中央研究院數學研究所

招考八十一學年度研習員筆試試題

Choose 4 among the following questions. Please give complete arguments to the questions. If you can not answer a whole question, you may try a special case.

1. Let $f : Z^3 \longrightarrow Z^3$

$$f(x,y,z) = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

compute $\mid Z^3/f(Z^3) \mid$. Explain your method.

2. On d-dimensional space, find all the α's and β's so that the following integrals converge

(1)

$$\int_{|x|>1} \frac{1}{\mid x \mid^{\alpha}} dx, x \in R^d.$$

(2)

$$\int_{|x|<1} \frac{1}{\mid x \mid^{\beta}} dx, x \in R^d.$$

3. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 + x^2} dx, a > 0.$$

4. Find the maximum of the function $f(x,y,z) = x + y + z$ on the surface $2x^2 + y^2 + z^2 = 1$. Sketch the proof of the theorem you used.

5. Let V be a finite dimension vector space over R, $B : V \times V \longrightarrow R$ is bilinear, show that if
(1)$B(v,v) = 0, \forall v \in V$ (2)$B(v,w) = 0, \forall w \in V \Longrightarrow v = 0$
then dim V must be even.

6. Given a positive integer $p \geq 2$ and variables $x_1, x_2, ..., x_p$. Let $S = \{x_{i_1} x_{i_2} ... x_{i_p} \mid \{i_1, i_2, ... i_p\} = \{1, 2, ..., p\}\}$. Define an equivalence relation "$\sim$" on S by imposing the condition :

$$x_{i_1} x_{i_2} ... x_{i_k} x_{i_{k+1}} ... x_{i_p} \sim x_{i_1} x_{i_2} ... x_{i_{k+1}} x_{i_k} ... x_{i_p}$$

whenever $i_k - i_{k+1} \not\equiv \pm 1 \pmod{p}$. Show that S contains exactly $2^p - 2$ equivalence classes.

7. Let

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Find P such that $P^{-1}MP$ is the Jordan Cononical form of M .

8. (1) What is the definition of analytic function defined on C.
(2) Suppose $f : C \longrightarrow C$ is analytic, with Re$f(z)$ = constant. What can you say about f? Verify your statements.
(3) Suppose $f : C \longrightarrow C$ is analytic, with Arg$f(z)$ = constant. What can you say about f? Verify your statements.
(4) Suppose $f : C \cup \{\infty\} \longrightarrow C$ is analytic. What can you say about f.
(5) Suppose $f : C \cup \{\infty\} \longrightarrow C \cup \{\infty\}$ is analytic. What can you say about f? Verify your statements.