上期徵答問題

優勝名單

11201 優勝名單

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問題詳解

11201 「用鴿籠原理解數論問題」解答（張鎮華提供）

這個問題分為六部分，從簡單到難，事實上只要回答（已）部份，找出g（m，n）就回答了所有其他部份。問題所以這樣安排，是希望讀者從特例的經驗，一步步走到最後總答案，因此我們的解答也就不厭其煩，從簡單的寫起。

從另一個角度來看，先會解答簡單情況以後，下一步才有可能去問更難的問題。事實上當我寄出這個問題時，正在授課中討論(丁)，這本是Brualdi 書中的一個習題，回答了這個習題以後，很自然的，就問了(戊)和(己)這樣的問題。很高興可以告訴大家的是，最近這些問題的完整答案都已寫出來，由逢甲大學郝新生教授和我發表在ORDER 3（1987）355—357，文章名稱為The minimum of the antichains in the factor poset。（請參見「附錄」）

(甲)假設a_1，…，a_{101}是從1，…，200 中選出來的101個數，將他們寫成

$$a_1 = 2^{n_1} b_1$$
$$a_2 = 2^{n_2} b_2$$
$$\vdots$$
$$a_{101} = 2^{n_{101}} b_{101} \qquad \text{（共101個）}$$

其中各n_i為非負整數，各b_i為奇數，這些奇數(共101個)必是從1，3，5，……，199這100個數目中選出來的，所以由鴿籠原理，必存在某個$b_i = b_j$（其中$i \neq j$），假設$n_i \leq n_j$，我們可以取$x = a_i = 2^{n_i} b_i$，$y = a_j = 2^{n_j} b_j$，則x整除y。

(乙)將上述解答中的100用n替換，101用$n+1$，200用$2n$代替，就可以得到解答。

(丙) $n+1$，$n+2$，$n+3$，……，$2n$ 這 n 個數兩兩不互相整除。

(丁)假設 a_1，……，a_{100} 是從 1，………，200 中選出來的 100 個數，將它們寫成

$$a_2 = 2^{n_1} \cdot b_2$$

$$\vdots$$

$$a_{100} = 2^{n_{100}} b_{100}$$

其中各 n_i 爲負整數，各 b_i 爲奇數，如果有某個 $b_i = b_j$（其中 $i \neq j$），則如(甲)所述，可得某 x 整除 y。所以可以假設這 100 個奇數 b_i 均相異，也就是 1，3，……，199 全部出現在這些 b_i 中，且每一個恰好出現一次。爲簡化計，乾脆假設我們選出來的 100 個數是

$$c_1 = 2^{m_1} \cdot 1$$

$$c_3 = 2^{m_3} \cdot 1$$

$$\vdots$$

$$c_{199} = 2^{m_{199}} \cdot 199$$

其中各 m_i 爲非負整數，如果存在某個奇數 i 整除 j（其中 $1 \leq i < j \leq 199$）使得 $m_i \leq m_j$，則 c_i 整除 c_j，證明完畢，所以假設 $m_i < m_j$ 對於 i 整除 j 恒成立。考慮 c_1，c_3，c_9，c_{27}，c_{81}，因爲 1 整除 3，3 整除 9，……，所以 $m_1 > m_3 > m_9 > m_{27} > m_{81} \geq 0$，也就是 $m_1 \geq 4$，$m_3 \geq 3$，$m_9 \geq 2$。同理，考慮 c_5，c_{15}，c_{135}，可以得到 $m_5 \geq 3$，$m_{15} \geq 2$。考慮 c_7，c_{21}，c_{63}，c_{189}，可以得到 $m_7 \geq 3$。考慮 c_{11}，c_{33}，c_{99}，得到 $m_{11} \geq 2$。考慮 c_{13}，c_{39}，c_{117}，得到 $m_{13} \geq 2$。綜合而言，

$m_1 \geq 4$ 可得 $c_1 \geq 16$；
$m_3 \geq 3$ 可得 $c_3 \geq 24$；
$m_5 \geq 3$ 可得 $c_5 \geq 40$；
$m_7 \geq 3$ 可得 $c_7 \geq 56$；
$m_9 \geq 2$ 可得 $c_9 \geq 36$；
$m_{11} \geq 2$ 可得 $c_{11} \geq 44$；
$m_{13} \geq 2$ 可得 $c_{13} \geq 52$；
$m_{15} \geq 2$ 可得 $c_{15} \geq 60$；
$c_i \geq 17$ 對於 $i \geq 17$。

