$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n}x_{ij}^{m}\right) \geqq \left[\sum_{j=1}^{n}\left(\prod_{i=1}^{m}x_{ij}\right)\right]^{m}$$

葉東進

定理：設實數 $x_{ij} \geq 0$ ， $i=1,2,\cdots\cdots,m$；$j=1,2,\cdots\cdots,n$ ，則

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n}x_{ij}^{m}\right) \geq \left[\sum_{j=1}^{n}\left(\prod_{i=1}^{m}x_{ij}\right)\right]^{m}$$

，且等號成立的充分必要條件是 $\bigwedge_{i=1}^{m} x_{i1} = \bigwedge_{i=1}^{m} x_{i2} = \cdots\cdots = \bigwedge_{i=1}^{m} x_{in}$ （註）

（註：①符號 $\bigwedge_{i=1}^{m} a_i$ 表示 $a_1, a_2, \cdots\cdots, a_m$ 的連比 $a_1 : a_2 : \cdots : a_m$。

②$m=2$ 時，即爲熟知的柯西不等式，此時不作 $x_{ij} \geq 0$ 的限制。）

證明：取 $X_i = \sum_{j=1}^{n} x_{ij}^{m}$ ， $i=1,2,\cdots,m$

令 $a_{ij} = \dfrac{x_{ij}^{m}}{X_i}$ ， $i=1,2,\cdots\cdots,m$ ；

$j=1,2,\cdots\cdots,n$

由算術幾何平均不等式，我們有：

$$(*)\begin{cases} \dfrac{1}{m}\sum_{i=1}^{m} a_{i1} \geq \prod_{i=1}^{m}(a_{i1})^{\frac{1}{m}} \\ \dfrac{1}{m}\sum_{i=1}^{m} a_{i2} \geq \prod_{i=1}^{m}(a_{i2})^{\frac{1}{m}} \\ \vdots \\ \dfrac{1}{m}\sum_{i=1}^{m} a_{in} \geq \prod_{i=1}^{m}(a_{in})^{\frac{1}{m}} \end{cases}$$

以上諸式相加，得：

$$\frac{1}{m}[\sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots\cdots + \sum_{i=1}^{m} a_{in}]$$

$$\geq \prod_{i=1}^{m}(a_{i1})^{\frac{1}{m}} + \prod_{i=1}^{m}(a_{i2})^{\frac{1}{m}} + \cdots\cdots\cdots\cdots + \prod_{i=1}^{m}(a_{in})^{\frac{1}{m}}$$

但是 $\frac{1}{m}[\sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots\cdots\cdots + \sum_{i=1}^{m} a_{in}]$

$$= \frac{1}{m}[\sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots\cdots\cdots\cdots + \sum_{j=1}^{n} a_{mj}]$$

$$= \frac{1}{m}[\underbrace{1+1+\cdots\cdots\cdots+1}_{m\text{個}}]$$

$$= 1$$

而 $\prod_{i=1}^{m}(a_{i1})^{\frac{1}{m}} + \prod_{i=1}^{m}(a_{i2})^{\frac{1}{m}} + \cdots\cdots + \prod_{i=1}^{m}(a_{in})^{\frac{1}{m}}$

$$= \frac{\prod_{i=1}^{m} x_{i1} + \prod_{i=1}^{m} x_{i2} + \cdots\cdots + \prod_{i=1}^{m} x_{in}}{(\mathbf{X}_1\mathbf{X}_2\cdots\cdots\cdots\mathbf{X}_m)^{\frac{1}{m}}}$$

$$= \frac{\sum_{j=1}^{n}(\prod_{i=1}^{m} x_{ij})}{[\prod_{i=1}^{m}(\sum_{j=1}^{n} x_{ij}^{m})]^{\frac{1}{m}}}$$

$$\therefore \quad 1 \geq \frac{\sum_{j=1}^{n}(\prod_{i=1}^{m} x_{ij})}{[\prod_{i=1}^{m}(\sum_{j=1}^{n} x_{ij}^{m})]^{\frac{1}{m}}}$$

故 $\prod_{i=1}^{m}(\sum_{j=1}^{n} x_{ij}^{m}) \geq [\sum_{j=1}^{n}(\prod_{i=1}^{m} x_{ij})]^{m}$

可以看出，上面式子在等號成立時，其充分必要條件是(*)中的諸式的等號都成立，也就是：

$$\begin{cases} a_{11} = a_{21} = \cdots\cdots = a_{m1} \\ a_{12} = a_{22} = \cdots\cdots = a_{m2} \\ \vdots \\ a_{1n} = a_{2n} = \cdots\cdots = a_{mn} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{x_{11}^{m}}{\mathbf{X}_1} = \frac{x_{21}^{m}}{\mathbf{X}_2} = \cdots\cdots = \frac{x_{m1}^{m}}{\mathbf{X}_m} \\ \frac{x_{12}^{m}}{\mathbf{X}_1} = \frac{x_{22}^{m}}{\mathbf{X}_2} = \cdots\cdots = \frac{x_{m2}^{m}}{\mathbf{X}_m} \\ \vdots \\ \frac{x_{1n}^{m}}{\mathbf{X}_1} = \frac{x_{2n}^{m}}{\mathbf{X}_2} = \cdots\cdots = \frac{x_{mn}^{m}}{\mathbf{X}_m} \end{cases}$$

$$\Leftrightarrow \bigwedge_{i=1}^{m} x_{i1}^{m} = \bigwedge_{i=1}^{m} x_{i2}^{m} = \cdots\cdots = \bigwedge_{i=1}^{m} x_{in}^{m} = \bigwedge_{i=1}^{m} X_i$$

$$(x_{ij} \geq 0) \Leftrightarrow \bigwedge_{i=1}^{m} x_{i1} = \bigwedge_{i=1}^{m} x_{i2} = \cdots\cdots\cdots = \bigwedge_{i=1}^{m} x_{in} = \bigwedge_{i=1}^{m} X_i^{\frac{1}{m}}$$

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