14402　六角星形配號問題(一)

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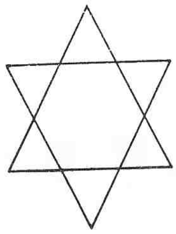
參考答案（張國男提供）

可設原圖所示之六角星形爲正六角星形（參閱評註第二段）。



因一解經旋轉及反映所得12解可併爲一類，故爲方便計，對於同類12解，本文均僅列出一解以爲代表（可選擇將1配予上一條水平直線左起第一個或第三個交點之標準解）。

因每一交點恰有二條直線經過，若對六條直線上之交點分條計算其號數和再相加，則每數皆計算二次，故知任一解之共線四點之號數和均爲 $(1+2+\cdots+12)\times2\div6=26$。

此後，對於四數組，均約定其數全異，且由小而大排之。

和爲26之四數組，計有下列33個：

(1,2,11,12), (1,3,10,12), (1,4,9,12), (1,4,10,11), (1,5,8,12), (1,5,9,11), (1,6,7,12), (1,6,8,11), (1,6,9,10), (1,7,8,10), (2,3,9,12), (2,3,10,11), (2,4,8,12), (2,4,9,11), (2,5,7,12), (2,5,8,11), (2,5,9,10), (2,6,7,11), (2,6,8,10), (2,7,8,9), (3,4,7,12), (3,4,8,11), (3,4,9,10), (3,5,6,12), (3,5,7,11), (3,5,8,10), (3,6,7,10), (3,6,8,9), (4,5,6,11), (4,5,7,10), (4,5,8,9), (4,6,7,9), (5,6,7,8)。

由上列33個，易知經過1之二條直線可選配之數組共有下列11對：

(a) (1,2,11,12)+(1,6,9,10),　(b) (1,2,11,12)+(1,7,8,10),　(c) (1,3,10,12)+(1,5,9,11),
(d) (1,3,10,12)+(1,6,8,11),　(e) (1,4,9,12)+(1,6,8,11),　(f) (1,4,9,12)+(1,7,8,10),
(g) (1,4,10,11)+(1,5,8,12),　(h) (1,4,10,11)+(1,6,7,12),　(i) (1,5,8,12)+(1,6,9,10),
(j) (1,5,9,11)+(1,6,7,12),　(k) (1,5,9,11)+(1,7,8,10)。

茲對情形(a)詳細討論如次：

(A)可設上一條水平直線 l_1 配以數組 (1,2,11,12)，並設直線 l_2 配以數組 (1,6,9,10)。先由前列33個找出平行於 l_1 之直線可選配之數組（即找出1,2,11,12均不出現且6,9,10恰出現一數之四數組），得 (3,5,8,10), (4,5,7,10), (4,5,8,9) 與 (5,6,7,8) 四個，再找出平行於 l_2 之直線可選配之數組，得 (3,4,7,12), (3,4,8,11) 與 (3,5,7,11) 三個，並由前四個與後三個中各取一個配對（有唯一共同數者始可配成一對），遂有下列6種可能情形：

(i)　(1,2,11,12)+(1,6,9,10)+(3,5,8,10)+(3,4,7,12)，

(ii)　(1,2,11,12)+(1,6,9,10)+(4,5,7,10)+(3,4,8,11)，

(iii)　(1,2,11,12)+(1,6,9,10)+(4,5,8,9)+(3,4,7,12)， (iv)　(1,2,11,12)+(1,6,9,10)+(4,5,8,9)+(3,5,7,11)，

(v)　(1,2,11,12)+(1,6,9,10)+(5,6,7,8)+(3,4,7,12)，

(vi)　(1,2,11,12)+(1,6,9,10)+(5,6,7,8)+(3,4,8,11)。

上列(i)至(vi)，均各有二組平行線而得四個交點，此等交點合計爲 1,3,4,5,6,7,8,9,10,11,12，故對於(i)至(vi)各種情形，皆可考慮經過 2，與原有四條直線均各交於一點，且不經過原有四個交點之直線可選配之數組。由前列 33 個，依上述條件求之，結果如下：

(i)　(1,2,11,12)+(1,6,9,10)+(3,5,8,10)+(3,4,7,12)+(2,7,8,9)：可得圖A1至A4。

(ii)　(1,2,11,12)+(1,6,9,10)+(4,5,7,10)+(3,4,8,11)+(2,7,8,9)：可得圖A5至A8。

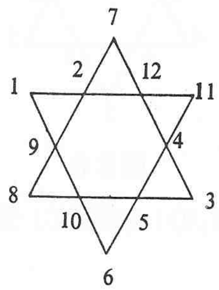
(iii)　(1,2,11,12)+(1,6,9,10)+(4,5,8,9)+(3,4,7,12)：無合乎條件之第五個數組可選配，故無解。

(iv)　(1,2,11,12)+(1,6,9,10)+(4,5,8,9)+(3,5,7,11)：同理，無解。

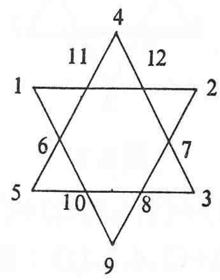
(v)　(1,2,11,12)+(1,6,9,10)+(5,6,7,8)+(3,4,7,12)：同理，無解。

(vi)　(1,2,11,12)+(1,6,9,10)+(5,6,7,8)+(3,4,8,11)：同理，無解。

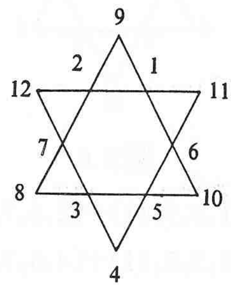
由是可知：若經過 1 之二條直線配以數組 (1,2,11,12)+(1,6,9,10)，則可得圖A1至A8所示八個代表。



