國立中正大學八十一學年度應用數學研究所
碩士班研究生招生考試試題

基礎數學

I.(20%) Test for convergence or divergence of the following infinite series

$$(a) \sum_{n=1}^{\infty} \frac{\cos(\frac{\pi}{n})}{n} \qquad (b) \sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{n})}{n}$$

$$(c) \sum_{n=2}^{\infty} \frac{1}{n(\log\, n)^p} \quad (p > 1) \qquad (d) \sum_{n,m=1}^{\infty} \frac{1}{n^2 + m^2}$$

II.(15%) Compute the following integrals and differentiation

$$(a) \int_0^{a^{12}} \frac{dt}{a^2 - \sqrt{t}} \quad (a < 1) \qquad (b) \int_0^{2\pi} \sin\, x^{2n+1} dx \quad (n \geq 0 \text{ integer})$$

$$(c) \frac{\partial f}{\partial z} \text{ where } f(x, y, z) = \phi(xe^{-z}, ye^{-2z}) \cdot e^{-3z}, \quad \phi(u, v) = e^{uv}$$

III.(15%) Find the maximum and minimum of $f(x) = 3x - 2y + z$ subject to the condition $x^2 + 3y^2 + 6z^2 = 1$.

IV.(10%) Let A, B be compact subsets of R^n. $f : A \to B$ is 1-1, onto and continuous. Show that f^{-1} is continuous.

V.(5%) Given 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with det $A < 0$. Show that A is diagonalizable.

VI.(5%) Let A be an $n \times n$ matrix with the property $A^k = 0$ for some $k > 0$, integer. Show that both $I - A$, $I + A$ are not invertible.

VII.(10%) Let $A = \begin{pmatrix} 1 & a_1 & \cdots & & a_n \\ -a_1 & 1 & & & \\ & & & 0 & \\ \vdots & & \ddots & & \\ & 0 & & & \\ -a_n & & & & 1 \end{pmatrix}$

(a) Find A^{-1} (b) Find all eigenvalues of A.

VIII.(10%) (a) Define an "inner product" space.

(b) State and prove the Cauchy-Schwarz inegaulity for an inner product space.

IX.(10%) Prore or disprove following statement:

Let V be any vector space $T : V \to V$ is a linear map. If T is 1-1, then T is onto!

統計學

(20%)**1.** Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the distribution with p.d.f. $f(x; \theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$.

(5%)(a) Find the method of moments estimator of θ.

(5%)(b) Find the maximam likelihood estimator of θ.

(10%)(c) Let $n = 1$, find the most powerful test with significant level $\alpha = .05$ for testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(20%)**2.** Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. For testing $H_0 : \sigma^2 = \sigma_0^2$ verses $H_1 : \sigma^2 \neq \sigma_0^2$, show that the likelihood ratio test is equivalent to the χ^2 (Chi-squared) test for variances.

(25%)**3.** Consider the simple linear regression model:

$$Y_i \ = \ \beta_i + \beta_1 x_i + \epsilon_i \quad i = 1, 2, \cdots, n.$$

$$\text{where } var(\epsilon_i) = \sigma^2 \text{ and } cov(\epsilon_i, \epsilon_j) = 0 \quad i \neq j.$$

(10%)(a) Derive the least squares estimates of β_0 and β_1 (denoted by $\hat{\beta}_0$ and $\hat{\beta}_1$ respectively).

(10%)(b) Show that $E(\hat{\beta}_1) = \beta_1$ and find $Var(\hat{\beta}_1)$

(5%)(c) Show that $cov(\overline{Y}, \hat{\beta}_1) = 0$

(25%)**4.** Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $U(0, \theta)$, the uniform distribution over $(0, \theta)$.

(5%)(a) Show that $X_{(n)}$ is a sufficient statistic for θ, where $X_{(n)} = \max(X_1, \cdots, X_n)$.

(10%)(b) Construct a $100(1-\alpha)\%$ confidence interval for θ based on the sufficient statistic $X_{(n)}$.

(10%)(c) Find the best unbiased estimator of θ.

(10%)**5.** Let X be a random variable with continuous distribution function F, and let F^{-1} be the inverse of F. Show that the random variable $Y = F(X)$ is distributed as $U(0, 1)$.

計算統計

1 Part I:數值計算方法

Reminder: The answer will not be accepted without proper explanation.

