中央研究院數學研究所
招考八十四年度研習員筆試試題
及錄取名單

試題:

1. (a) Let $p_i > 0, q_i > 0, i = 1, \cdots, n$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Then

$$-\sum_{i=1}^{n} p_i \log p_i \leq -\sum_{i=1}^{n} p_i \log q_i$$

with equality iff $p_i = q_i$ for all i.

(b) Use (a) to prove the inequality between the arithmetic and geometric means: Let $x_1, \cdots, x_n$ be arbitrary positive numbers, let $a_1, \cdots, a_n > 0$ and $\sum_{i=1}^n a_i = 1$. Then

$$x_1^{a_1} x_2^{a2} \cdots x_n^{a_n} \leq \sum_{i=1}^{n} a_i x_i$$

with equality iff all x_i are equal.

(c) Let $f : [0,1] \longrightarrow R$ be a continuous function. Prove the following continuous version of the inequality between the arithmetic and geometric means

$$\exp \int_0^1 f(x)dx \leq \int_0^1 \exp f(x)dx.$$

2. Find the conditions for α, β, such that

$$\int_0^\infty \frac{e^{-x}dx}{x^\alpha + x^\beta}, \text{ converge.}$$

3. Prove or disprove (give a counterexample) the following statements:

   (a) $\Sigma_{n=1}^{\infty} a_n < +\infty, a_n \geq 0$ implies $\lim_{n\to\infty} a_n = 0$.

   (b) $\int_0^\infty f(x)dx < +\infty$ where $f : [0,\infty) \longrightarrow R$ be a nonnegative continuous function implies $\lim_{x\to\infty} f(x) = 0$.

4. Let f_n be a bounded sequence of holomorphic functions on the unit disk $\triangle$ in C such that
$$\lim_{n\to\infty} f_n^{(k)}(0) = 0 \text{ for all } k.$$
Show that $f_n \to 0$ uniformly on any compact subset of $\triangle$.

5. Let M be an $n \times n$ matrix over C

   (a) State a necessary and sufficient condition for M to be diagonalizable over C.

   (b) Prove your statement.

錄取名單:

林昭廷: 國立台灣大學數學研究所

林秀穎: 國立清華大學數學研究所