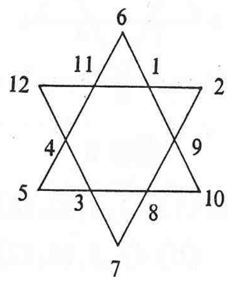
圖A 1



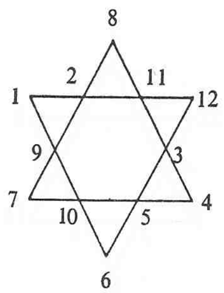
圖A 2



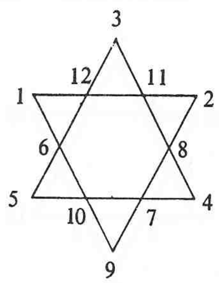
圖A 3



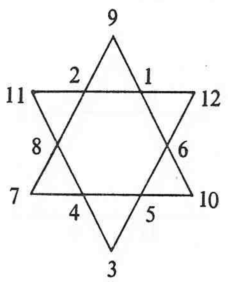
圖A 4



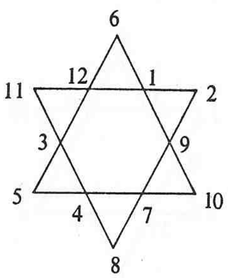
圖A 5



圖A 6



圖A 7



圖A 8

仿(A)之討論，對於(b)至(k)各種情形，可先找出前四條直線所配之數組，再依(b)，(c)，…，(k)之次序，考慮經過 2,3,3,3,2,5,5,2,3,2，與原有四條直線均各交於一點，且不經過原有四個交點之直線可選配之數組，故在(B)至(K)，不再敍述過程，僅列出結果。〔附記：爲配合評註第一段將互補對一一標出，圖B1至B4，D1至D4，F1至F4，I5至I8，J5至J8，K3,K4,K7與K8中之1已由標準位置（即上一條水平直線左起第一個及第三個交點）隨標準解作適當之反映或旋轉而配至該處。〕

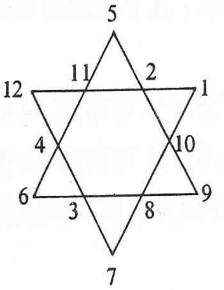
(B)(i)　(1,2,11,12)+(1,7,8,10)+(3,4,9,10)+(3,5,6,12)：無解。

(ii) (1,2,11,12)+(1,7,8,10)+(3,4,9,10)+(4,5,6,11)：無解。

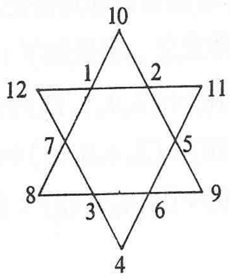
(iii) (1,2,11,12)+(1,7,8,10)+(3,6,8,9)+(4,5,6,11)+(2,5,9,10)：可得圖B1至B4。

(iv) (1,2,11,12)+(1,7,8,10)+(4,5,8,9)+(3,5,6,12)：無解。

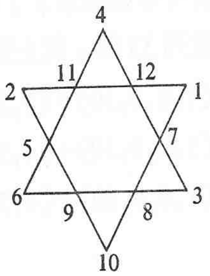
(v) (1,2,11,12)+(1,7,8,10)+(4,6,7,9)+(3,5,6,12)+(2,5,9,10)：可得圖B5至B8。



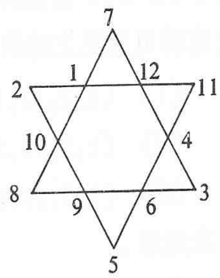
圖B 1



圖B 2



圖B 3



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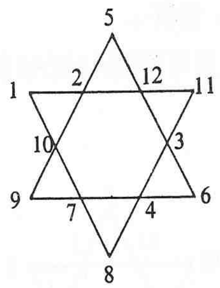

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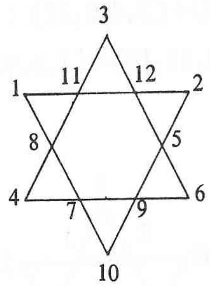
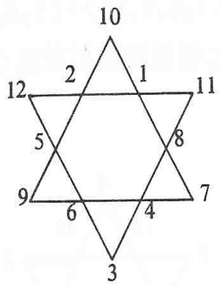
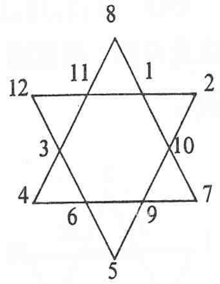


圖B 6



圖B 7



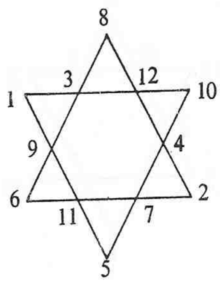
圖B 8

(C)(i) (1,3,10,12)+(1,5,9,11)+(2,6,7,11)+(2,4,8,12)+(3,6,8,9)：可得圖C1至C4。

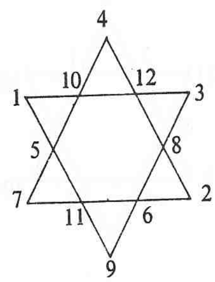
(ii) (1,3,10,12)+(1,5,9,11)+(4,6,7,9)+(2,4,8,12)：無解。

(iii) (1,3,10,12)+(1,5,9,11)+(4,6,7,9)+(2,6,8,10)+(3,4,8,11)：可得圖C5至C8。

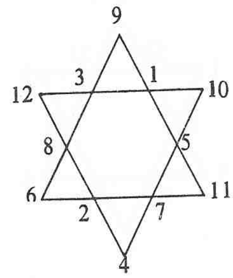
(iv) (1,3,10,12)+(1,5,9,11)+(5,6,7,8)+(2,4,8,12)：無解。



