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無理有理一線牽

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摘要: 本文分別探討以 $\left\{\left(2\cos\frac{2\pi}{7}\right)^n,\ \left(2\cos\frac{4\pi}{7}\right)^n,\ \left(2\cos\frac{6\pi}{7}\right)^n\right\}$, $\left\{\left(2\cos\frac{2\pi}{7}2\cos\frac{4\pi}{7}\right)^n,\ \left(2\cos\frac{4\pi}{7}2\cos\frac{6\pi}{7}\right)^n,\ \left(2\cos\frac{6\pi}{7}2\cos\frac{2\pi}{7}\right)^n\right\}$, $\left\{2\cos\frac{2k\pi}{2n+1}\ \middle|\ 1\le k\le n\right\}$ 或 $\left\{2\cos\frac{2k\pi}{2n}\ \middle|\ 1\le k\le n\right\}$ 等四組無理數爲根的整係數多項式, 進而探討該諸多項式的係數間的關係。這些研究顯示透過整係數多項式, 我們可以把由三角函數所形成的一些無理數一線牽起來。

引理 1: $\left(x-2\cos\dfrac{2\pi}{7}\right)\left(x-2\cos\dfrac{4\pi}{7}\right)\left(x-2\cos\dfrac{6\pi}{7}\right)=x^3+x^2-2x-1$

令 $T_n=\left(2\cos\dfrac{2\pi}{7}\right)^n+\left(2\cos\dfrac{4\pi}{7}\right)^n+\left(2\cos\dfrac{6\pi}{7}\right)^n$,

$Q_n=\left(2\cos\dfrac{2\pi}{7}2\cos\dfrac{4\pi}{7}\right)^n+\left(2\cos\dfrac{4\pi}{7}2\cos\dfrac{6\pi}{7}\right)^n+\left(2\cos\dfrac{6\pi}{7}2\cos\dfrac{2\pi}{7}\right)^n$,

則 $T_n\in Z$, $(T_1,T_2,T_3)=(-1,5,-4)$, $T_{n+3}=-T_{n+2}+2T_{n+1}+T_n$,

$Q_n=\dfrac{1}{2}[T_n^2-T_{2n}]$。

證: (1). $\alpha=2\cos\dfrac{2\pi}{7}$, $\beta=2\cos\dfrac{4\pi}{7}$, $\gamma=2\cos\dfrac{6\pi}{7}$ 是 $x^3=-x^2+2x+1$ 的三根

$\Rightarrow \alpha,\beta,\gamma$ 也是 $x^{n+3}=-x^{n+2}+2x^{n+1}+x^n$ 的三根

$$\Rightarrow\begin{cases}\alpha^{n+3}=-\alpha^{n+2}+2\alpha^{n+1}+\alpha^n\\ \beta^{n+3}=-\beta^{n+2}+2\beta^{n+1}+\beta^n\\ \gamma^{n+3}=-\gamma^{n+2}+2\gamma^{n+1}+\gamma^n\end{cases}$$

$\Rightarrow(\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3})=-(\alpha^{n+2}+\beta^{n+2}+\gamma^{n+2})+2(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1})$
$+(\alpha^n+\beta^n+\gamma^n)$

$\Rightarrow T_{n+3}=-T_{n+2}+2T_{n+1}+T_n.$

(2). $Q_n=\alpha^n\beta^n+\beta^n\gamma^n+\gamma^n\alpha^n=\frac{1}{2}\Big[\Big(\alpha^n+\beta^n+\gamma^n\Big)^2-\Big((\alpha^n)^2+(\beta^n)^2+(\gamma^n)^2\Big)\Big]$

$$=\frac{1}{2}\Big[T_n^2-T_{2n}\Big].$$

命題 2: 令 $f(x)=\prod_{j=1}^{m}(x-\alpha_j)$, 則 $x^{n+1}f'(x)\div f(x)$ 之商的常數項爲 $\sum_{j=1}^{m}\alpha_j^n$,
特別是當 $(\alpha_1,\alpha_2,\alpha_3)=\Big(2\cos\frac{2\pi}{7},2\cos\frac{4\pi}{7},2\cos\frac{6\pi}{7}\Big)$ 時, $(3x^{n+3}+2x^{n+2}-2x^{n+1})\div(x^3+x^2-2x-1)$ 之商的常數項爲 T_n 。

