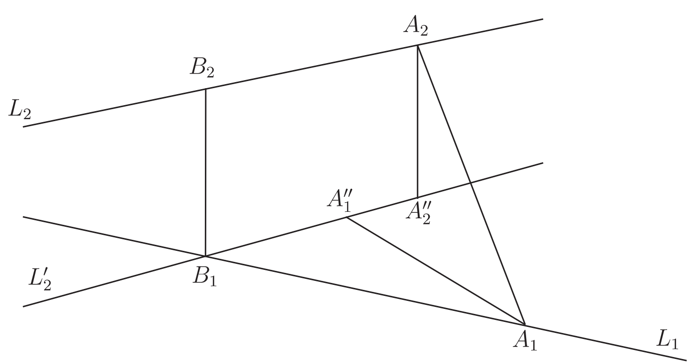
研究目的：

延續投稿在龍騰數亦優第9刊主題爲 “利用正射影及外積的概念求兩歪斜線的公垂線段的距離及兩端點座標”的結論 “$\overrightarrow{OB_2}=\overrightarrow{OA_2}+\dfrac{\overrightarrow{A_2A_1}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_2}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_2}$”, 利用同樣的推理方法求得 “$\overrightarrow{OB_1}=\overrightarrow{OA_1}+\dfrac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_1}$”, 並簡化向量三重積 $\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})$ 及 $\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})$ 求得兩歪斜線的公垂線段的兩端點座標的公式解法。

研究過程:

一、利用同樣的推理方法求得 $\overrightarrow{OB_1}=\overrightarrow{OA_1}+\dfrac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_1}$



如上圖所示, 已知空間直角座標系中, O 爲原點, 兩歪斜線 L_1 與 L_2 分別通過點 A_1、點

A_2, L_1 與 L_2 的方向向量分別爲 $\overrightarrow{d_1}$ 與 $\overrightarrow{d_2}$, 四邊形 $A_2A_2''B_1B_2$ 爲矩形, $\angle A_1A_1''B_1=90°$, $\overline{B_1B_2}\perp L_1$ 與 L_2, 試求點 B_1 的座標。

解: $\because\ \overrightarrow{A_1A_1''}\perp\overrightarrow{B_1A_2}\Rightarrow\overrightarrow{A_1A_1''}\cdot\overrightarrow{B_1A_2}=0$,

又 $\overrightarrow{A_1A_1''}\parallel\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})$ 且 $\overrightarrow{B_1A_2}=\overrightarrow{A_1A_2}-\overrightarrow{A_1B_1}=\overrightarrow{A_1A_2}-t\overrightarrow{d}_1$,

$$\overrightarrow{A_1A_1''}\cdot\overrightarrow{B_1A_2}=0\Rightarrow[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]\cdot(\overrightarrow{A_1A_2}-t\overrightarrow{d_1})=0,$$

解得 $t=\dfrac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}$

因此

$$\overrightarrow{OB_1}=\overrightarrow{OA_1}+t\overrightarrow{d_1}=\overrightarrow{OA_1}+\frac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_1}。$$

二、已知 $\overrightarrow{\alpha}=(l_1,m_1,n_1)$, $\overrightarrow{\beta}=(l_2,m_2,n_2)$, $\overrightarrow{\gamma}=(l_3,m_3,n_3)$,

則 $\overrightarrow{\alpha}\times(\overrightarrow{\beta}\times\overrightarrow{\gamma})=(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})\overrightarrow{\beta}-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})\overrightarrow{\gamma}$。

證明: $\because\ \overrightarrow{\beta}\times\overrightarrow{\gamma}=(m_2n_3-m_3n_2,n_2l_3-n_3l_2,l_2m_3-l_3m_2)$,

$$\begin{aligned}
&\therefore\ \overrightarrow{\alpha}\times(\overrightarrow{\beta}\times\overrightarrow{\gamma})\\
&=(m_1(l_2m_3-l_3m_2)-n_1(n_2l_3-n_3l_2),n_1(m_2n_3-m_3n_2)-l_1(l_2m_3-l_3m_2),\\
&\quad l_1(n_2l_3-n_3l_2)-m_1(m_2n_3-m_3n_2))\\
&=((\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})l_2-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})l_3,(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})m_2-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})m_3,(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})n_2-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})n_3)\\
&=(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})(l_2,m_2,n_2)-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})(l_3,m_3,n_3)\\
&=(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})\overrightarrow{\beta}-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})\overrightarrow{\gamma},\ \text{得證。}
\end{aligned}$$