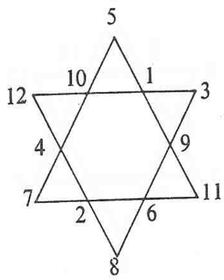
圖C 1



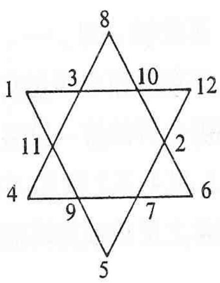
圖C 2



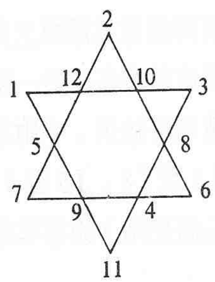
圖C 3



圖C 4



圖C 5



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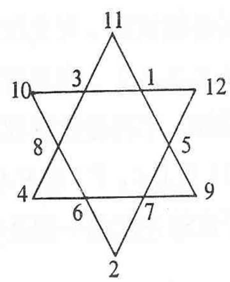
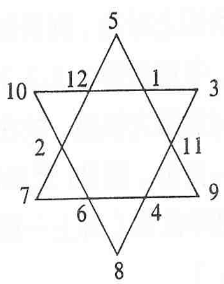


圖C 7



圖C 8

(D)(i) (1,3,10,12)+(1,6,8,11)+(2,4,9,11)+(2,5,7,12)：無解。

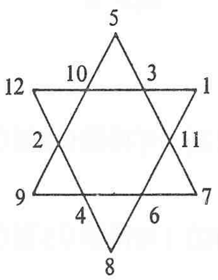
(ii) (1,3,10,12)+(1,6,8,11)+(2,4,9,11)+(4,5,7,10)：無解。

(iii) (1,3,10,12)+(1,6,8,11)+(2,7,8,9)+(4,5,7,10)：無解。

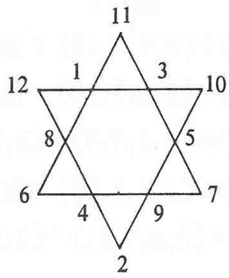
(iv) (1,3,10,12)+(1,6,8,11)+(4,5,8,9)+(2,5,7,12)：無解。

(v) (1,3,10,12)+(1,6,8,11)+(4,6,7,9)+(2,5,7,12)：無解。

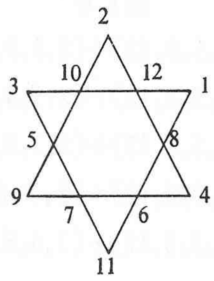
(vi) (1,3,10,12)+(1,6,8,11)+(4,6,7,9)+(2,5,9,10)+(3,5,7,11):可得圖D1至D4。



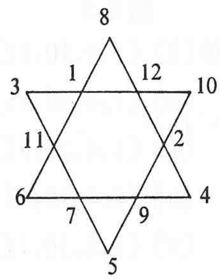
圖D 1



圖D 2



圖D 3



圖D 4

(E)(i) (1,4,9,12)+(1,6,8,11)+(2,3,10,11)+(2,5,7,12)：無解。

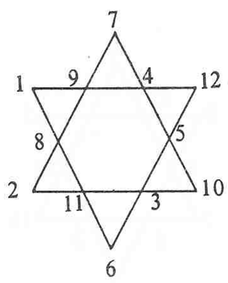
(ii) (1,4,9,12)+(1,6,8,11)+(2,3,10,11)+(4,5,7,10)+(3,5,6,12):可得圖E1至E4。

(iii) (1,4,9,12)+(1,6,8,11)+(3,5,7,11)+(2,5,9,10)：無解。

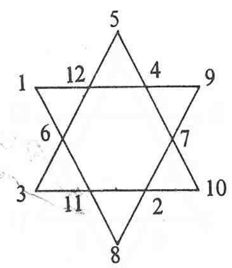
(iv) (1,4,9,12)+(1,6,8,11)+(3,5,8,10)+(2,5,7,12)：無解。

(v) (1,4,9,12)+(1,6,8,11)+(3,6,7,10)+(2,5,7,12)：無解。

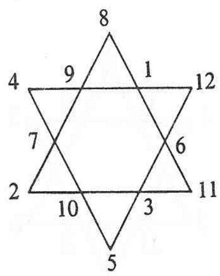
(vi) (1,4,9,12)+(1,6,8,11)+(3,6,7,10)+(2,5,9,10)：無解。



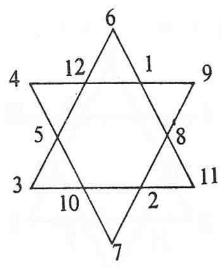
圖E 1



圖E 2



圖E 3



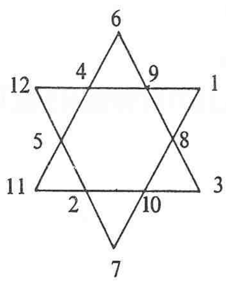
圖E 4

(F)(i) (1,4,9,12)+(1,7,8,10)+(2,3,10,11)+(3,5,6,12)：無解。

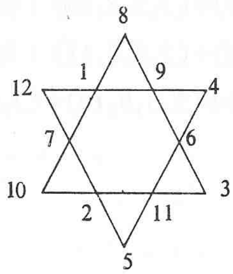
(ii) (1,4,9,12)+(1,7,8,10)+(2,3,10,11)+(4,5,6,11)+(2,5,7,12):可得圖F1至F4。

(iii) (1,4,9,12)+(1,7,8,10)+(2,5,8,11)+(3,5,6,12)：無解。

(iv) (1,4,9,12)+(1,7,8,10)+(2,6,7,11)+(3,5,6,12)+(2,5,9,10):可得圖F5至F8。



圖F 1



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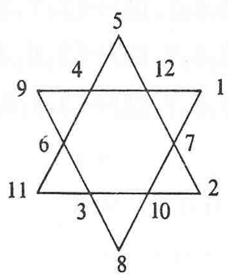

