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THE QUANTUM GROUP $\dot{U}$ AND FLAG MANIFOLDS OVER THE SEMIFIELD Z

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Abstract

We give a parametrization of the canonical basis of the modified quantum group corresponding to a root datum in terms of the flag manifold over the semifield Z associated to the reductive group corresponding to the dual root datum.

Introduction

0.1. Let $\mathbf{f}$ be the $+$ part of the Drinfeld-Jimbo quantized enveloping algebra U (over $\mathbf{Q}(v)$) attached to a root datum of simply laced type and let $\dot{U}$ be the modified form of U (see [9, §23]). Let $\mathbf{B}$ (resp. $\dot{\mathbf{B}}$) be the canonical basis of $\mathbf{f}$ (resp. $\dot{U}$) defined in [6] (resp. in [8], see also [9, 25.2]). In [6] it was shown that $\mathbf{B}$ is naturally parametrized by something which later [10] was interpreted in terms of certain objects attached to the semifield $\mathbf{Z}$. In this paper we want to find an analogous parametrization for $\dot{\mathbf{B}}$ (see 5.12), compatible with the involution ω of $\dot{U}$ interchanging the $+$ part and the $-$ part (see 3.3) which preserves $\dot{\mathbf{B}}$ (see 3.8). In particular we show that $\dot{\mathbf{B}}$ is preserved by ω.

Let X be the lattice of weights of our root datum and let X^+ be the set of dominant weights in X. We will put $\dot{\mathbf{B}}$ in bijection with $\sqcup_{\lambda\in X^+}(B_\lambda \times B_\lambda)$ where B_λ is the canonical basis [6] of the simple finite dimensional U-module

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Λ_λ with highest weight λ. We are reduced to finding a parametrization of B_λ in terms of objects attached to the semifield $\mathbf{Z}$. Such a parametrization was given in [6, §8], but this is still not enough for our purpose.

We will give another parametrization of B_λ based on the following observation of [12]. According to [6, §8], the set B_λ can be parametrized in two different ways: by regarding Λ_λ as a highest weight module or as a lowest weight module. The rather complicated combinatorics relating these two parametrizations was shown in [12] to be expressible in terms of a remarkable involution $\phi_{\mathbf{Z}}$ (defined in [12]) of a certain object $\mathcal{B}_{\mathbf{Z}}$ attached to the semifield $\mathbf{Z}$. We can parametrize B_λ in terms of this involution $\phi_{\mathbf{Z}}$. More precisely, if G is a reductive group corresponding to the dual root datum, then $\mathcal{B}_{\mathbf{Z}}$ is a variant of the flag manifold of G over the semifield $\mathbf{Z}$. Then $\mathcal{B}_{\mathbf{Z}}$ has two remarkable subsets $\mathcal{B}^+_{\mathbf{N}}, \mathcal{B}^-_{\mathbf{N}}$ (interchanged by $\phi_{\mathbf{Z}}$) in which certain parameters in $\mathbf{Z}$ are assumed to be in $\mathbf{N}$. As in [12], $\mathcal{B}_{\mathbf{Z}}$ has an action of the group X; this is the $\mathbf{Z}$-variant of the conjugation action of a maximal torus of G on the flag manifold of G. Now, for $\lambda \in X$ we can consider the intersection of $\mathcal{B}^+_{\mathbf{N}}$ with the λ translate (in the sense of the X-action) of $\mathcal{B}^-_{\mathbf{N}}$. Although $\mathcal{B}^+_{\mathbf{N}}$ and $\mathcal{B}^-_{\mathbf{N}}$ are in general infinite, we show that the intersection above is finite; moreover, it is nonempty if and only if λ is dominant, in which case it naturally parametrizes B_λ. This parametrization is an immediate consequence of [12, 4.9], except for the nonemptiness criterion above. (The proof of 4.9 in [12] contained some misprints and we reprove it in 5.5.)

The resulting parametrization of $\dot{\mathbf{B}}$ has the property that the action of ω has a simple description in terms of the involution $\phi_{\mathbf{Z}}$.

Another parametrization of B_λ (which again uses ideas in [12]) is given in [3].