證: 觀察綜合除法 $x^{n+1}\div(x-\alpha_1)$:

$$\begin{array}{ccccccc|l} 1 & +\ 0 & +\ 0 & + \cdots\cdots\cdots & +\ 0 & +\ 0 & & \alpha_1 \\ & \alpha_1 & +\alpha_1^2 & + \cdots\cdots\cdots & +\alpha_1^n & +\alpha_1^{n+1} & & \\ \hline 1 & +\alpha_1 & +\alpha_1^2 & + \cdots\cdots\cdots & +\alpha_1^n & +\alpha_1^{n+1} & & \end{array}$$

表成 $\frac{x^{n+1}}{x-\alpha_1}=x^n+\sum_{i=1}^{n-1}\alpha_1^i x^{n-i}+\alpha_1^n+\frac{\alpha_1^{n+1}}{x-\alpha_1}$

同理 $\frac{x^{n+1}}{x-\alpha_2}=x^n+\sum_{i=1}^{n-1}\alpha_2^i x^{n-i}+\alpha_2^n+\frac{\alpha_2^{n+1}}{x-\alpha_2}$

$$\vdots$$

$$\frac{x^{n+1}}{x-\alpha_m}=x^n+\sum_{i=1}^{n-1}\alpha_m^i x^{n-i}+\alpha_m^n+\frac{\alpha_m^{n+1}}{x-\alpha_m}$$

此 m 個式子相加: $\sum_{j=1}^{m}\frac{x^{n+1}}{x-\alpha_j}=mx^n+\sum_{j=1}^{m}\sum_{i=1}^{n-1}\alpha_j^i x^{n-i}+\sum_{j=1}^{m}\alpha_j^n+\sum_{j=1}^{m}\frac{\alpha_j^{n+1}}{x-\alpha_j}$

左式 $=x^{n+1}\sum_{j=1}^{m}\frac{1}{x-\alpha_j}=x^{n+1}\frac{\sum_{k=1}^{m}\prod_{j\neq k}^{m}(x-\alpha_j)}{\prod_{j=1}^{m}(x-\alpha_j)}=x^{n+1}\frac{f'(x)}{f(x)}$ 。

命題 3: 令 $a_n\in Z$, $(a_1,a_2,a_3)\equiv(1,1,0)\pmod 2$, 且 $a_{n+3}=a_{n+2}+a_n$,

則 $(a_{7m+1},a_{7m+2},\ldots,a_{7m+7})\equiv(1,1,0,1,0,0,1)\pmod 2$

$(a_n,a_{2n})\equiv(1,1)$ 或 $(0,0)\pmod 2$

特別是 $(T_{7m+1},T_{7m+2},\ldots,T_{7m+7})\equiv(1,1,0,1,0,0,1)\pmod 2$

$(T_n,T_{2n})\equiv(1,1)$ 或 $(0,0)\pmod 2$ $\qquad Q_n\in Z$ 。

證: (1).① 當 $m=0$, $(a_1,a_2,\ldots,a_7)\equiv(1,1,0,a_3+a_1,a_4+a_2,a_5+a_3,a_6+a_4)$

$$\equiv (1,1,0,1,0,0,1) \pmod 2$$

② 設 $m=k$ 成立, 即設 $(a_{7k+1},a_{7k+2},\ldots,a_{7k+7})\equiv(1,1,0,1,0,0,1) \pmod 2$

③ 當 $m=k+1$, $a_{7k+8}=a_{7k+7}+a_{7k+5}\equiv 1+0\equiv 1 \pmod 2$

$$a_{7k+9}=a_{7k+8}+a_{7k+6}\equiv 1+0\equiv 1 \pmod 2$$
$$a_{7k+10}=a_{7k+9}+a_{7k+7}\equiv 1+1\equiv 0 \pmod 2$$
$$a_{7k+11}=a_{7k+10}+a_{7k+8}\equiv 0+1\equiv 1 \pmod 2$$
$$a_{7k+12}=a_{7k+11}+a_{7k+9}\equiv 1+1\equiv 0 \pmod 2$$
$$a_{7k+13}=a_{7k+12}+a_{7k+10}\equiv 0+0\equiv 0 \pmod 2$$
$$a_{7k+14}=a_{7k+13}+a_{7k+11}\equiv 0+1\equiv 1 \pmod 2,$$