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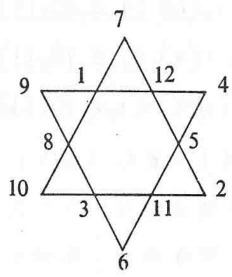
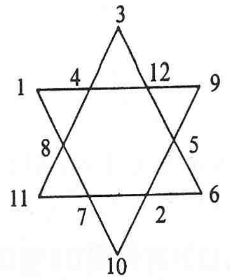


圖F 4



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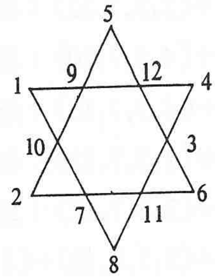
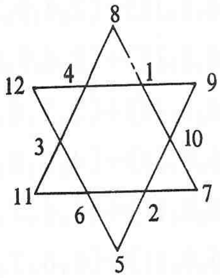
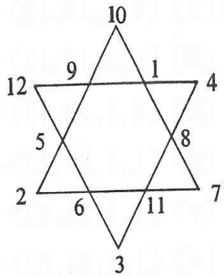


圖F 6



圖F 7



圖F 8

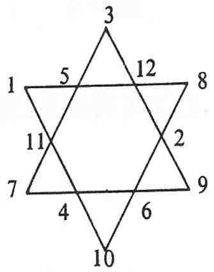
(G)(i) (1,4,10,11)+(1,5,8,12)+(2,3,9,12)+(2,6,7,11)：無解。

(ii) (1,4,10,11)+(1,5,8,12)+(2,3,9,12)+(3,6,7,10)：無解。

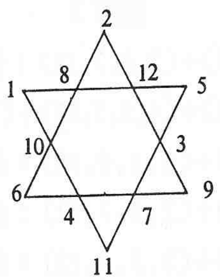
(iii) (1,4,10,11)+(1,5,8,12)+(2,3,9,12)+(4,6,7,9)+(3,5,7,11)：可得圖G1至G4。

(iv) (1,4,10,11)+(1,5,8,12)+(2,7,8,9)+(3,6,7,10)：無解。

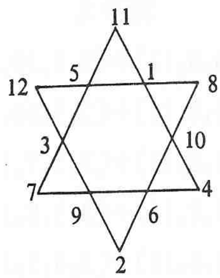
(v) (1,4,10,11)+(1,5,8,12)+(3,6,8,9)+(2,6,7,11)+(2,5,9,10)：可得圖G5至G8。



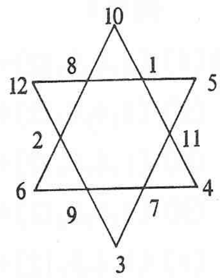
圖G 1



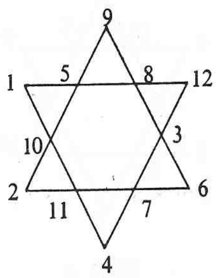
圖G 2



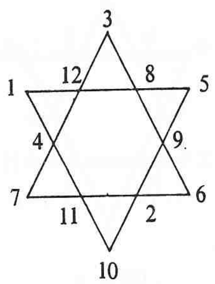
圖G 3



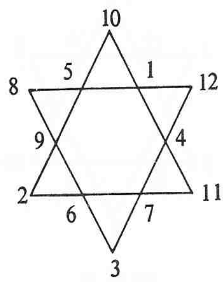
圖G 4



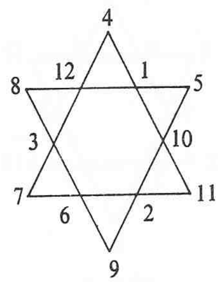
圖G 5



圖G 6



圖G 7



圖G 8

(H)(i) (1,4,10,11)+(1,6,7,12)+(2,3,9,12)+(2,5,8,11)：無解。

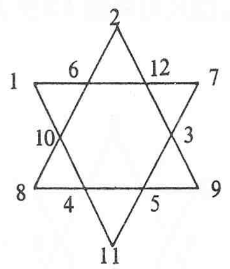
(ii) (1,4,10,11)+(1,6,7,12)+(2,3,9,12)+(3,5,8,10)：無解。

(iii) (1,4,10,11)+(1,6,7,12)+(2,3,9,12)+(4,5,8,9)+(3,5,7,11)：可得圖H1至H4。

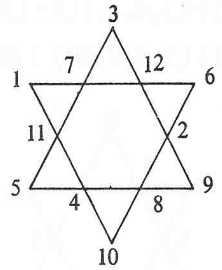
(iv) (1,4,10,11)+(1,6,7,12)+(2,7,8,9)+(3,5,8,10)：無解。

(v) (1,4,10,11)+(1,6,7,12)+(3,6,8,9)+(2,5,8,11)：無解。

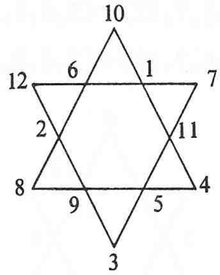
(vi) (1,4,10,11)+(1,6,7,12)+(3,6,8,9)+(2,5,9,10)+(3,5,7,11)：可得圖H5至H8。



