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RICCI SOLITON ON A CLASS OF RIEMANNIAN MANIFOLDS UNDER $\mathcal{D}$-ISOMETRIC DEFORMATION

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Abstract

In this article, we investigate the behavior of Ricci solitons under $\mathcal{D}$-isometric deformations on a class of Riemannian manifolds. A $\mathcal{D}$-isometry is a diffeomorphism that preserves the distance function induced by a Riemannian metric up to a constant factor. We consider a family of Riemannian metrics g on a manifold M that are related by $\mathcal{D}$-isometric deformations, and we study the Ricci soliton equation on each metric g. We show that under certain conditions on the deformation function, the solutions to the Ricci soliton equation on each metric g are invariant. In particular, we obtain a family of Ricci solitons that are related by a scaling factor under $\mathcal{D}$-isometric deformations. We also provide explicit examples of $\mathcal{D}$-isometric deformations and compute the corresponding Ricci solitons.

1. Introduction

The study of geometry on Riemannian manifolds is a vast research field, and the construction of structures such as the Ricci soliton has recently gained much attention in works like [2, 6, 8, 12, 11, 22]. These structures are essential for building geometric tools like special vector fields, metric deformations, and manifold products.

In 1982, Hamilton [14] introduced the notion of Ricci flow to obtain a canonical metric on a smooth manifold. Ricci flow has since become a powerful tool for studying Riemannian manifolds, particularly those with

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positive curvatures. The vector field V generates the Ricci soliton, which is a special solution of the Ricci flow. A Ricci soliton is said to be a gradient Ricci soliton if the generating vector field V is the gradient of a potential function.

In this paper, we focus on the geometry of transformations related to the $(n-1)$-dimensional distribution $\mathcal{D}$ of an n-dimensional Riemannian manifold M. Specifically, we aim to construct a Ricci soliton from a unit global vector field, which will be the primary tool for deforming the metric. The paper is organized as follows: Section 2 presents the background formulas to be used in subsequent sections. In Section 3, we establish an interesting deformation of metrics and prove some basic properties. In Section 4, we investigate Ricci soliton rigidity under $\mathcal{D}$-isometric deformations.

2. Review of Needed Notions

By R, Q, S and r we denote respectively the Riemannian curvature tensor, the Ricci operator, the Ricci tensor and the scalar curvature of a Riemannian manifold (M^n, g). Then R, Q S and r are defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.1}$$

$$QX = \sum_{i=1}^{n} R(X, e_i)e_i, \tag{2.2}$$

$$S(X,Y) = g(QX,Y) = \sum_{i=1}^{n} g\big(R(X,e_i)e_i, Y\big), \tag{2.3}$$

$$r = \sum_{i=1}^{n} S(e_i, e_i), \tag{2.4}$$

where ∇ is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. For all vector field X on M, we have

$$X = \sum_{i=1}^{n} g(X, e_i)e_i, \tag{2.5}$$

the divergence of a vector field V is defined by:

$$\mathrm{div}V = \sum_{i=1}^{n} g(\nabla_{e_i} V, e_i), \tag{2.6}$$

and for a unit closed 1-form η (i.e., $\mathrm{d}\eta = 0$) one can get

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \qquad g(\nabla_X \xi, \xi) = 0 \qquad \text{and} \qquad \nabla_\xi \xi = 0. \tag{2.7}$$

where η be the g-dual of ξ which means $\eta(X) = g(X, \xi)$ for all vector field X on M. On the other hand, ξ is said a Jacobi-Type vector field if and only if [3],

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi - R(X, \xi)Y = 0, \tag{2.8}$$

Substituting $Y = \xi$ in (2.8) and using (2.7), one can obtain,

$$\nabla_{\nabla_X \xi} \xi + R(X, \xi)\xi = 0, \tag{2.9}$$

From [9], we have the following important result:

Theorem 2.1. *Every Jacobi-type vector field on a compact Riemannian manifold is a Killing vector field.*

We recall that (M^n, g), $n \geq 3$, is said to be Einstein if at every $x \in M$ its Ricci tensor S has the form

$$S = \rho g, \qquad \rho \in \mathbb{R}. \tag{2.10}$$

(For more details, see for example [21]).