由數學歸納法故得證。

(2). $\big(a_{2(7k+1)},a_{2(7k+2)},a_{2(7k+3)},a_{2(7k+4)},a_{2(7k+5)},a_{2(7k+6)},a_{2(7k+7)}\big)$

$$=\big(a_{7h+2},a_{7h+4},a_{7h+6},a_{7\ell+1},a_{7\ell+3},a_{7\ell+5},a_{7\ell+7}\big)$$
$$\equiv(1,1,0,1,0,0,1)\equiv(a_{7k+1},a_{7k+2},\ldots,a_{7k+7}) \pmod 2$$

命題 4: $(T_{3m+1},T_{3m+2},T_{3m+3})\equiv(6,5,3) \pmod 7$

證: (1). 當 $m=0$, $(T_1,T_2,T_3)\equiv(-1,5,-4)\equiv(6,5,3) \pmod 7$

(2). 設 $m=k$ 成立, 即設 $(T_{3k+1},T_{3k+2},T_{3k+3})\equiv(6,5,3) \pmod 7$

(3). 當 $m=k+1$, $T_{3k+4}=-T_{3k+3}+2T_{3k+2}+T_{3k+1}\equiv -3+2\cdot 5+6\equiv 6 \pmod 7$

$$T_{3k+5}=-T_{3k+4}+2T_{3k+3}+T_{3k+2}\equiv -6+2\cdot 3+5\equiv 5 \pmod 7$$
$$T_{3k+6}=-T_{3k+5}+2T_{3k+4}+T_{3k+3}\equiv -5+2\cdot 6+3\equiv 3 \pmod 7$$

$\Rightarrow$ 由數學歸納法故得證。

定理 5: 1. 令 $f_n(x)=\Big(x-\Big(2\cos\dfrac{2\pi}{7}\Big)^n\Big)\Big(x-\Big(2\cos\dfrac{4\pi}{7}\Big)^n\Big)\Big(x-\Big(2\cos\dfrac{6\pi}{7}\Big)^n\Big)$, 則 $f_n(x)\in Z[x]$, 且 $f_n(x)=x^3-T_nx^2+Q_nx-1$。

2. 令 $g_n(x)=\Big(x-\Big(2\cos\dfrac{2\pi}{7}\Big)^{2^n}\Big)\Big(x-\Big(2\cos\dfrac{4\pi}{7}\Big)^{2^n}\Big)\Big(x-\Big(2\cos\dfrac{6\pi}{7}\Big)^{2^n}\Big)$,

則 $g_n(x)\in Z[x]$, 且 $g_n(x)=x^3-r_nx^2+s_nx-1$, 式中 $r_{n+1}=r_n^2-2s_n$, $s_{n+1}=s_n^2-2r_n$, 當 $n\geq 2$ 時 r_n, s_n 皆爲 13 的倍數。

證: (1). 由 $\alpha = 2\cos\frac{2\pi}{7}$, $\beta = 2\cos\frac{4\pi}{7}$, $\gamma = 2\cos\frac{6\pi}{7}$

$$\Rightarrow \begin{cases} \alpha^n + \beta^n + \gamma^n = T_n \\ \alpha^n\beta^n + \beta^n\gamma^n + \gamma^n\alpha^n = \frac{1}{2}\Big[(\alpha^n+\beta^n+\gamma^n)^2 - \Big((\alpha^n)^2+(\beta^n)^2+(\gamma^n)^2\Big)\Big] \\ \qquad\qquad\qquad\qquad\qquad = \frac{1}{2}[T_n^2 - T_{2n}] = Q_n \in Z \\ \alpha^n\cdot\beta^n\cdot\gamma^n = (\alpha\beta\gamma)^n = 1 \end{cases}$$

(2). $r_{n+1} = \alpha^{2^{n+1}} + \beta^{2^{n+1}} + \gamma^{2^{n+1}} = (\alpha^{2^n} + \beta^{2^n} + \gamma^{2^n})^2 - 2(\alpha^{2^n}\beta^{2^n} + \beta^{2^n}\gamma^{2^n} + \gamma^{2^n}\alpha^{2^n})$
$= r_n^2 - 2s_n$

