與三角形高有關的幾何性質

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摘要: 本文獲得了與三角形高有關的三個有趣的幾何性質, 並進行推廣。

關鍵詞: 三角形、邊長、高線、半周長、外接圓半徑、內切圓半徑。

本文約定: $\triangle ABC$ 的三邊長、半周長、面積、外接圓半徑、內切圓半徑、三邊上的高分別爲 a、b、c、P、S、R、r、h_a、h_b、h_c

經過探討, 筆者現已得到:

定理1: $\frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r}=\frac{4R}{r}+1$

證明: $\because S=\frac{1}{2}ah_a=\frac{1}{2}bh_b=\frac{1}{2}ch_c=rP$
$\therefore h_a=\frac{2rP}{a}$, $h_b=\frac{2rP}{b}$, $h_c=\frac{2rP}{c}$
由海倫公式 $S=\sqrt{P(P-a)(P-b)(P-c)}$ 便可得到

$$(P-a)(P-b)(P-c)=r^2P$$

又 $\because abc=4RrP$

$\therefore abc+(P-a)(P-b)(P-c)=4RrP+r^2P$

$\because abc+(P-a)(P-b)(P-c)$

$=abc+[P^3-(a+b+c)P^2+(ab+bc+ca)P-abc]$

$=P^3-2P^3+(ab+bc+ca)P$

$=(ab+bc+ca)P-P^3$

$\therefore (ab+bc+ca)P-P^3=4RrP+r^2P$

$\therefore ab+bc+ca=P^2+4Rr+r^2$

$\therefore \dfrac{1}{P-a}+\dfrac{1}{P-b}+\dfrac{1}{P-c}$

$$
\begin{aligned}
&= \frac{ab+bc+ca-P^2}{(P-a)(P-b)(P-c)} \\
&= \frac{P^2+4Rr+r^2-P^2}{r^2P} \\
&= \frac{4R+r}{rP} \\
&\therefore \frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r} \\
&= \frac{\frac{2rP}{a}}{\frac{2rP}{a}-2r}+\frac{\frac{2rP}{b}}{\frac{2rP}{b}-2r}+\frac{\frac{2rP}{c}}{\frac{2rP}{c}-2r} \\
&= \frac{P}{P-a}+\frac{P}{P-b}+\frac{P}{P-c} \\
&= P\left(\frac{1}{P-a}+\frac{1}{P-b}+\frac{1}{P-c}\right) \\
&= P\cdot\frac{4R+r}{rP} \\
&= \frac{4R}{r}+1
\end{aligned}
$$

故: $\frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r}=\frac{4R}{r}+1$

由 Euler 不等式 $R\geq 2r$, 便可得到:

推論1: $\frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r}\geq 9$

這便是著名的 Bokov 不等式。

由 Wlombier-Doncet 不等式 $3P^2\leq(4R+r)^2$ 便可推出: $P\leq\frac{4R+r}{\sqrt{3}}$

於是又可得到:

推論2: $\frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r}\geq\frac{\sqrt{3}P}{r}$

由於

$$
\begin{aligned}
\frac{h_a}{h_a-2r} &= 1+2r\cdot\frac{1}{h_a-2r} \\
\frac{h_b}{h_b-2r} &= 1+2r\cdot\frac{1}{h_b-2r} \\
\frac{h_c}{h_c-2r} &= 1+2r\cdot\frac{1}{h_c-2r}
\end{aligned}
$$

$$
\begin{aligned}
&\therefore \frac{h_a}{h_a-2r}+\frac{h_b}{h_b-2r}+\frac{h_c}{h_c-2r} \\
&= 3+2r\left(\frac{1}{h_a-2r}+\frac{1}{h_b-2r}+\frac{1}{h_c-2r}\right)
\end{aligned}
$$

再由定理1知:

$$3+2r\left(\frac{1}{h_a-2r}+\frac{1}{h_b-2r}+\frac{1}{h_c-2r}\right)=\frac{4R}{r}+1$$