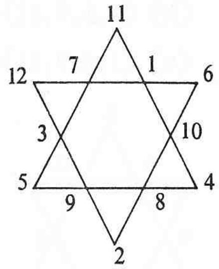
圖H 1



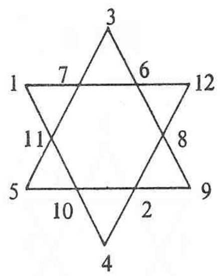
圖H 2



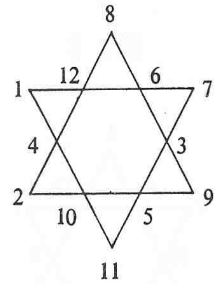
圖H 3



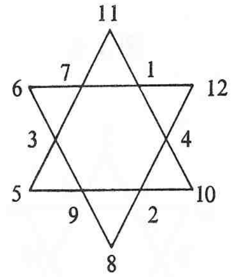
圖H 4



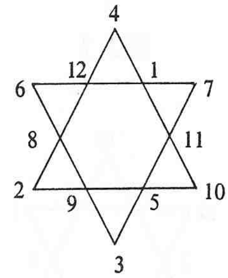
圖H 5



圖H 6



圖H 7



圖H 8

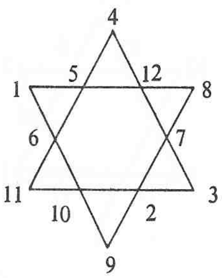
(I)(i) (1,5,8,12)+(1,6,9,10)+(2,3,10,11)+(3,4,7,12)+(2,7,8,9): 可得圖 I1至I4。

(ii) (1,5,8,12)+(1,6,9,10)+(2,4,9,11)+(3,4,7,12) ：無解。

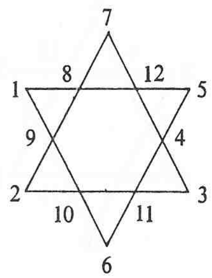
(iii) (1,5,8,12)+(1,6,9,10)+(2,4,9,11)+(3,5,7,11) ：無解。

(iv) (1,5,8,12)+(1,6,9,10)+(2,6,7,11)+(3,4,7,12) ：無解。

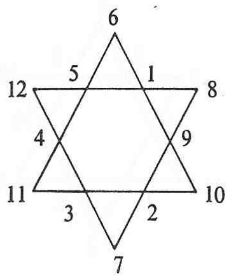
(v) (1,5,8,12)+(1,6,9,10)+(2,6,7,11)+(3,4,8,11)+(2,3,9,12): 可得圖 I5至I8。