A smooth vector field V on a Riemannian manifold (M, g) is said to define a Ricci soliton if it satisfies the following Ricci soliton equation:

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) = 2\lambda g(X, Y), \tag{2.11}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of g along a vector field V and λ is a constant. We shall denote a Ricci soliton by (Mg, V, λ). If λ is a smooth function on M, we say that (M, V, λ, g) is an almost Ricci soliton. We call the vector field V the potential field. A Ricci soliton (M, g, V, λ) is called shrinking, steady or expanding according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A trivial Ricci soliton is one for which V is zero or Killing, in which case the metric is Einstein (see, for instance, [8, 13, 19]).

3. Deformations of Metrics

Let M^n be a Riemannian manifold equipped with Riemannian metric g, and η a unit global closed 1-form on M. We denote by ξ the vector field corresponding to η, i.e., for all X vector field on M,

$$\eta(X) = g(\xi, X) \qquad \textit{and} \qquad \eta(\xi) = 1.$$

We define on M a Riemannian metric, denoted $\tilde{g}$, by

$$\tilde{g}(X,Y) = g(X,Y) + \eta(X)\eta(Y). \tag{3.1}$$

The equation $\eta = 0$ defines a $(n-1)$-dimensional distribution $\mathcal{D}$ on M. Then, we have

$$\begin{cases} \tilde{g}(\xi,\xi) = 2, \\ \tilde{g}(X,X) = g(X,X) \qquad \forall X \in \mathcal{D}. \end{cases}$$

That is why, we refer to this construction as $\mathcal{D}$-isometric deformation.

Note that the simplest case for this deformation is for $\eta = \mathrm{d}f$ where $f \in \mathcal{C}^\infty(M)$. This case has been studied in [4] and [5].

In [15], Innami proved that M admits a non-constant affine function if and only if M splits as a Riemannian product $M = N \times \mathbb{R}$. In our situation, since $\mathrm{d}\eta = 0$ then ξ is locally of type gradient, what means at each point p on M there exists a function f such that $\xi = \nabla f$ on a neighborhood at p, where ∇f denotes the gradient vector field of f.

Proposition 3.1. *Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g and $\tilde{g}$ respectively. Then, for all X, Y vector fields on M, we have the relation:*

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2} g(\nabla_X \xi, Y)\xi. \tag{3.2}$$

Proof. Using Koszul's formula for the metric $\tilde{g}$,

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) \;=\; X\tilde{g}(Y,Z) + Y\tilde{g}(Z,X) - Z\tilde{g}(X,Y)$$

$$-\tilde{g}(X,[Y,Z])+\tilde{g}(Y,[Z,X])+\tilde{g}(Z,[X,Y],$$

one can obtain

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X Y,Z) &= \tilde{g}(\nabla_X Y,Z)\\ &\quad+\frac{1}{2}\big((\nabla_X\eta)Y+(\nabla_Y\eta)X\big)\eta(Z)\end{aligned}$$

Knowing that $\mathrm{d}\eta=0$, we get

$$\tilde{g}(\tilde{\nabla}_X Y,Z)=\tilde{g}(\nabla_X Y,Z)+(\nabla_X\eta)(Y)\eta(Z),$$

with

$$(\nabla_X\eta)(Y)=g(\nabla_X\xi,Y).$$

On the other hand, we have

$$\begin{aligned}\eta(Z)&=g(\xi,Z)\\ &=\tilde{g}(\xi,Z)-\eta(Z),\end{aligned}$$

which gives

$$\eta(Z)=\frac{1}{2}\tilde{g}(\xi,Z).$$

Therefore

$$\tilde{\nabla}_X Y=\nabla_X Y+\frac{1}{2}g(\nabla_X\xi,Y)\xi. \qquad \square$$

Theorem 3.1. *Let $\tilde{R}$, $\tilde{Q}$ and $\tilde{S}$ be the Riemannian curvature tensor, the Ricci operator and the Ricci tensor of the Riemannian manifold $(M^n,\tilde{g})$ respectively. For all X,Y and Z vector fields on M, we have*

$$\begin{aligned}2\tilde{R}(X,Y)Z&=2R(X,Y)Z-g\big(R(X,Y)Z,\xi\big)\xi\\ &\quad+g(\nabla_Y\xi,Z)\nabla_X\xi-g(\nabla_X\xi,Z)\nabla_Y\xi,\end{aligned} \tag{3.3}$$