1. Let $P(x) = 9.5x^{20} + 8.1x^{16} + 7.2x^{12} + 6.5x^8$. What is the least number of multiplications required for evaluating $P(x)$? (10%)

2. What is the polynomial $P(x)$ with the least degree which satisfies $P(0) = 1$, $P'(0) = 0$, $P(1) = 4$ and $P'(1) = 9$? (10%)

3. Let $f(x) = (x-1)^3$, $x \in R$. Suppose that the initial $x^{(0)} = 0$ and $x^{(n)}$, $n \geq 1$, is defined by the Newton's method. Will the sequence $\{x^{(n)}\}$ converges to 1? If so, what is the order of convergence? (15%)

4. Let the system $Ax = b$ be nonsingular where $A \in R^{n\times n}$; $x, b \in R^n$. In particular, we may actually solve the perturbed system $Ay = b + \Delta b$ with $\|\Delta b\|$ small under some vector norm. Let cond (A) be the condition number of A under some matrix norm. Show that $\frac{1}{\mathrm{cond}(A)}\frac{\|\Delta b\|}{\|b\|} \leq \frac{\|y-x\|}{\|x\|} \leq \mathrm{cond}(A)\frac{\|\Delta b\|}{\|b\|}$. (15%)

2 Part II: 計算機系統概念

1. 試用您熟悉的一種程式語言 (譬如 C, Fortran, or Pascal, etc.) 把計算機系 統是如何地來計算出e^x寫成一個副程式。(10%)

2. 就您所熟悉或使用過的兩種計算機系統 (譬如 IBM PC and SUN Work Station, etc.), 簡述他們的特性以及比較他們之間的異同 (可以從軟、硬體和相關方面來回答這個問題)。(15%)

3. 您知道計算機系統中有那些硬體部份可以用來儲存資料呢? 如何的歸類? 並依您的歸類方式略述他們的特性和差異性。進一步我們要透過計算機系統來儲存和找尋資料的時候, 則系統是如何地來幫助我們呢? (可以就您所熟悉的 File and Data Structures 說明之)。(15%)

4. 一個 Computer Word (譬如說有4 bytes) 可能存放著一個指令 (Instruction), 也有可能被解釋成放的是一組資料 (Data), 計算機系統是如何地來區別呢? 並請您略述一下他們各有那些歸類方式? 例如有那些 Instruction Formats 以及那些不同型態的 Data? (可以就您所熟悉的概念略述之)。(10%)

線性代數

1. For vectors $x = [x_1, \ldots, x_n]$ and $y = [y_1, \ldots, y_n]$ in the vector space R^n, the length and the inner product are given by the following:

$$\|x\|^2 = x_1^2 + \ldots + x_n^2, \qquad \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j.$$

Suppose that $v_1, \ldots, v_m$, $m \leq n$, is an orthonomal set of R^n, i.e.

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Prove that for any vector g in R^n,

$$\sum_{j=1}^{m} \langle g, v_j \rangle^2 \geq \|g\|^2. \qquad (15\%)$$

2. Let W and V be vector subspaces of R^n. Prove that

$$\dim W + \dim V = \dim (W + V) + \dim(W \cap V). \qquad (15\%)$$

Here $\dim X$ denotes the dimension of X.

3. Find real constants c_0, c_1 and c_2 so that the following integral has minimal value.

$$\int_0^1 (e^x - c_0 - c_1 x - c_2 x^2)^2 dx. \qquad (20\%)$$

4. For any $n \times n$ matrix A, we define $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$.

(a) Prove that $e^{A+B} = e^A e^B$ if $AB = BA$. (10%)

(b) Find e^A if $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. (5%)

(c) Find e^B if $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (5%)

(d) Find the general solution to $\frac{du}{dt} = Au$ if $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (10%)

5. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$. Find $\max_{\|x\|=1} \|Ax\|$ and $\min_{\|x\|=1} \|Ax\|$. (20%)

高等微積分

(20%)#**1.** Let $f(s)=\sum\limits_{n=1}^{\infty} n^{-s}$. Show that f is continuous on $[2,\infty)$.

(20%)#**2.** Let $f(x)=3x^2+x+100, \quad \forall x\in R'$. Show that f is not uniformly continuous on R^1.