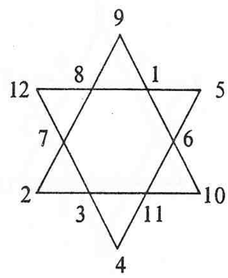
圖I 1



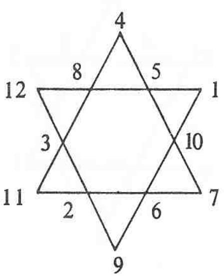
圖I 2



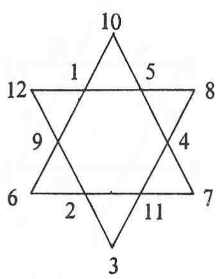
圖I 3



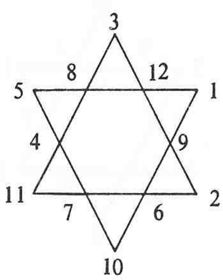
圖I 4



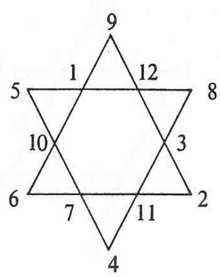
圖I 5



圖I 6



圖I 7



圖I 8

(J)(i) (1,5,9,11)+(1,6,7,12)+(2,4,8,12)+(2,3,10,11)+(3,6,8,9): 可得圖 J1至J4。

(ii) (1,5,9,11)+(1,6,7,12)+(2,4,8,12)+(3,4,9,10) ：無解。

(iii) (1,5,9,11)+(1,6,7,12)+(2,4,8,12)+(3,5,8,10)：無解。

(iv) (1,5,9,11)+(1,6,7,12)+(2,6,8,10)+(3,4,8,11)+(2,3,9,12)：可得圖 J5 至 J8。

(v) (1,5,9,11)+(1,6,7,12)+(2,6,8,10)+(3,4,9,10)：無解。

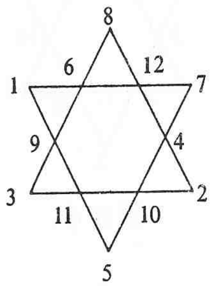


圖 J 1

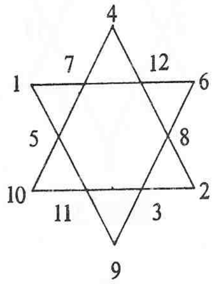


圖 J 2

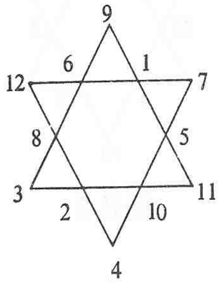


圖 J 3

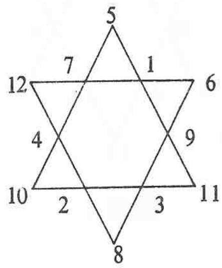


圖 J 4

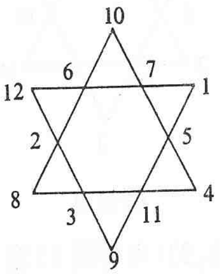


圖 J 5

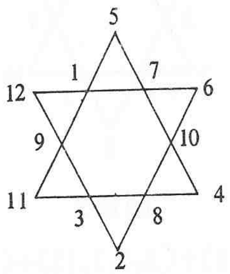


圖 J 6

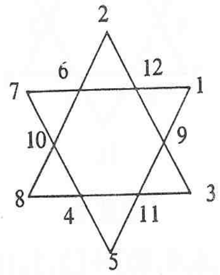


圖 J 7

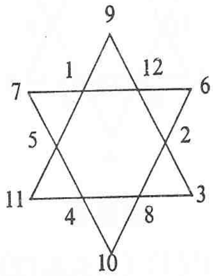


圖 J 8

(K)(i) (1,5,9,11)+(1,7,8,10)+(2,4,8,12)+(3,5,6,12)+(2,3,10,11)：可得圖 K1 至 K4。

(ii) (1,5,9,11)+(1,7,8,10)+(2,4,8,12)+(3,5,6,12)+(2,6,7,11)：可得圖 K5 至 K8。

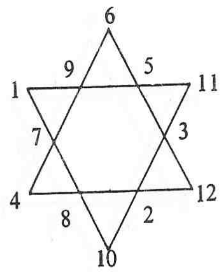


圖 K 1

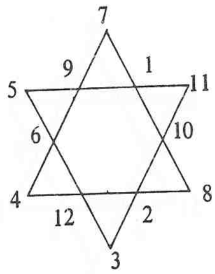


圖 K 2

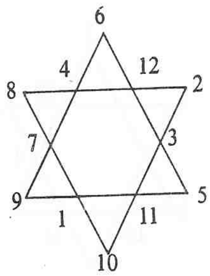


圖 K 3

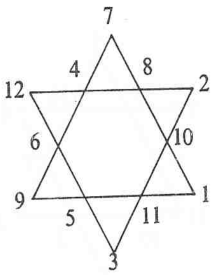


圖 K 4

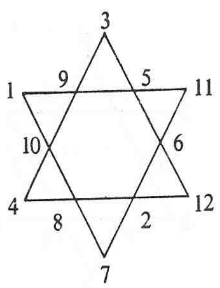


圖 K 5

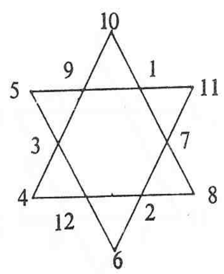


圖 K 6

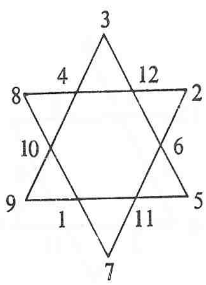


圖 K 7

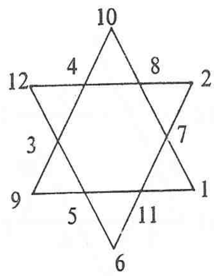


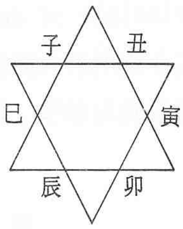
圖 K 8

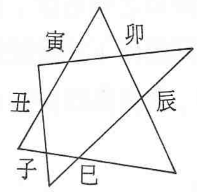
結論：將上列 80 個代表作旋轉及反映，即得所有解；共有 960 種配號法。

評　註

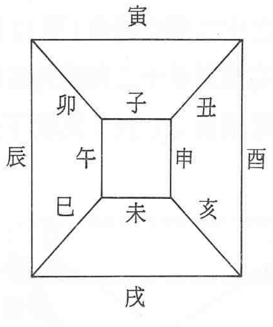
對於任一解，若以13為被減數，減去各交點之號數，以作為新配之號數（即各交點所配新舊二號數之和均為13），則所得者亦為解；如此二解，稱為互補解，所配成之對，稱為互補對。茲將上列80個解配成之40個互補對標出如下：(A1,A4),(A2,A3),(A5,B1),(A6,B2),(A7,B3),(A8,B4),(B5,B8),(B6,B7),(C1,C4),(C2,C3),(C5,D1),(C6,D2),(C7,D3),(C8,D4),(E1,F1),(E2,F2),(E3,F3),(E4,F4),(F5,F8),(F6,F7),(G1,G4),(G2,G3),(G5,I5),(G6,I6),(G7,I7),(G8,I8),(H1,H4),(H2,H3),(H5,J5),(H6,J6),(H7,J7),(H8,J8),(I1,I4),(I2,I3),(J1,J4),(J2,J3),(K1,K4),(K2,K3),(K5,K8),(K6,K7)。

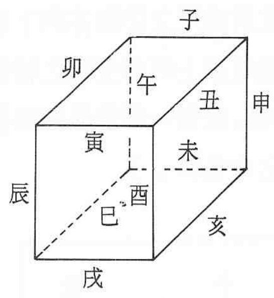
若將正斜二個六角星形之六邊形頂點同依順時針方向命名（參見下圖），易知斜六角星形之配號問題與正六角星形之配號問題合而為一。





再者，若採用下列三圖所示之命名法，顯然可知本文所處理之問題與下述問題相當：㈠將下左所示平面圖形之十二邊由1至12配號，使四邊形（共有六個）四邊之號數和均相等，試求所有配號法。㈡將下右圖所示六面體之十二條稜由1至12配號，使外表每面四條稜之號數和均相等，試求所有配號法。

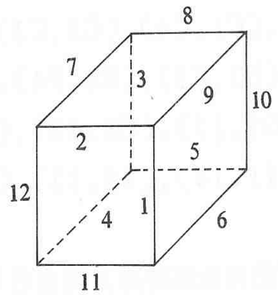




茲將問題㈡之直接解法簡述如下：

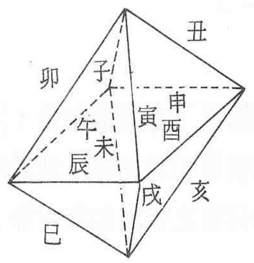
任一解經過適當之變換均可產生48個解（包括該解在內），說明如下：設ㄅ為已予任一解，而ㄆ為該六面體之一個複製品，其稜猶待配號。注意ㄅ之表面包含1之兩個四邊形區域，設$A=1$而此二個四邊形區域與1相對之邊所配之號數為B與C。首先，可將A配予ㄆ中任一稜，故A之配稜法共有12種。若A配予ㄆ中之稜甲，而ㄆ之表面包含甲之兩個四邊形區域與甲相對之邊為乙與丙，則可將B與C配予乙與丙，但順序不拘，故B，C之配稜法有2種，遂知A，B與C之配稜法有24種。A，B與C配定於ㄆ後，再將ㄅ之表面包含A與B之四邊形區域另外二邊所配之號數配予ㄆ之