三、利用二的恆等式 $\overrightarrow{\alpha}\times(\overrightarrow{\beta}\times\overrightarrow{\gamma})=(\overrightarrow{\alpha}\cdot\overrightarrow{\gamma})\overrightarrow{\beta}-(\overrightarrow{\alpha}\cdot\overrightarrow{\beta})\overrightarrow{\gamma}$,

得 $\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})=[(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_1}-(\overrightarrow{d_1}\cdot\overrightarrow{d_1})\overrightarrow{d_2}]=-[(\overrightarrow{d_1}\cdot\overrightarrow{d_1})\overrightarrow{d_2}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_1}]$,

$\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})=[(\overrightarrow{d_2}\cdot\overrightarrow{d_2})\overrightarrow{d_1}-(\overrightarrow{d_2}\cdot\overrightarrow{d_1})\overrightarrow{d_2}]$。

四、利用三的等式, 向量三重積 $\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})=-[(\overrightarrow{d_1}\cdot\overrightarrow{d_1})\overrightarrow{d_2}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_1}]$,

將 $\dfrac{\overrightarrow{A_2A_1}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_2}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}$ 化簡為

$$\frac{\overrightarrow{A_2A_1}\cdot-[(\overrightarrow{d_1}\cdot\overrightarrow{d_1})\overrightarrow{d_2}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_1}]}{\overrightarrow{d_2}\cdot-[(\overrightarrow{d_1}\cdot\overrightarrow{d_1})\overrightarrow{d_2}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_1}]}=\frac{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{A_2A_1})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{A_2A_1})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}$$

$$\begin{aligned}\text{因此 }\overrightarrow{OB_2}&=\overrightarrow{OA_2}+\frac{\overrightarrow{A_2A_1}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_2}\cdot[\overrightarrow{d_1}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_2}\\&=\overrightarrow{OA_2}+\frac{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{A_2A_1})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{A_2A_1})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}\overrightarrow{d_2}\text{。}\end{aligned}$$

五、利用三的等式, 向量三重積 $\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})=[(\overrightarrow{d_2}\cdot\overrightarrow{d_2})\overrightarrow{d_1}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_2}]$,

將 $\dfrac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}$ 化簡為

$$\frac{\overrightarrow{A_1A_2}\cdot[(\overrightarrow{d_2}\cdot\overrightarrow{d_2})\overrightarrow{d_1}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_2}]}{\overrightarrow{d_1}\cdot[(\overrightarrow{d_2}\cdot\overrightarrow{d_2})\overrightarrow{d_1}-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})\overrightarrow{d_2}]}=\frac{(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}$$

$$\begin{aligned}\text{因此 }\overrightarrow{OB_1}&=\overrightarrow{OA_1}+\frac{\overrightarrow{A_1A_2}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}{\overrightarrow{d_1}\cdot[\overrightarrow{d_2}\times(\overrightarrow{d_1}\times\overrightarrow{d_2})]}\overrightarrow{d_1}\\&=\overrightarrow{OA_1}+\frac{(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}\overrightarrow{d_1}\text{。}\end{aligned}$$

六、結論

$$\overrightarrow{OB_1}=\overrightarrow{OA_1}+\frac{(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{A_1A_2}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}\overrightarrow{d_1}\text{。}$$

$$\overrightarrow{OB_2}=\overrightarrow{OA_2}+\frac{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{A_2A_1})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{A_2A_1})}{(\overrightarrow{d_1}\cdot\overrightarrow{d_1})(\overrightarrow{d_2}\cdot\overrightarrow{d_2})-(\overrightarrow{d_1}\cdot\overrightarrow{d_2})(\overrightarrow{d_1}\cdot\overrightarrow{d_2})}\overrightarrow{d_2}\text{。}$$

參考文獻

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