$$\begin{aligned} s_{n+1} &= \alpha^{2^{n+1}}\beta^{2^{n+1}} + \beta^{2^{n+1}}\gamma^{2^{n+1}} + \gamma^{2^{n+1}}\alpha^{2^{n+1}} = (\alpha^{2^n}\beta^{2^n})^2 + (\beta^{2^n}\gamma^{2^n})^2 + (\gamma^{2^n}\alpha^{2^n})^2 \\ &= (\alpha^{2^n}\beta^{2^n} + \beta^{2^n}\gamma^{2^n} + \gamma^{2^n}\alpha^{2^n})^2 - 2(\alpha^{2^n}\beta^{2^n}\beta^{2^n}\gamma^{2^n} + \beta^{2^n}\gamma^{2^n}\gamma^{2^n}\alpha^{2^n} \\ &\quad + \gamma^{2^n}\alpha^{2^n}\alpha^{2^n}\beta^{2^n}) \\ &= (\alpha^{2^n}\beta^{2^n} + \beta^{2^n}\gamma^{2^n} + \gamma^{2^n}\alpha^{2^n})^2 - 2\alpha^{2^n}\beta^{2^n}\gamma^{2^n}(\beta^{2^n} + \gamma^{2^n} + \alpha^{2^n}) \\ &= s_n^2 - 2\cdot 1\cdot r_n = s_n^2 - 2r_n. \end{aligned}$$

(3). 由 $r_2 = 13$ 與 $s_2 = 26$, 知當 $n \geq 2$ 時 r_n 與 s_n 都是 13 的倍數。

定理 6: 令

$$\begin{aligned} f_n(x) = &\Big(x - \Big(2\cos\frac{2\pi}{7}2\cos\frac{4\pi}{7}\Big)^n\Big)\Big(x - \Big(2\cos\frac{4\pi}{7}2\cos\frac{6\pi}{7}\Big)^n\Big) \\ &\times\Big(x - \Big(2\cos\frac{6\pi}{7}2\cos\frac{2\pi}{7}\Big)^n\Big), \end{aligned}$$

則 $f_n(x) \in Z[x]$, 且 $f_n(x) = x^3 - Q_nx^2 + T_nx - 1$。

證: 由 $\alpha = 2\cos\frac{2\pi}{7}$, $\beta = 2\cos\frac{4\pi}{7}$, $\gamma = 2\cos\frac{6\pi}{7}$

$$\Rightarrow \begin{cases} (\alpha\beta)^n + (\beta\gamma)^n + (\gamma\alpha)^n = \alpha^n\beta^n + \beta^n\gamma^n + \gamma^n\alpha^n = Q_n \\ (\alpha\beta)^n(\beta\gamma)^n + (\beta\gamma)^n(\gamma\alpha)^n + (\gamma\alpha)^n(\alpha\beta)^n = \alpha^n\beta^n\gamma^n\cdot(\beta^n+\gamma^n+\alpha^n) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1\cdot T_n = T_n \\ (\alpha\beta)^n\cdot(\beta\gamma)^n\cdot(\gamma\alpha)^n = (\alpha^n\beta^n\gamma^n)^2 = 1^2 = 1 \end{cases}$$

引理 7: 1. $\displaystyle\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1} = -\frac{1}{2}$ 2. $\displaystyle\sum_{k=1}^{n}\cos\frac{2k\pi}{2n} = -1$

證: (1). 令 $\omega=\cos\dfrac{2\pi}{2n+1}+i\sin\dfrac{2\pi}{2n+1}$, 則 $\omega^{2n+1}=1$ 且 $\omega^{2n}+\omega^{2n-1}+\cdots+\omega^1+1=0$

$$\begin{aligned}\text{左式}\ &=\frac{1}{2}(\omega^1+\omega^{-1})+\frac{1}{2}(\omega^2+\omega^{-2})+\cdots+\frac{1}{2}(\omega^n+\omega^{-n})\\&=\frac{1}{2}(\omega^1+\omega^{2n}+\omega^2+\omega^{2n-1}+\cdots+\omega^n+\omega^{n+1})=\frac{1}{2}(-1)=-\frac{1}{2}\end{aligned}$$

(2). 令 $\omega=\cos\dfrac{2\pi}{2n}+i\sin\dfrac{2\pi}{2n}$, 則 $\omega^{2n}=1$ 且 $\omega^{2n-1}+\omega^{2n-2}+\cdots+\omega^1+1=0$ 且 $\omega^n=-1$

$$\begin{aligned}\text{左式}\ &=\frac{1}{2}(\omega^1+\omega^{-1})+\frac{1}{2}(\omega^2+\omega^{-2})+\cdots+\frac{1}{2}(\omega^n+\omega^{-n})\\&=\frac{1}{2}(\omega+\omega^{2n-1}+\omega^2+\omega^{2n-2}+\cdots+\omega^n+\omega^n)=\frac{1}{2}(-1+\omega^n)=\frac{1}{2}(-1-1)=-1\end{aligned}$$