$$2\tilde{Q}X=2QX-S(X,\xi)\xi+(\mathrm{div}\xi)\nabla_X\xi, \tag{3.4}$$

$$2\tilde{S}(X,Y)=2S(X,Y)+\mathrm{div}\xi g(\nabla_X\xi,Y). \tag{3.5}$$

$$2\tilde{r}=2r-\frac{1}{2}S(\xi,\xi)+(\mathrm{div}\,\xi)^2. \tag{3.6}$$

Proof. By the definition of the curvature tensor $\tilde{R}$

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z, \tag{3.7}$$

using formula 3.2, the first term of (3.7) is given by

$$\begin{aligned}
\tilde{\nabla}_X\tilde{\nabla}_Y Z &= \tilde{\nabla}_X\Big(\nabla_Y Z + \frac{1}{2}g(\nabla_Y\xi, Z)\xi\Big)\\
&= \tilde{\nabla}_X\nabla_Y Z + \frac{1}{2}g(\nabla_X\nabla_Y\xi, Z)\xi\\
&\quad+\frac{1}{2}g(\nabla_Y\xi,\nabla_X Z)\xi + \frac{1}{2}g(\nabla_Y\xi, Z)\tilde{\nabla}_X\xi\\
&= \nabla_X\nabla_Y Z + \frac{1}{2}g(\nabla_X\xi,\nabla_Y Z)\xi\\
&\quad+\frac{1}{2}g(\nabla_X\nabla_Y\xi, Z)\xi + \frac{1}{2}g(\nabla_Y\xi,\nabla_X Z)\xi\\
&\quad+\frac{1}{2}g(\nabla_Y\xi, Z)\nabla_X\xi.
\end{aligned}\tag{3.8}$$

With the similar method, the second term of (3.7) is given by

$$\begin{aligned}
\tilde{\nabla}_Y\tilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + \frac{1}{2}g(\nabla_Y\xi,\nabla_X Z)\xi\\
&\quad+\frac{1}{2}g(\nabla_Y\nabla_X\xi, Z)\xi + \frac{1}{2}g(\nabla_X\xi,\nabla_Y Z)\xi\\
&\quad+\frac{1}{2}g(\nabla_X\xi, Z)\nabla_Y\xi.
\end{aligned}\tag{3.9}$$

Knowing that $\mathrm{d}\eta = 0$, the last term of (3.7) becomes

$$\begin{aligned}
\tilde{\nabla}_{[X,Y]}Z &= \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_{[X,Y]}\xi, Z)\xi\\
&= \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_{\nabla_X Y}\xi, Z)\xi - \frac{1}{2}g(\nabla_{\nabla_Y X}\xi, Z)\xi\\
&= \nabla_{[X,Y]} + \frac{1}{2}g(\nabla_Z\xi,\nabla_X Y)\xi - \frac{1}{2}g(\nabla_Z\xi,\nabla_Y X)\xi.
\end{aligned}\tag{3.10}$$

Formula (3.3) follows from equations (3.7)-(3.10).

To show the second formula (3.4), consider $\{\xi, e_i\}_{2\leq i\leq n}$ the orthonormal basis on M with respect to the metric g, it is easy to prove that $\{\frac{1}{\sqrt{2}}\xi, e_i\}_{2\leq i\leq n}$ is an orthonormal frame on M with respect to the metric $\tilde{g}$. Using formula

(2.2), we have

$$\begin{aligned}\tilde{Q}X &= \frac{1}{2}\tilde{R}(X,\xi)\xi + \sum_{i=2}^{n}\tilde{R}(X,e_i)e_i \\ &= -\frac{1}{2}\tilde{R}(X,\xi)\xi + \sum_{i=1}^{n}\tilde{R}(X,e_i)e_i \\ &= -\frac{1}{2}R(X,\xi)\xi + \sum_{i=1}^{n}\Big(R(X,e_i)e_i - \frac{1}{2}g\big(R(X,e_i)e_i,\xi\big)\xi \\ &\quad +\frac{1}{2}g(\nabla_{e_i}\xi,e_i)\nabla_X\xi - \frac{1}{2}g(\nabla_X\xi,e_i)\nabla_{e_i}\xi\Big). \qquad (3.11)\end{aligned}$$