這和存在某個 $c_i < 16$ 矛盾。所以證明完畢。

(戊)和(己)的證明只是將上述的方法寫得更一般化就可，有興趣的讀者可以參考我們在 ORDER 上的文章。在這裡我只將答案寫出來，證明省略。

定理 $g(m,n) = 2^i$，對於 $i \leq \log_3 m$，$1 + s(m, i-1) < n \leq 1 + s(m, i)$ 恒成立，其中 $s(m, i)$ 是滿足 $m/3^i < x \leq m$ 的奇數個數。

〔例〕 $m = 200$ 時，$\log_3 m = 4.82$……

$s(m, 1) = 67$，$g(m, n) = 2$
其中 $2 \leq n \leq 68$；
$s(m, 2) = 89$，$g(m, n) = 4$
其中 $69 \leq n \leq 90$，
$s(m, 3) = 96$，$g(m, n) = 8$
其中 $91 \leq n \leq 97$；
$s(m, 4) = 99$，$g(m, n) = 16$
其中 $98 \leq n \leq 100$。

附錄：

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The Minimum of the Antichains in the Factor Poset

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Abstract. Denote $g(m, n)$ the minimum of min A, where A is a subset of $\{1, 2, \ldots, m\}$ of size n and there do not exist two distinct x and y in A such that x divides y. We use a method of poset to prove that $g(m, n) = 2^i$ for positive integer $i \leqslant \log_3 m$ and $1 + s(m, i-1) < n \leqslant 1 + s(m, i)$, where $s(m, i)$ is the number of odd integers x such that $m/3^i < x \leqslant m$.

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1. Introduction

Consider the poset $N(m) = \{1, 2, \ldots, m\}$ in which $x \propto y$ if and only if x is a proper factor of y; we call this poset a *factor poset*. A trivial linear extension of the factor poset $N(m)$ is to view $x < y$ as x less than y in the usual sense of integers. In this paper, the 'minimum' of a subset A of the factor poset $N(m)$ always means the minimum of A in this linear extension.

It is a well-known application of the pigeonhole principle to prove that the factor poset $N(2n)$ has no antichain of size $n+1$. A slightly more complicated argument proves that the factor poset $N(200)$ has no antichain of size 100 containing an element less than 16 in the usual sense (see page 22 of [1]). Let $g(m, n)$ be the minimum of the minimal element of an antichain of size n in the factor poset $N(m)$. The above results are then equivalent to (i) $g(2n, k)$ is

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defined only for $k \leqslant n$ and (ii) $g(200, 100) \geqslant 16$. The purpose of this paper is to generalize these results. $g(m, 1) = 1$ is clear. In general we have

MAIN THEOREM. *$g(m, n) = 2^i$ for positive integer $i \leqslant \log_3 m$ and $1 + s(m, i-1) < n \leqslant 1 + s(m, i)$, where $s(m, i)$ is the number of odd integers x such that $m/3^i < x \leqslant m$.*

2. The Proof of Main Theorem

The set $O(m)$ of all odd numbers in $N(m)$ induces a subposet of $N(m)$. Denote d the *depth function* of $O(m)$, i.e., for any $x \in O(m)$, $d(x)$ is the maximum r such that there exist $x = x_r \propto x_{r-1} \propto \cdots \propto x_1$ in $O(m)$. It is easy to see that $3^{d(x)-1}x \leqslant m < 3^{d(x)}x$; consequently, $d(x) = 1 + \lfloor \log_3(m/x) \rfloor$. Denote $L(m, i)$ the set of all elements of $O(m)$ whose depth is no more than i. Then $L(m, i) = \{x \in O(m): m < 3^i x\}$ and has size $s(m, i) = \lceil m/2 \rceil - \lceil \lfloor m/3^i \rfloor / 2 \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x and $\lfloor x \rfloor$ the largest integer less than or equal to x.

LEMMA 1. *If A is an antichain of the factor poset $N(m)$, then for any two distinct elements $2^r x$ and $2^s y$ of A, where $x, y \in O(m)$, we have* (i) *$x \neq y$ and* (ii) *$x \propto y$ implies $r > s$. Consequently, $A^* = \{x \in O(m): 2^r x \in A$ for some $r\}$ has the same size as A.*

LEMMA 2. *If an antichain A of the factor poset $N(m)$ contains the element 2^i with $i \geqslant 0$, then $|A| \leqslant 1 + s(m, i)$.*

Proof. The case of $i = 0$ is trivial, so we can assume that $i \geqslant 1$. By Lemma 1, every $x = 2^r y \in A' = A - \{2^i\}$ with $y \in O(m)$ must have $r < i$ and $y \neq 1$. Define f: $A' \to L(m, i)$ by $f(x) = 3^{\max\{0, d(y)-r-1\}} y$. Since $d(a) = 1 + \lfloor \log_3(m/a) \rfloor$ for any $a \in O(m)$, we have

$$d(f(x)) = d(y) - \max\{0, d(y) - r - 1\} = \min\{d(y), r+1\} \leqslant i,$$

which proves that f is well-defined.