接下來的證明中, $\displaystyle\prod_{k=1}^{n}\Big(x-2\cos\frac{2k\pi}{2n+1}\Big)=0$ 的根爲 $\Big\{x_k=2\cos\dfrac{2k\pi}{2n+1}\ \Big|\ 1\le k\le n\Big\}$, 而 $z^{2n+1}=1$ 的前 n 個根爲 $\Big\{z_k=\omega^k=\cos\dfrac{2k\pi}{2n+1}+i\sin\dfrac{2k\pi}{2n+1}\ \Big|\ 1\le k\le n\Big\}$, 其關係爲 $x_k=z_k+z_k^{-1}$, 即 $x=z+z^{-1}$ 。

定理 8: 1. 令 $f_n(x)=\displaystyle\prod_{k=1}^{n}\Big(x-2\cos\frac{2k\pi}{2n+1}\Big)$, 則 $f_n(x)\in Z[x]$, $f_{n+2}(x)=xf_{n+1}(x)-f_n(x)$, 且 $f_n(x)=x^n+x^{n-1}-(n-1)x^{n-2}-(n-2)x^{n-3}+\dfrac{1}{2}(n-2)(n-3)x^{n-4}+\dfrac{1}{2}(n-3)(n-4)x^{n-5}+\cdots+(-1)^{[\frac{n}{2}]}$

2. 令 $g_n(x)=\displaystyle\prod_{k=1}^{n}\Big(x-2\cos\frac{2k\pi}{2n}\Big)$, 則 $g_n(x)\in Z[x]$, $g_{n+2}(x)=xg_{n+1}(x)-g_n(x)$, 且 $g_n(x)=x^n+2x^{n-1}-(n-2)x^{n-2}-2(n-2)x^{n-3}+\dfrac{1}{2}(n-3)(n-4)x^{n-4}+\dfrac{2}{2}(n-3)(n-4)x^{n-5}+\cdots+\Big((-1)^{[\frac{n}{2}]}+(-1)^{[\frac{n+3}{2}]}\Big)$

證: (1). ① 令 $\omega=\cos\dfrac{2\pi}{2n+1}+i\sin\dfrac{2\pi}{2n+1}$, 則 $\omega^{2n+1}=1$ 且$\omega^{2n}+\omega^{2n-1}+\cdots+\omega^1+1=0$

$$\begin{aligned}f(x)&=\prod_{k=1}^{n}\Big(x-2\cos\frac{2k\pi}{2n+1}\Big)=\prod_{k=1}^{n}\Big(x-(\omega^k+\omega^{-k})\Big)\\&=\prod_{k=1}^{n}\Big(x-(\omega^k+\omega^{2n+1-k})\Big)=\prod_{k=1}^{n}\Big(z+z^{-1}-(\omega^k+\omega^{2n+1-k})\Big)\\&=z^{-n}\prod_{k=1}^{n}\Big(z^2-(\omega^k+\omega^{2n+1-k})z+1\Big)=z^{-n}\prod_{k=1}^{n}(z-\omega^k)(z-\omega^{2n+1-k})\end{aligned}$$

$$= z^{-n}\prod_{k=1}^{2n}(z-\omega^k) = z^{-n}(z^{2n}+z^{2n-1}+\cdots+1) = z^n+z^{n-1}+\cdots+z^{-n}.$$

② $x\cdot f_{n+1}(x)-f_n(x)$

$$= (z+z^{-1})(z^{n+1}+z^n+\cdots+z^{-n-1})-(z^n+z^{n-1}+\cdots+z^{-n})$$

$$= (z^{n+2}+z^{n+1}+\cdots+z^{-n})+(z^n+z^{n-1}+\cdots+z^{-n-2})-(z^n+z^{n-1}+\cdots+z^{-n})$$

$$= (z^{n+2}+z^{n+1})+(z^n+z^{n-1}+\cdots+z^{-n-2}) = f_{n+2}(x).$$

(2).① 當 $n=6$, $f_6(x)=x^6+x^5-5x^4-4x^3+6x^2+3x-1$ $\Rightarrow f_6(x)$ 降冪前 6 項成立