於是, 我們便可得到:

推論3: $\frac{1}{h_a-2r}+\frac{1}{h_b-2r}+\frac{1}{h_c-2r}=\frac{2R-r}{r^2}$

再由 Euler 不等式 $R\geq 2r$, 便可得到:

推論4: $\frac{1}{h_a-2r}+\frac{1}{h_b-2r}+\frac{1}{h_c-2r}\geq\frac{3}{r}$

定理2: $\frac{b+c}{h_b+h_c}+\frac{c+a}{h_c+h_a}+\frac{a+b}{h_a+h_b}\geq 2\sqrt{3}$

證明: $\because S=\frac{1}{2}ah_a=\frac{1}{2}bh_b=\frac{1}{2}ch_c=\frac{abc}{4R}$

$\therefore h_a=\dfrac{bc}{2R},\quad h_b=\dfrac{ca}{2R},\quad h_c=\dfrac{ab}{2R}$

$\therefore \dfrac{b+c}{h_b+h_c}+\dfrac{c+a}{h_c+h_a}+\dfrac{a+b}{h_a+h_b}$

$=\dfrac{b+c}{\frac{a}{2R}(c+b)}+\dfrac{c+a}{\frac{b}{2R}(a+c)}+\dfrac{a+b}{\frac{c}{2R}(b+a)}$

$=2R\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$ ··(1)

$\because \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

$\geq 3\sqrt[3]{\dfrac{1}{abc}}$

$=\dfrac{3}{2R}\sqrt[3]{\dfrac{1}{\sin A\sin B\sin C}}$

又 $\because \sin A\cdot\sin B\cdot\sin C\leq\dfrac{3\sqrt{3}}{8}$

$\therefore \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\geq\dfrac{\sqrt{3}}{R}$ ··(2)

由 (1) 、(2) 得:

$\dfrac{b+c}{h_b+h_c}+\dfrac{c+a}{h_c+h_a}+\dfrac{a+b}{h_a+h_b}\geq 2\sqrt{3}$

又由於: $\dfrac{b+c}{h_b+h_c}+\dfrac{c+a}{h_c+h_a}+\dfrac{a+b}{h_a+h_b}$

$=\dfrac{2P-a}{h_b+h_c}+\dfrac{2P-b}{h_c+h_a}+\dfrac{2P-c}{h_a+h_b}$

$=2P\left(\dfrac{1}{h_b+h_c}+\dfrac{1}{h_c+h_a}+\dfrac{1}{h_a+h_b}\right)-\left(\dfrac{a}{h_b+h_c}+\dfrac{b}{h_c+h_a}+\dfrac{c}{h_a+h_b}\right)$

$$\begin{aligned}
&\therefore \frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\\
&\le 2P\left(\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a}+\frac{1}{h_a+h_b}\right)-2\sqrt{3}\\
&=2P\left(\frac{1}{\frac{2rP}{b}+\frac{2rP}{c}}+\frac{1}{\frac{2rP}{c}+\frac{2rP}{a}}+\frac{1}{\frac{2rP}{a}+\frac{2rP}{b}}\right)-2\sqrt{3}\\
&=\frac{1}{r}\left(\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\right)-2\sqrt{3}\\
&\le \frac{1}{r}\left(\frac{b+c}{4}+\frac{c+a}{4}+\frac{a+b}{4}\right)-2\sqrt{3}\\
&=\frac{1}{r}\cdot\frac{1}{2}(a+b+c)-2\sqrt{3}\\
&=\frac{P}{r}-2\sqrt{3}
\end{aligned}$$

又 $\because P=\frac{1}{2}(a+b+c)$

$\quad\quad = R(\sin A+\sin B+\sin C)$

且 $\sin A+\sin B+\sin C\le \frac{3\sqrt{3}}{2}$

$\therefore P\le \frac{3\sqrt{3}}{2}R$

於是, 我們便可得到:

推論: $\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\le \frac{3\sqrt{3}R}{2r}-2\sqrt{3}$

定理3: $\frac{9R^2}{4R^2+2r^2}\le \frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}\le \frac{R}{r}$

證明: $\because S=\frac{1}{2}ah_a=\frac{1}{2}bh_b=\frac{1}{2}ch_c$

$\therefore h_a=\frac{2S}{a},\ h_b=\frac{2S}{b},\ h_c=\frac{2S}{c}$

$$\begin{aligned}
&\therefore \frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}\\
&=\frac{a^2}{4S^2(\frac{1}{b^2}+\frac{1}{c^2})}+\frac{b^2}{4S^2(\frac{1}{c^2}+\frac{1}{a^2})}+\frac{c^2}{4S^2(\frac{1}{a^2}+\frac{1}{b^2})}\\
&=\frac{a^2b^2c^2}{4S^2(b^2+c^2)}+\frac{a^2b^2c^2}{4S^2(c^2+a^2)}+\frac{a^2b^2c^2}{4S^2(a^2+b^2)}\\
&=\frac{a^2b^2c^2}{4S^2}\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\right)\\
&=\frac{(4RS)^2}{4S^2}\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\right)
\end{aligned}$$

$$= 4R^2 \cdot \left(\frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} + \frac{1}{a^2+b^2}\right) \cdots\cdots (3)$$

又 $\because b^2+c^2 \geq 2bc, \quad c^2+a^2 \geq 2ca, \quad a^2+b^2 \geq 2ab$

$$\therefore \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} + \frac{1}{a^2+b^2}$$
$$\leq \frac{1}{2bc} + \frac{1}{2ca} + \frac{1}{2ab}$$
$$= \frac{1}{2}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)$$
$$= \frac{a+b+c}{2abc}$$
$$= \frac{2P}{2 \cdot 4RrP}$$
$$= \frac{1}{4Rr} \cdots\cdots (4)$$

由 (3)、(4) 得

$$\frac{a^2}{h_b^2+h_c^2} + \frac{b^2}{h_c^2+h_a^2} + \frac{c^2}{h_a^2+h_b^2} \leq \frac{R}{r}$$

$$\because [(a^2+b^2)+(b^2+c^2)+(c^2+a^2)]\left(\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2}\right) \geq 9$$

$$\therefore \frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{9}{2(a^2+b^2+c^2)}$$

又由於 $a^2+b^2+c^2 = 2(P^2-4Rr-r^2)$

且由 Gerretsen 不等式知:

$$P^2 \leq 4R^2 + 4Rr + 3r^2$$

$$\therefore a^2+b^2+c^2 \leq 2[(4R^2+4Rr+3r^2) - 4Rr - r^2]$$
$$= 2(4R^2+2r^2)$$

$$\therefore \frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{9}{4(4R^2+2r^2)} \cdots\cdots (5)$$

由 (3)、(5) 得

$$\frac{a^2}{h_b^2+h_c^2} + \frac{b^2}{h_c^2+h_a^2} + \frac{c^2}{h_a^2+h_b^2} \geq \frac{9R^2}{4R^2+2r^2}$$

故 $\dfrac{9R^2}{4R^2+2r^2} \leq \dfrac{a^2}{h_b^2+h_c^2} + \dfrac{b^2}{h_c^2+h_a^2} + \dfrac{c^2}{h_a^2+h_b^2} \leq \dfrac{R}{r}$

又 $\because \dfrac{9R^2}{4R^2+2r^2} = \dfrac{8R^2+R^2}{4R^2+2r^2}$

$$\geq \frac{8R^2+(2r)^2}{4R^2+2r^2}$$
$$= 2$$

於是, 我們又可得到:

Cordon 不等式:

$$\frac{a^2}{h_b^2+h_c^2}+\frac{b^2}{h_c^2+h_a^2}+\frac{c^2}{h_a^2+h_b^2}\geq 2$$

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