Using equations (2.2), (2.3), (2.5) and (2.6), we get (3.4).
For the third formula (3.5), just use the two formulas (2.3) and (3.4). Finally, we compute the trace of S to get (3.6). □

4. Ricci soliton

In this section, we will investigate Ricci soliton on such manifolds. First, we have the following immediate result,

Theorem 4.1. *Let (M,g) be a Riemannian manifold endowed with a unit closed vector field ξ admitting Ricci soliton (g,ξ,λ). The following holds:*

- $r = n\lambda - \operatorname{div}\xi$,
- *λ is eigenvalue of Ricci operator Q with ξ its associated eigenvector.*

In addition, if (M^n,g) is compact and ξ is of Jacobi-type, then:

- *(M^n,g) is Einstein.*
- $r = n\lambda$.

Proof. Since (g,ξ,λ) is a Ricci soliton then equation (2.11) along with (2.7) gives,

$$g(\nabla_X\xi,Y) + S(X,Y) = \lambda g(X,Y), \qquad (4.1)$$

Computing the trace of S from equation (4.1) yields:

$$r = \sum_{i=1}^{n} S(e_i,e_i) = \sum_{i=1}^{n}\lambda g(e_i,e_i) - \sum_{i=1}^{n} g(\nabla_{e_i}\xi,e_i) = n\lambda - \operatorname{div}\xi.$$

Again taking $X = \xi$ in (4.1) and using equations (2.2) and (2.7), we can obtain

$$S(\xi, Y) = \lambda\eta(Y) \Rightarrow Q\xi = \lambda\xi.$$

The second part is immediately obtained using **Theorem** 2.1. □

Proposition 4.2. *Let (M^n, g) be a Riemannian manifold endowed with a unit closed vector field ξ admitting (g, ξ, λ) Ricci soliton. Then $(g, f\xi, \lambda)$ is a Ricci soliton if and only if f is constant.*

Proof. Using the definition of the Lie derivative, we have:

$$(\mathcal{L}_V g)(X, Y) = f(\mathcal{L}_\xi g)(X, Y) + X(f)\eta(Y) + Y(f)\eta(X).$$

Hence, from equation (2.1) we obtain:

$$\begin{aligned}
&(\mathcal{L}_{f\xi} g)(X, Y) + 2S(X, Y) - 2\lambda g(X, Y) \\
&= f(\mathcal{L}_\xi g)(X, Y) + X(f)\eta(Y) + Y(f)\eta(X) + 2S(X, Y) - 2\lambda g(X, Y) \\
&= 0
\end{aligned} \tag{4.2}$$

Consider $\{\xi, e_i\}_{2\leq i\leq n}$ an orthonormal frame with respect to g and since (g, ξ, λ) is a Ricci soliton ($S(e_i, \xi) = 0$), substituting $X = Y = \xi$ in (4.2) gives:

$$\xi(f) = 0,$$

Again, putting $X = \xi$ and $Y = e_i$ in (4.2) yields:

$$e_i(f) = 0,$$ □

In the following, we will study Ricci soliton behaviour under $\mathcal{D}$-isometric deformation (3.1). First, we consider the case where the potential field V is pointwise collinear with the vector field ξ (i.e., $V = f\xi$ f is a function on M^n). With direct computations we have:

$$\begin{aligned}
(\mathcal{L}_V \tilde{g})(X, Y) &= \tilde{g}(\tilde{\nabla}_X V, Y) + \tilde{g}\big(\tilde{\nabla}_Y V, X\big) \\
&= \tilde{g}\big(\tilde{\nabla}_X (f\xi), Y\big) + \tilde{g}\big(\tilde{\nabla}_Y (f\xi), X\big) \\
&= 2fg(\nabla_X \xi, Y) + 2X(f)\eta(Y) + 2Y(f)\eta(X) \\
&= 2f(\mathcal{L}_\xi g)(X, Y) + 2X(f)\eta(Y) + 2Y(f)\eta(X).
\end{aligned}$$

Thus,

$$(\mathcal{L}_V\tilde{g})(X,Y) = 2(\mathcal{L}_V g)(X,Y) = (\mathcal{L}_{2V} g)(X,Y), \tag{4.3}$$