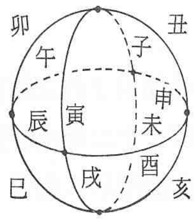
表包含過A與B之四邊形區域之另外二邊，其順序亦不拘，故亦有2種配稜法，遂知此五號數之配稜法共有48種。此五號數配定於ㄆ後，其他號數可依表面四邊形區域與邊之相互關係而由ㄅ配至ㄆ之其他稜，其配法唯一。由是可知：任一解經如此變換，均可產生48個解（包括該解在內）。如此48個解，可併爲一類，而僅列出其中一解以爲代表。例如由(1,2,11,12)+(1,6,9,10)+(3,5,8,10)+(3,4,7,12)+(2,7,8,9)所產生之48個解，可以



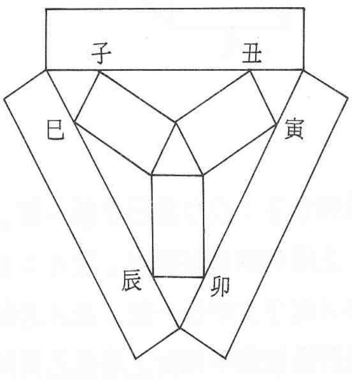
爲代表。職是之故，可知：若仿照本文所用之方法討論之，則代表圖解畫出20個即足矣！

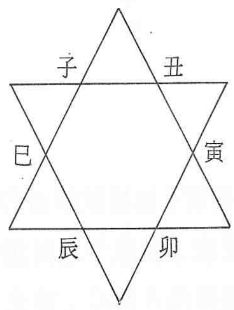
直接比較下列三圖所採用之命名法，或將對偶原理（principle of duality）應用於上述問題㈡，均可知本文所處理之問題與下述問題同義：將下左（右）圖所示八面體（球面）之十二條稜（四分圓周）由1至12配號，使共點四條稜（四分圓周）之號數和均相等，試求所有配號法。

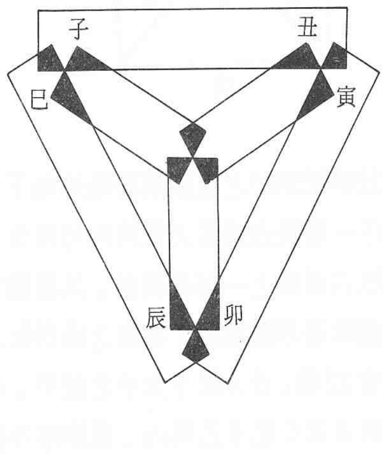


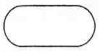
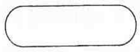


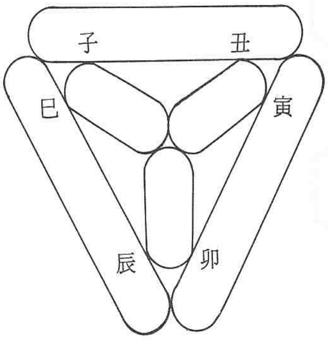
本文所處理之問題亦與下述問題等價：㈠將下左圖所示六個矩形之十二個交點由1至12配號，使每一個矩形上四個交點之號數和均相等，試求所有配號法。㈡將下右圖所示十二個黑色區域由1至12配號，使每一個矩形內四個黑色區域之號數和均相等，試求所有配號法。（註：比較下列三圖所採用之命名法。）

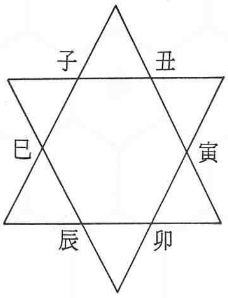


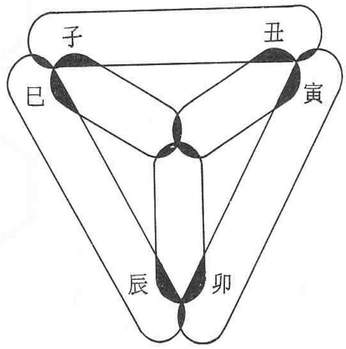




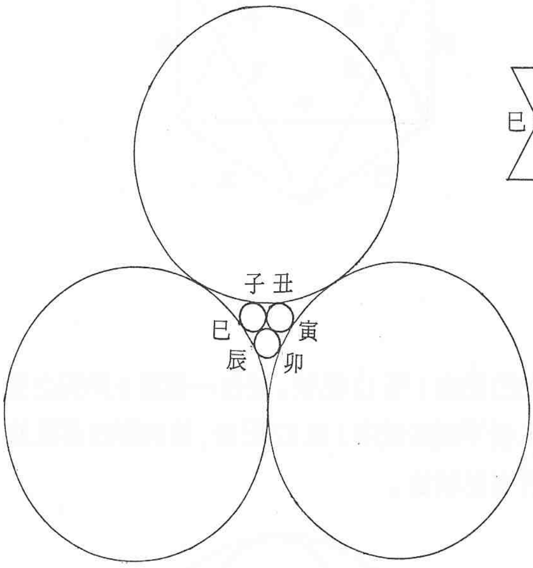
㈢將下左圖所示三個形如  及三個形如  之曲線之十二個切點由1至12配號，使如此形狀每一條曲線上四個切點之號數和均相等，試求所有配號法。㈣將下右圖所示十二個黑色區域由1至12配號，使如㈢所述形狀每一條曲線內四個黑色區域之號數和均相等，試求所有配號法。（註：比較下列三圖所採用之命名法。）