當 $n=7$, $f_7(x)=x^7+x^6-6x^5-5x^4+10x^3+6x^2-4x-1 \Rightarrow f_7(x)$ 降冪前 6 項成立

② 設 $n=k$ 成立, 即設 $f_k(x)$ 降冪前 6 項

爲 $x^k+x^{k-1}-(k-1)x^{k-2}-(k-2)x^{k-3}+\frac{1}{2}(k-2)(k-3)x^{k-4}+\frac{1}{2}(k-3)(k-4)x^{k-5}$

設 $n=k+1$ 成立, 即設 $f_{k+1}(x)$ 降冪前 6 項

爲 $x^{k+1}+x^k-kx^{k-1}-(k-1)x^{k-2}+\frac{1}{2}(k-1)(k-2)x^{k-3}+\frac{1}{2}(k-2)(k-3)x^{k-4}$

③ 當 $n=k+2$, $x\cdot f_{k+1}(x)-f_k(x)$ 降冪前『8』項

$$\begin{aligned}
\text{爲}\ & x^{k+2}+x^{k+1}-kx^k-(k-1)x^{k-1}+\frac{1}{2}(k-1)(k-2)x^{k-2} \\
&+\frac{1}{2}(k-2)(k-3)x^{k-3}-\Big[x^k+x^{k-1}-(k-1)x^{k-2}-(k-2)x^{k-3} \\
&+\frac{1}{2}(k-2)(k-3)x^{k-4}+\frac{1}{2}(k-3)(k-4)x^{k-5}\Big] \\
=\ & x^{k+2}+x^{k+1}-(k+1)x^k-kx^{k-1}+\frac{1}{2}k(k-1)x^{k-2} \\
&+\frac{1}{2}(k-1)(k-2)x^{k-3}-\frac{1}{2}(k-2)(k-3)x^{k-4}-\frac{1}{2}(k-3)(k-4)x^{k-5}
\end{aligned}$$

$\Rightarrow f_{k+2}(x)$ 降冪前 6 項成立, 由數學歸納法故得證。

(3). 設 $f_n(x)$ 常數項爲 c_n, 由 $f_1(x)$ 與 $f_2(x)$, 知 $c_1=1$ 與 $c_2=-1$,

由 $f_{n+2}(x)=x\cdot f_{n+1}(x)-f_n(x)$, 知 $c_n=-c_{n-2}$,

故 c_n 爲 $1,-1,-1,1,\ldots\ldots$ 的週期數列, 相同於 $(-1)^{[\frac{n}{2}]}$ 。

(以下爲 $g_n(x)$ 的證明, 類似 $f_n(x)$ 的證明)

(4).① 令 $\omega=\cos\dfrac{2\pi}{2n}+i\sin\dfrac{2\pi}{2n}$, 則 $\omega^{2n}=1$ 且 $\omega^{2n-1}+\omega^{2n-2}+\cdots+\omega^1+1=0$ 且 $\omega^n=-1$

$$\begin{aligned}
g(x)&=\prod_{k=1}^{n}\Big(x-2\cos\frac{2k\pi}{2n}\Big)=\prod_{k=1}^{n}\Big(x-(\omega^k+\omega^{-k})\Big)\\
&=\prod_{k=1}^{n}\Big(x-(\omega^k+\omega^{2n-k})\Big)=\prod_{k=1}^{n}\Big(z+z^{-1}-(\omega^k+\omega^{2n-k})\Big)\\
&=z^{-n}\prod_{k=1}^{n}\Big(z^2-(\omega^k+\omega^{2n-k})z+1\Big)=z^{-n}\prod_{k-1}^{n}(z-\omega^k)(z-\omega^{2n-k})\\
&=z^{-n}(z-\omega^n)\prod_{k=1}^{2n-1}(z-\omega^k)=z^{-n}(z+1)(z^{2n-1}+z^{2n-2}+\cdots+1)\\
&=(z+1)(z^{n-1}+z^{n-2}+\cdots+z^{-n})
\end{aligned}$$