Replacing (3.1), (3.5) and (4.3) in (2.11) along with Proposition 4.2, we obtain,

$$\begin{aligned}(\mathcal{L}_V\tilde{g})(X,Y) &+ 2\tilde{S}(X,Y) - 2\lambda\tilde{g}(X,Y)\\ =&\ (\mathcal{L}_{2V} g)(X,Y)+2S(X,Y)+\mathrm{div}\xi g(\nabla_X\xi,Y)-2\lambda g(X,Y)\\ &-2\lambda\eta(X)\eta(Y).\end{aligned} \tag{4.4}$$

Thus $(M,\tilde{g},f\xi,\lambda)$ is a Ricci soliton if and only if

$$\mathrm{div}\,\xi g(\nabla_X\xi,Y) - \lambda\eta(X)\eta(Y) = 0, \tag{4.5}$$

Therefore, summing up the arguments above, we have the following theorem:

Theorem 4.2. *Let (M^n,g) be a Riemannian manifold endowed with a unit closed Jacobi-Type vector field ξ admitting Ricci soliton $(g,2f\xi,\lambda)$. Then, under $\mathcal{D}$-isometric deformation $(\tilde{g},f\xi,\lambda)$ is a Ricci soliton if and only if,*

$$\mathrm{div}\,\xi g(\nabla_X\xi,Y) - \lambda\eta(X)\eta(Y) = 0,$$

and $(g,2f\xi,\lambda)$, $(\tilde{g},f\xi,\lambda)$ are steady.

Corollary 4.1. *Let (M^n,g) be a compact Riemannian manifold endowed with a unit closed Jacobi-Type vector field ξ. Then, under $\mathcal{D}$-isometric deformation η-Einstein Ricci soliton $(g,f\xi,\lambda)$ deforms to an Einstein metric.*

Next, we consider the case where V is orthogonal to ξ i.e., $\eta(V)=0$. We compute

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X V,Y) &= \tilde{g}\Big(\nabla_X V + \frac{1}{2}g(\nabla_X\xi,V)\xi,Y\Big)\\ &= \tilde{g}(\nabla_X V,Y) + \frac{1}{2}g(\nabla_X\xi,V)\tilde{g}(\xi,Y),\end{aligned}$$

knowing that $\tilde{g} = g+\eta\otimes\eta$ and $g(\nabla_X\xi,V) = -\eta\big(\nabla_X V\big)$, we get

$$\tilde{g}(\tilde{\nabla}_X V,Y) = g(\nabla_X V,Y),$$

then

$$(\mathcal{L}_V\tilde{g})(X,Y) = g(\nabla_X V,Y)+g(\nabla_Y V,X) = (\mathcal{L}_V g)(X,Y). \qquad (4.6)$$

Replacing (3.1), (3.5) and (4.6) in (2.11), we obtain

$$\big(\mathcal{L}_V\tilde{g}+2\tilde{S}-2\lambda\tilde{g}\big)(X,Y) = \big(\mathcal{L}_V g+2S-2\lambda g\big)(X,Y) + g\Big(\mathrm{div}\xi\nabla_X\xi-2\lambda\eta(X)\xi,Y\Big). \qquad (4.7)$$

If (M,g,V,λ) is a Ricci soliton, the above equation takes the form

$$\big(\mathcal{L}_V\tilde{g}+2\tilde{S}-2\lambda\tilde{g}\big)(X,Y) = +g\Big(\mathrm{div}\xi\nabla_X\xi-2\lambda\eta(X)\xi,Y\Big).$$

Thus $(M,\tilde{g},V,\lambda)$ is a Ricci soliton if and only if

$$\mathrm{div}\xi\nabla_X\xi-2\lambda\eta(X)\xi=0. \qquad (4.8)$$

By taking $X=\xi$ in (4.8), we obtain

$$\lambda=0, \qquad (4.9)$$

Substituting equations (4.8) and (4.9) in (3.5), we obtain

$$\tilde{S}=S. \qquad (4.10)$$

Then we get

$$\mathcal{L}_V\tilde{g}+2\tilde{S}-2\lambda\tilde{g}=\mathcal{L}_V g+2S-2\lambda g-2\lambda\,\eta\otimes\eta.$$

Thus we have the following:

Theorem 4.3. *Let (M,g,V,λ) be a Ricci soliton with the potential vector field V orthogonal to ξ. Then $(M,\tilde{g},V,\lambda)$ is a steady Ricci soliton.*