Next we will show that f is one to one and so the lemma holds. Suppose there exist two distinct elements $x = 2^r y$ and $z = 2^s w$, where y and w are in $O(m)$, such that $f(x) = f(z)$. By the definition of f we know that

$$3^{\max\{0, d(y)-r-1\}} y = 3^{\max\{0, d(w)-s-1\}} w.$$

Without loss of generality, we have that $y \propto w$ and then $\max\{0, d(y) - r - 1\} > \max\{0, d(w) - s - 1\}$. The former implies $r > s$ by Lemma 1 and the latter implies $d(y) > r + 1$. So

$$\begin{aligned} d(f(x)) &= \min\{d(y), r+1\} \\ &= r + 1 > s + 1 \geqslant \min\{d(w), s+1\} = d(f(z)), \end{aligned}$$

which contradicts $f(x) = f(z)$.

LEMMA 3. *If an antichain A of the factor poset $N(m)$ contains $x < 2^{i+1}$ for some i, then $|A| \leqslant 1 + s(m, i)$.*

Proof. Suppose $x = 2^r y$, where $y \in O(m)$. Consider $B = \{z \in N(n) : zy \in A\}$, where $n = \lfloor m/y \rfloor$. It is clear that B is an antichain of $N(n)$. Since $2^r \in B$, by Lemma 2, $|B| \leqslant 1 + s(n, r)$. Let $A^* \subseteq O(m)$ and $B^* \subseteq O(n)$ be the sets corresponding to A and B, respectively, as in Lemma 1. Then $|A^*| = |A|$ and $|B^*| = |B|$. Also

$$\begin{aligned} |O(m)| - |A^*| &\geqslant |O(n)| - |B^*| \\ &\geqslant \lceil n/2 \rceil - 1 - s(n, r) = \lceil \lfloor n/3^r \rfloor / 2 \rceil - 1. \end{aligned}$$

Note that $3^{i-r} \geqslant 2^{i+1-r} - 1 \geqslant y$, so $m/(y3^r) \geqslant m/3^i$ and then

$$\begin{aligned} |A| = |A^*| &\leqslant \lceil m/2 \rceil - \lceil \lfloor n/3^r \rfloor / 2 \rceil + 1 \\ &\leqslant \lceil m/2 \rceil - \lceil \lfloor m/3^i \rfloor / 2 \rceil + 1 = 1 + s(m, i). \end{aligned}$$

Proof of Main Theorem. Let A be an antichain of size n in the factor poset $N(m)$ such that $g(m, n)$ is the minimum of A. Suppose $g(m, n) < 2^i$. By Lemma 3, we have $n = |A| \leqslant 1 + s(m, i-1)$, which is impossible. Thus $g(m, n) \geqslant 2^i$.

On the other hand, since $n \leqslant 1 + s(m, i)$, we can choose a subset A' of $L(m, i)$ of size $n - 1$. Note that $i \leqslant \log_3 m < d(1)$ and $1 \notin L(m, i)$. Let $A = \{2^i\} \cup \{2^{d(x)-1} x : x \in A'\}$. It is clear that A is an antichain of size n in $N(m)$. For any $x \in A'$ we have

$$3^{d(1)-1} \leqslant m < 3^{d(x)} x.$$

Then

$$3^{d(1)-d(x)-1} < x.$$

So

$$2^{d(1)-d(x)} \leqslant 3^{d(1)-d(x)-1} + 1 \leqslant x.$$

Therefore

$$2^i \leqslant 2^{d(1)-1} \leqslant 2^{d(x)-1} x.$$

This proves that $\min A = 2^i$ and then $g(m, n) \leqslant 2^i$. So $g(m, n) = 2^i$.

EXAMPLE. $m = 200$, $|O(m)| = 100$, $\log_3 200 = 4.82\ldots$

$$\begin{aligned} &s(m, 1) = 67, g(m, n) = 2 && \text{for } 2 \leqslant n \leqslant 68, \\ &s(m, 2) = 89, g(m, n) = 4 && \text{for } 69 \leqslant n \leqslant 90, \\ &s(m, 3) = 96, g(m, n) = 8 && \text{for } 91 \leqslant n \leqslant 97, \\ &s(m, 4) = 99, g(m, n) = 16 && \text{for } 98 \leqslant n \leqslant 100, \end{aligned}$$

Reference

1. R. A. Brualdi (1977) *Introductory Combinatorics*, North-Holland, New York, Oxford, Amsterdam.