② $x\cdot g_{n+1}(x)-g_n(x)$

$$\begin{aligned}
&=(z+z^{-1})(z+1)(z^n+z^{n-1}+\cdots+z^{-n-1})-(z+1)(z^{n-1}+z^{n-2}+\cdots+z^{-n})\\
&=(z+1)\Big[(z^{n+1}+z^n+\cdots+z^{-n})+(z^{n-1}+z^{n-2}+\cdots+z^{-n-2})\\
&\quad-(z^{n-1}+z^{n-2}+\cdots+z^{-n})\Big]\\
&=(z+1)\Big[(z^{n+2}+z^{n+1})+(z^n+z^{n-1}+\cdots+z^{-n-2})\Big]=g_{n+2}(x)
\end{aligned}$$

(5).① 當 $n=6$, $g_6(x)=x^6+2x^5-4x^4-8x^3+3x^2+6x\Rightarrow g_6(x)$ 降冪前 6 項成立

當 $n=7$, $g_7(x)=x^7+2x^6-5x^5-10x^4+6x^3+12x^2-x-2\Rightarrow g_7(x)$ 降冪前 6 項成立

② 設 $n=k$ 成立, 即設 $g_k(x)$ 降冪前 6 項

爲 $x^k+2x^{k-1}-(k-2)x^{k-2}-2(k-2)x^{k-3}+\dfrac{1}{2}(k-3)(k-4)x^{k-4}+\dfrac{2}{2}(k-3)(k-4)x^{k-5}$

設 $n=k+1$ 成立, 即設 $g_{k+1}(x)$ 降冪前 6 項

爲 $x^{k+1}+2x^k-(k-1)x^{k-1}-2(k-1)x^{k-2}+\dfrac{1}{2}(k-2)(k-3)x^{k-3}+\dfrac{2}{2}(k-2)(k-3)x^{k-4}$

③ 當 $n=k+2$, $x\cdot g_{k+1}(x)-g_k(x)$ 降冪前『8』項

$$\begin{aligned}
\text{爲}\,&x^{k+2}+2x^{k+1}-(k-1)x^k-2(k-1)x^{k-1}+\frac{1}{2}(k-2)(k-3)x^{k-2}\\
&+\frac{2}{2}(k-2)(k-3)x^{k-3}-\Big[x^k+2x^{k-1}-(k-2)x^{k-2}-2(k-2)x^{k-3}\\
&+\frac{1}{2}(k-3)(k-4)x^{k-4}+\frac{2}{2}(k-3)(k-4)x^{k-5}\Big]
\end{aligned}$$

$$= x^{k+2} + 2x^{k+1} - kx^k - 2kx^{k-1} + \frac{1}{2}(k-1)(k-2)x^{k-2}$$
$$+\frac{2}{2}(k-1)(k-2)x^{k-3} - \frac{1}{2}(k-3)(k-4)x^{k-4}$$
$$-\frac{2}{2}(k-3)(k-4)x^{k-5}$$

$\Rightarrow g_{k+2}(x)$ 降冪前 6 項成立, 由數學歸納法故得證。

(6). 設 $g_n(x)$ 常數項爲 c_n, 由 $g_1(x)$ 與 $g_2(x)$, 知 $c_1 = 2$ 與 $c_2 = 0$,

由 $g_{n+2}(x) = x \cdot g_{n+1}(x) - g_n(x)$, 知 $c_n = -c_{n-2}$,

故 c_n 爲 $2, 0, -2, 0, \ldots\ldots$ 的週期數列, 相同於 $(-1)^{[\frac{n}{2}]} + (-1)^{[\frac{n+3}{2}]}$ 。

在此之前的分子都是偶數個 π, 若把分子改成奇數個 π, 由

$$\Big(\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}\Big) = \Big(-\cos\frac{6\pi}{7}, -\cos\frac{4\pi}{7}, -\cos\frac{2\pi}{7}\Big),$$
$$\Big\{2\cos\frac{(2k-1)\pi}{2n+1}\Big| 1 \le k \le n\Big\} = \Big\{-2\cos\frac{2k\pi}{2n+1}\Big| 1 \le k \le n\Big\}$$

及

$$\Big\{2\cos\frac{(2k-1)\pi}{2n}\Big| 1 \le k \le n\Big\} = \Big\{-2\cos\frac{(2k-1)\pi}{2n}\Big| 1 \le k \le n\Big\},$$