5. A Class of Examples

Example 5.1. Let $M=\mathbb{H}^n\times\mathbb{R}=\{(x_i,z)\in\mathbb{R}^{n+1}/x_n>0\}$, where $(x_i,z)_{1\leq i\leq n}$ are standard co-ordinate in $\mathbb{R}^{n+1}$. Let $\{e_i,\xi\}_{1\leq i\leq n}$ be linearly independent vector fields given by

$$e_i = x_n \frac{\partial}{\partial x_i}, \qquad \xi = \frac{\partial}{\partial z}.$$

We define a Riemannian metric g by

$$g = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2 + dz^2.$$

Let ∇ be the Riemannian connection of g, then we have

$$[e_i, e_n] = -e_i, \quad [e_i, \xi] = [e_i, e_j] = 0, \quad \forall i \neq j \in \{1, \dots, n-1\}.$$

By using the Koszul formula for the Riemannian metric g,

$$2g(\nabla_{e_i} e_j, e_k) = -g(e_i, [e_j, e_k]) + g(e_j, [e_k, e_i]) + g(e_k, [e_i, e_j]),$$

the non-zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_i} e_i = e_n, \qquad \nabla_{e_i} e_n = -e_i, \qquad \forall i \in \{1, \dots, n-1\}.$$

The non-vanishing curvature tensor R components are computed as

$$R(e_i, e_j)e_j = -e_i \; R(e_i, e_n)e_i = e_n, \; R(e_i, e_n)e_n = -e_i, \; \forall i \neq j \in \{1, \dots, n-1\}.$$

The Ricci operator Q and the Ricci curvature S components are computed as

$$Qe_i = (1-n)e_i, \quad S(e_i, e_j) = (1-n)\delta_{ij}, \quad S(\xi, \xi) = 0, \quad \forall i \in \{1, \dots, n\}.$$

One can easily check that for f constant function on M, $(g, f\xi, \lambda)$ is a steady Ricci soliton. Since $\text{div}\xi = 0$, using **Theorem** 4.2, we obtain that $(\tilde{g}, f\xi, \lambda)$ is also a steady Ricci soliton such that

$$\tilde{g} = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2 + 2dz^2.$$

Considering Theorem 4.2, it is proved that there exists an infinite number of Ricci solitons $(M, g_m, f\xi, \lambda)$ where $g_m = g + m\eta \otimes \eta$.

Example 5.2. (3D cigar soliton)
Let $M = \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R} = \{(x, r, t) \in \mathbb{R}^3 / x > 0\}$, and let $\{e_1, e_2, e_3\}$ be linearly independent vector fields given by

$$e_1 = (1 + x^2)\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial r}, \qquad e_3 = \frac{1}{x}\frac{\partial}{\partial t}.$$

and $\{\theta^1, \theta^2, \theta^3\}$ be the dual frame of differential 1-forms such that

$$\theta^1 = \frac{1}{1 + x^2}dx, \qquad \theta^2 = dr, \qquad \theta^3 = xdt.$$

We define a Riemannian metric g by $g = \sum_{i=1}^{3} \theta^i \otimes \theta^i$, that is the form

$$g = \begin{pmatrix} \frac{1}{(1+x^2)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^2 \end{pmatrix}.$$

The potential vector field is given by $V = \text{grad} f = 2x(1 + x^2)e_1$ where the potential function is $f = \ln(1 + x^2)$.

With simple computations one can prove that the condition (4.2) is satisfied and we find

$$S = -2(1 + x^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \mathcal{L}_V g = 4(1 + x^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can easily notice that $\mathcal{L}_V g + 2S = 0$ which implies that (M, g, V) is a steady Ricci soliton.

We take $\xi = e_2$ to ensure the conditions $V \perp \xi$ and $\mathrm{d}\theta^2 = 0$ then we deform the metric as follows

$$\tilde{g} = g + \theta^2 \otimes \theta^2 = \begin{pmatrix} \frac{1}{(1+x^2)^2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & x^2 \end{pmatrix}.$$

Using formulas (4.6) and (4.10) we conclude that $(M, \tilde{g}, V)$ is a steady Ricci soliton too.

6. Declarations

Ethical Approval

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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