8402 計算極限值解答 （余文卿提供）

因級數 $\sum_{n=-\infty}^{\infty}\left(\frac{1}{i\left(n+\frac{1}{3}\right)}\right)^{1+\varepsilon}$ 絕對收斂

，故次序可以調整，而級數為

$$\sum_{n=0}^{\infty}\left(\frac{1}{i\left(n+\frac{1}{3}\right)}\right)^{1+\varepsilon}+\sum_{n=-1}^{\infty}\left(\frac{1}{i\left(n+\frac{1}{3}\right)}\right)^{1+\varepsilon}$$

$$=\sum_{n=0}^{\infty}\frac{\exp[-\pi i(1+\varepsilon)/2]}{(n+1/3)^{1+\varepsilon}}$$

$$+\sum_{n=0}^{\infty}\frac{\exp[\pi i(1+\varepsilon)/2]}{(n+2/3)^{1+\varepsilon}}$$

$$=\sum_{n=0}^{\infty}\frac{-2\sin\pi\varepsilon/2}{(n+1/3)^{1+\varepsilon}}+\exp[\pi i(1+\varepsilon)/2]$$

$$\cdot\sum_{n=0}^{\infty}\left(\frac{1}{(n+2/3)^{1+\varepsilon}}-\frac{1}{(n+1/3)^{1+\varepsilon}}\right)$$

這裏用到尤拉式 $\exp(i\theta)=\cos\theta+i\sin\theta$ 由泰勒展開式得出

$$\sin\pi\varepsilon/2=\frac{\pi\varepsilon}{2}-\frac{1}{3!}\left(\frac{\pi\varepsilon}{2}\right)+\cdots\cdots+$$

$$\frac{(-1)^{n+1}}{n!}\left(\frac{\pi\varepsilon}{2}\right)^{n}+\cdots\cdots$$

另一方面設

$$\phi_n(\varepsilon)=\frac{1}{(n+1/3)^{1+\varepsilon}}-\int_n^{n+1}\frac{dx}{(x+1/3)^{1+\varepsilon}}$$

則很容易可看出 $\phi_n(\varepsilon)>0$，$\sum_{n=1}^{\infty}\phi_n(\varepsilon)$ 收斂且

$$\sum_{n=0}^{\infty}\phi_n(\varepsilon)\leq 1\ ,\ \forall\ \varepsilon>0$$

而 $\sum_{n=0}^{\infty}\frac{1}{(n+1/3)^{1+\varepsilon}}=\sum_{n=0}^{\infty}\phi_n(\varepsilon)+\frac{1}{\varepsilon}$

由此得到第一項的極限值是 $-\pi$，而第二項的極限值是

$$i\sum_{n=0}^{\infty}\left(\frac{1}{n+2/3}-\frac{1}{n+1/3}\right)$$

$$=-3i\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\cdots\cdots+\frac{1}{3n+1}-\frac{1}{3n+2}+\cdots\cdots\right)$$

$$=-3i\frac{\pi}{3\sqrt{3}}=-\frac{\pi i}{\sqrt{3}}$$

故最後得出極限值是 $-\pi-\pi i/\sqrt{3}$