國立清華大學數學研究所
八十一學年度碩士班入學考試題

高等微積分

(10 points)

1. Find the volume of the solid region cut from the unit ball $x^2 + y^2 + z^2 \leq 1$ by the cylinder $x^2 + y^2 = x$.

(15 points)

2. Compute the surface integral

$$\iint_{\partial Q} (x, y^2, z^3) \cdot \nu \, dS$$

$(Q = [-1, 1]^3,\ \nu =$ the outer normal of $Q)$

(a) directly,

(b) by Gauss' divergence theorem.

(20 points)

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 + y^2 - 2x^2 y - \dfrac{4x^6 y^2}{(x^4 + y^2)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Show that f is continuous.

(b) Is f differentiable at $(0, 0)$? Why?

(c) Show that the restriction of f to each line through $(0, 0)$ has a local minimum at $(0, 0)$.

(d) Does f have a local minimum at $(0, 0)$? Explain why.

(15 points)

4. Let $U = \{(x, y) \mid x^2 + y^2 < 1\}$ be the open unit disc in the plane, and let $f : U \longrightarrow \mathbb{R}$ be a continuous function. For each $\theta \in \mathbb{R}$, define $f_\theta : U \longrightarrow \mathbb{R}$ by

$$f_\theta(x, y) = f(x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta).$$

Show that $f_\theta \longrightarrow f$ uniformly, as $\theta \longrightarrow 0$, on every compact subset of U.

(15 points)

5. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function and there is no $x \in \mathbb{R}$ such that $f(x) = 0 = f'(x)$. Let $Z_f = \{x \in \mathbb{R} \mid f(x) = 0\}$ be the zero set of f. Show that Z_f is at most countable. (Hint: Try to show that $Z_f \cap [a, b]$ is finite for any bounded interval $[a, b]$).

(15 points)

6. Evaluate the following integrals

(a) $\int_1^3 e^{-x}\, d[x]$.

(b) $\int_0^2 \int_0^2 [x + y]\, dx\, dy$, where $[x]$ is the greatest integer $\leq x$.

(c) $\int_0^\infty e^{-x^2} \cos(2xt)\, dx$, where $t \in \mathbb{R}$.

(15 points)

7. Suppose $f : [0, 1] \longrightarrow \mathbb{R}$ is a continuous function. Show that

$$\lim_{\epsilon \to 0^+} \int_0^1 \frac{\epsilon f(x)}{(x-a)^2 + \epsilon^2}\, dx$$

exists for every $a \in \mathbb{R}$, and find it in terms of f.

(15 points)

8. Show that for any continuous real–valued function $f(x)$ on $[0, 1]$, there exists a sequence of polynomials $\{p_n(x)\}_{n=1}^\infty$ with $p_n(0) = f(0)$, $p_n(1) = f(1)$ for all n such that $p_n \longrightarrow f$ uniformly on $[0, 1]$, as $n \longrightarrow \infty$.

代數及線性代數

$\mathbb{Z}$ = the ring of integers. $\mathbb{R}$ = the field of real numbers. $\mathbb{C}$ = the field of complex numbers.

$M_n(F)$ = the ring of all $n \times n$ matrices over F. $\mathbb{Z}_n$ = the ring of integers modulo n.

1. (20 points).

   Let $A = A^* \in M_n(\mathbb{C})$.

   (a) Show that the eigenvalues of A are real numbers.

   (b) Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Show that $\lambda_1 = \max\limits_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \dfrac{\langle Ax, x\rangle}{\|x\|^2}$.

2. (10 points).

   Find all orthogonal linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$ which carry the line $y = x$ to the line $y = 3x$.

3. (15 points)

   Solve the following system of differential equations

   $$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 3x_1 - x_2, \end{cases}$$

   where, for each i, $x_i = x_i(t)$ is a differentiable real–valued function of the real variable t.

4. (15 points).

   Let $G = M_2(\mathbb{Z}_3)$ and let $U = \{A \in G \mid A^t A = I\}$.