知其性質極爲類似, 如下所列。

命題 9: 1. $\Big(x - 2\cos\frac{\pi}{7}\Big)\Big(x - 2\cos\frac{3\pi}{7}\Big)\Big(x - 2\cos\frac{5\pi}{7}\Big) = x^3 - x^2 - 2x + 1$

2. 令 $K_n = \Big(2\cos\frac{\pi}{7}\Big)^n + \Big(2\cos\frac{3\pi}{7}\Big)^n + \Big(2\cos\frac{5\pi}{7}\Big)^n$

則 (1) $K_n = |T_n|$

(2) $K_{n+3} = K_{n+2} + 2K_{n+1} - K_n$

(3) $(K_{7m+1}, K_{7m+2}, \ldots, K_{7m+7}) \equiv (1, 1, 0, 1, 0, 0, 1) \pmod 2$

(4) $(K_n, K_{2n}) \equiv (1, 1)$ 或 $(0, 0) \pmod 2$

(5) $Q_n = \frac{1}{2}[T_n^2 - T_{2n}] = \frac{1}{2}[K_n^2 - K_{2n}]$

(6) $(K_{7m+1}, K_{7m+2}, \ldots, K_{7m+7}) \equiv (1, 5, 4, 6, 2, 3) \pmod 7$

3.(1) 令 $f_n(x) = \Big(x - \Big(2\cos\frac{\pi}{7}\Big)^n\Big)\Big(x - \Big(2\cos\frac{3\pi}{7}\Big)^n\Big)\Big(x - \Big(2\cos\frac{5\pi}{7}\Big)^n\Big)$,

則 $f_n(x) \in Z[x]$, 且 $f_n(x) = x^3 - |T_n|x^2 + Q_nx - 1$

(2) 令 $g_n(x)=\left(x-\left(2\cos\frac{\pi}{7}\right)^{2^n}\right)\left(x-\left(2\cos\frac{3\pi}{7}\right)^{2^n}\right),\left(x-\left(2\cos\frac{5\pi}{7}\right)^{2^n}\right)$,

則 $g_n(x)\in Z[x]$, 且 $g_n(x)=x^3-r_nx^2+s_nx-1$, 式中 $r_{n+1}=r_n^2-2s_n$, $s_{n+1}=s_n^2-2r_n$,

當 $n\geq 2$ 時 r_n, s_n 皆爲 13 的倍數。

4. 令

$$f_n(x)=\left(x-\left(2\cos\frac{\pi}{7}2\cos\frac{3\pi}{7}\right)^n\right)\left(x-\left(2\cos\frac{3\pi}{7}2\cos\frac{5\pi}{7}\right)^n\right)\left(x-\left(2\cos\frac{5\pi}{7}2\cos\frac{\pi}{7}\right)^n\right),$$

則 $f_n(x)\in Z[x]$, 且 $f_n(x)=x^3-Q_nx^2+|T_n|x-1$

5. (1)$\displaystyle\sum_{k=1}^{n}\cos\frac{(2k-1)\pi}{2n+1}=\frac{1}{2}$

(2)$\displaystyle\sum_{k=1}^{n}\cos\frac{(2k-1)\pi}{2n}=0$

6. (1) 令 $f_n(x)=\displaystyle\prod_{k=1}^{n}\left(x-2\cos\frac{(2k-1)\pi}{2n+1}\right)$, 則 $f_n(x)\in Z[x]$, $f_{n+2}(x)=xf_{n+1}(x)-f_n(x)$, 且 $f_n(x)=x^n-x^{n-1}-(n-1)x^{n-2}+(n-2)x^{n-3}+\frac{1}{2}(n-2)(n-3)x^{n-4}-\frac{1}{2}(n-3)(n-4)x^{n-5}+\cdots+(-1)^{[\frac{n+1}{2}]}$

(2) 令 $g_n(x)=\displaystyle\prod_{k=1}^{n}\left(x-2\cos\frac{(2k-1)\pi}{2n}\right)$, 則 $g_n(x)\in Z[x]$, $g_{n+2}(x)=xg_{n+1}(x)-g_n(x)$,

且 $g_n(x)=x^n-nx^{n-2}+\frac{1}{2}n(n-3)x^{n-4}+\cdots+\left((-1)^{[\frac{n}{2}]}+(-1)^{[\frac{n+1}{2}]}\right)$

本文中的式子, 都經過撰寫 Mathematca 程式作預測與驗證, 不但省時與精確, 更能有效分析與歸納。

參考文獻

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