(15%)#**3.** $S\subseteq R^n$. Suppose for each x in S there exists an open set $N(x)$ such that $N(x)\cap S$ is countable. Show that S is countable.

(15%)#**4.** Let f be an one to one and real-valued continuous function on $[0,1]$. Show that f is strictly monotonic on $[0,1]$.

(15%)#**5.** Let f be a positive continuous real-valued function on $[0,1]$. Suppose $M=\max\limits_{0\le x\le 1} f(x)$. Show that

$$\lim_{n\to\infty}\left(\int_0^1 f^n(x)dx\right)^{\frac{1}{n}}=M.$$

(15%)#**6.** f and the derivative f' are continuous on $[0,\infty)$. Suppose that $\int_0^\infty |f'(x)|dx<\infty$. Show that the limit of $f(x)$ exists as x tends to ∞.

微分方程

1. Solve the following Differential Equations (50%)

a. $y'=\dfrac{x+4y-2}{4x-y+1}$

b. $y'=\dfrac{y}{ye^y-2x}$

c. $y'=\dfrac{3y}{x+y}$

d. $y'=\dfrac{x}{x^2y+y+y^3}$ (hint: let $u=x^2+1$)

e. $x^2y''+xy'+y=0$

2. Solve the following system: $Y' = AY + B$ where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ t \end{pmatrix} \qquad (15\%)$$

3. By the method of infinite series, find two linealy independent solutions for $y''_2 xy' + 2y = 0$ (15%)

4. Let $y = f(x)$ satisfy $y'' = xy$, $y(0) = 0$ $y'(0) = 1$.

(a) Show that $f(x)$ is strictly positive in $(0, \infty)$.

(b) What is $\lim_{x\to\infty} f(x)$? (10%)

5. Prove the uniqueness of the solution for the differential equation $y' = \sin y$, $y(o) = 1$. (10%)

數值分析

Reminder: The answer without the proper explanation will not be accepted.

1. Suppose a simple zero α of a C^2 function $f : IR \to IR$ is to be approximated by applying the Newton's method under the tolerance ϵ. We may have two possible stopping criteria:

$$\text{(A) } |f(x_n)| \leq \epsilon, \text{ or (B) } |x_{n+1} - x_n| \leq \epsilon,$$

where $\{x_n\}$ is the sequence of Newton's iterates in the program. Which criterion is better? Why?

(15%)

2. Given a data table as follows:

x	-3	-2	-1	0	1	2
$p(x)$	-62	-15	0	1	6	33

,

where $p(x)$ is a polynomial with $\deg(p) \leq 5$. What is the expression of $p(x)$? (10%)

3. Let $\mathbf{I}(f) = \int_0^1 f(x)dx$ where $f \in C[0,1]$. A quadrature of $\mathbf{I}(f)$ is defined by $\mathbf{I}_n(f) = \sum_{i=1}^n a_i f(x_i)$ for some nodes $x_i \in [0,1]$ and coefficients a_i. Also let $\mathbf{P}_3 = \{p(x) : p(x)$ is a polynomial on $[0,1]$ with $\deg(p) \leq 3\}$. Show that the quadrature $\mathbf{I}_n(f)$ derived from the Simpson's rule is exact for all p in $\mathbf{P}_3$. Hint: $\mathbf{I}(p) = \mathbf{I}_n(p)$. (20%)

4. Given an initial value problem (IVP)

$$dy/dx = f(x,y), \quad x \in [0,1],\ y(0) = y_0 \in \mathbf{R},$$

where f is Lipschitz continuous in y. Derive a weakly stable numerical method for solving (IVP). (15%)

5. For any matrix $A \in \mathbf{R}^{n\times n}$, it is known that $A = Q \cdot R$ where Q is orthonormal and R is upper triangular in $\mathbf{R}^{n\times n}$. Suppose

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -4 & 5 \end{bmatrix}.$$

What are Q and R? (15%)

6. Given a linear system $A \cdot x = b$ where $A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}$ and $b = [1,1,1,]^T$. Please derive an iterative method for solving the system whose iterates convege for any choice of initial guess in $\mathbf{R}^3$. (15%)

7. Let

$$B = \begin{bmatrix} 5 & -1 & 0 & 0 \\ -1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Show that all the eigenvalues of B must lie in the interval $[0,6]$. (6%)