   (a) Compute the order of the multiplicative group U.

   (b) Find A, B in U such that A is of order 2, B is of order 4 and $ABA = B^{-1}$.

5. (20 points).

   Let $A \in SL_2(\mathbb{R}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{R},\ ad - bc = 1\}$.

   (a) Explain why there exists a complex matrix P such that $P^{-1}AP$ is either $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, where λ is a nonzero complex number.

   (b) Assume moreover that $A \neq \pm I$ and A has distinct eigenvalues λ, μ.

   Show that $|\lambda| = |\mu|$ if and only if $|\text{trace}(A)| < 2$.

6. (15 points).

(a) Classify all abelian groups of order 99.

(b) Show that every group of order 99 is abelian.

7. (15 points).

(a) If a,b are integers and $5 \mid a^2 - 2b^2$, show that $5 \mid a$ and $5 \mid b$.

(b) Let $R = \{a + b\sqrt{2} \mid a,b \in \mathbb{Z}\}$ and let $M = \{a + b\sqrt{2} \mid 5 \mid a \text{ and } 5 \mid b\}$.

Show that R/M is a field having 25 elements.

8. (10 points).

True or false. Justify your answers.

(a) Let σ be an element of order 5 in S_{15}. Then there exists $s \in S$ such that $O(s)$ contains exactly three elements, where $O(s) = \{\sigma^i(s) \mid i \in \mathbb{Z}\}$.

(b) There is an integral domain containing exactly 15 elements .

複變數函數論

(5 points)

1. For the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(x + iy) = (x^3 + xy^2 + 1) + i(x^2 + y^2)$$

Determine where f is holomorphic and give $f'(x + iy)$ at those points.

(12 points)

2. Evaluate the following integrals

(a) $\displaystyle\int_0^\infty \cos(x^2)\, dx$

(b) $\displaystyle\int_0^\pi \frac{d\theta}{(2+\cos\theta)^2}$

(9 points)

3. Does there exist a holomorphic function $f: D = \{z \,;\, |z| < 1\} \longrightarrow \mathbb{C}$ satisfying the following? Give one if it exists, explain why if it does not.

(a) $f(\frac{1}{n}) = \frac{n}{n+1}$ for all positive integers n.

(b) $f(\frac{1}{2n}) = 0,\ f(\frac{1}{2n-1}) = 1$ for all positive integers n.

(c) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$ for all positive integers n.

(10 points)

4. Determine the number of zeros for the function $f(z) = z^4 - 6z + 3$ in $\{z \,;\, |z| < 1\}$, and in $\{z \,;\, 1 < |z| < 2\}$ respectively.

(12 points)

5. Let γ be a rectifiable simple closed curve in $\mathbb{C}$, γ divides $\mathbb{C}$ into two disjoint connected regions D_1, D_2 with D_1 bounded. Assume that $f: \overline{D}_2 = D_2 \cup \gamma \longrightarrow \mathbb{C}$ is continuous and $f|_{D_2}$ holomorphic, $\lim_{z\to\infty} f(z) = A(<\infty)$. Show that

(a) $$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \begin{cases} A - f(z) & z \in D_2 \\ A & z \in D_1 \end{cases}$$

(b) if $0 \in D_1$, then $$\frac{1}{2\pi i}\int_\gamma \frac{z\,f(w)}{zw-w^2}\,dw = \begin{cases} f(z) & z \in D_2 \\ 0 & z \in D_1 \end{cases}$$

(12 points)

6. For the domains G_1 and G_2 given as follows, does there exist a conformal map sending G_1 onto G_2? Give one if it exists, explain why if it does not.

(a) $G_1 = \{z \in \mathbb{C} \,;\, |z-1| < 1,\ |z-i| < 1\}$, $G_2 = \{z \,;\, |z| < 1\}$

(b) $G_1 = \{z \in \mathbb{C} \,;\, 0 < |z| < 1\}$, $G_2 = \{z \in \mathbb{C} \,;\, 1 < |z| < 2\}$