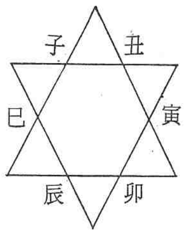


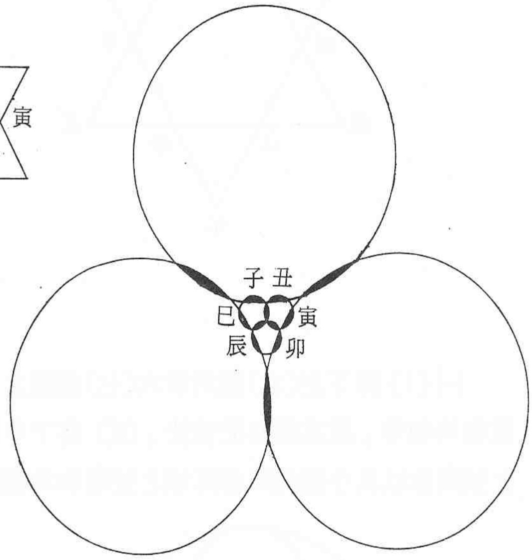




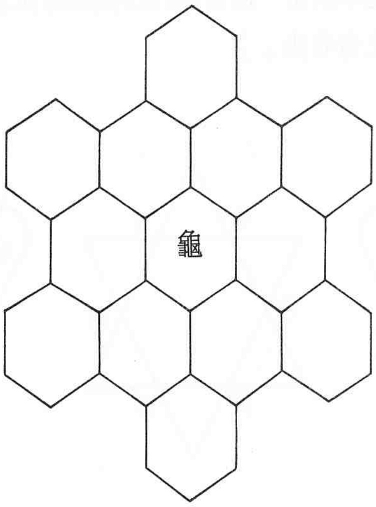
㈤將下左圖所示六個圓之十二個切點由1至12配號，使每一個圓上四個切點之號數和均相等，試求所有配號法。㈥將下右圖所示十二個黑色區域由1至12配號，使每一個圓內四個黑色區域之號數和均相等，試求所有配號法。（註：比較下列三圖所採用之命名法。）



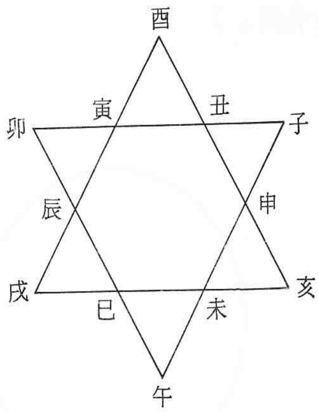


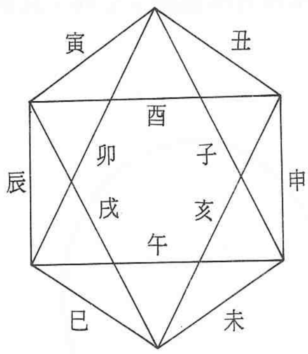


最後，易知本文所處理之問題與下述問題無別：㈠將下圖所示十二個正六邊形區域由1至12配號，使中心共線四個區域之號數和均相等，試求所有配號法。



㈡將構成下右圖之十二條線段由1至12配號，使經過大六邊形每一頂點四條線段之號數和均相等，試求所有配號法。(註：比較下列二圖所採用之命名法或應用對偶原理。)





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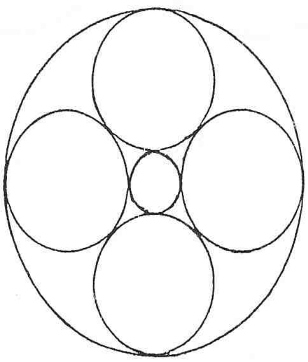
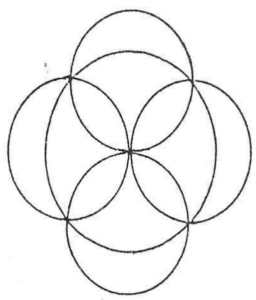
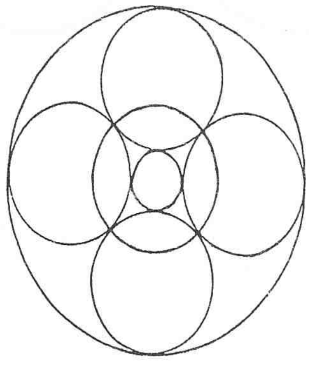
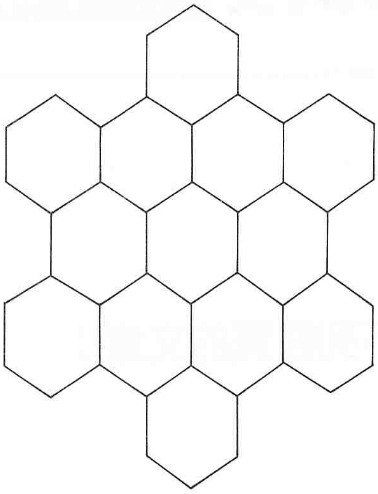
㈠(i)將下左(右)圖所示六(七)個圓之十二個切(交)點由1至12配號，使每一個圓上四點之號數和均相等，試求所有配號法。(ii)將下中圖所示十二個平面區域由1至12配號，使同形四個區域之號數和以及小圓內四個區域之號數和均相等，試求所有配號法。



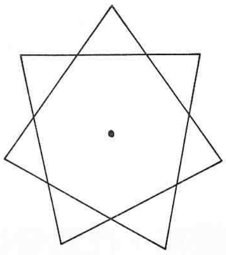




㈡將下圖所示龜形之十三個正六邊形區域由1至13配號，使中心共線之四個正六邊形區域之號數和均相等，試求所有配號法。〔提示：中央區域配以13之情形，已於本文討論。中央區域配以1之情形，與上述情形互補，可利用互補對應求解。（註：亦可以1配心，並將心外各區加1）其他情形，可仿上（以本文所用之解法及互補對應）處理之。〕



㈢將下圖所示正七角星形之十四個交點及其中心由1至15適當配號，可使共線四點之號數和均相等，試求一解。〔提示：可模仿本文所用之解法試配之。〕



㈣將下圖所示正八角星形之十六個交點及其中心由1至17適當配號，可使共線四點之號數和均相等，試求一解。