0.2. One can show that similar results hold when our root datum is not assumed to be simply laced, by using folding to reduce to the simply laced case (as was done for $\mathbf{B}$ in [14] where again the group G corresponding to the dual root datum was used.)

1. The Flag Manifold $\mathcal{B}_K$ for a Semifield K

1.1. We fix a finite set I and a simply laced Cartan matrix (a_{ij}) indexed by $I \times I$, that is a symmetric, positive definite matrix with integer entries such that $a_{ii} = 2$ for $i \in I$, $a_{ij} \in \{0, -1\}$ for $i \neq j$ in I. We denote the obvious $\mathbf{Z}$-basis of $\mathbf{Z}[I]$ as $\{i'; i \in I\}$. For $i \in I$ we define a homomorphism $s_i : \mathbf{Z}[I] \to \mathbf{Z}[I]$ by $j' \mapsto j' - a_{ij} i'$ $(j \in I)$. Let W be the subgroup of the automorphism group of $\mathbf{Z}[I]$ generated by $\{s_i; i \in I\}$. This is a finite Coxeter group with length function $w \mapsto |w|$.

Let w_0 be the unique element of W with $|w_0|$ maximal and let $\nu = |w_0|$.

For $i \in I$ we define $i^! \in I$ by $s_{i^!} = w_0 s_i w_0$; then $i \mapsto i^!$ is an involution of I.

Let $\mathcal{I}$ be the set of sequences $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$ such that $s_{i_1} s_{i_2} \dots s_{i_\nu} = w_0$. For example, if $I = \{i, j\}, a_{ij} = -1$, we have $\mathcal{I} = \{(i, j, i), (j, i, j)\}$.

1.2. Let K be a semifield. Let $(\mathbf{i}, \mathbf{a}), (\mathbf{i}', \mathbf{a}')$ in $\mathcal{I} \times K^\nu$ be given by

$$\begin{aligned} \mathbf{i} &= (i_1, i_2, \dots, i_\nu), & \mathbf{i}' &= (i'_1, i'_2, \dots, i'_\nu), \\ \mathbf{a} &= (a_1, a_2, \dots, a_\nu), & \mathbf{a}' &= (a'_1, a'_2, \dots, a'_\nu). \end{aligned}$$

We say that $(\mathbf{i}, \mathbf{a}), (\mathbf{i}', \mathbf{a}')$ are adjacent if one of (i),(ii) below holds.

(i) For some $l \in [1, \nu - 3]$ we have $i'_k = i_k$ for $k \notin \{l, l+1, l+2\}$ and $(i_l, i_{l+1}, i_{l+2}) = (i, j, i)$, $(i'_l, i'_{l+1}, i'_{l+2}) = (j, i, j)$ where i, j in I satisfy $a_{ij} = -1$; moreover, $(a_l, a_{l+1}, a_{l+2}) = (a, b, c)$, $(a'_l, a'_{l+1}, a'_{l+2}) = (a', b', c')$ where

$$a' = bc/(a+c), \quad b' = a + c, \quad c' = ab/(a+c),$$

or equivalently

$$a = b'c'/(a'+c'), \quad b = a' + c', \quad c = a'b'/(a'+c');$$

(ii) for some $l \in [1, \nu - 2]$ we have $i'_k = i_k$ for $k \notin \{l, l+1\}$ and $(i_l, i_{l+1}) = (i, j)$, $(i'_l, i'_{l+1}) = (j, i)$ where i, j in I satisfy $a_{ij} = 0$; moreover, $a'_l = a_{l+1}$, $a'_{l+1} = a_l$.

Let $\mathcal{U}_K$ be the set of equivalence classes on $\mathcal{I} \times K^\nu$ for the equivalence relation on $\mathcal{I} \times K^\nu$ generated by the adjacency relation.

We shall sometime denote $(\mathbf{i}, \mathbf{a}) \in \mathcal{I} \times K^\nu$, (or its equivalence class) as $i_1^{a_1} i_2^{a_2} \dots i_\nu^{a_\nu}$, where $\mathbf{i} = (i_1, i_2, \dots, i_\nu), \mathbf{a} = (a_1, a_2, \dots, a_\nu)$.

The assignment

$$i_1^{a_1} i_2^{a_2} \dots i_\nu^{a_\nu} \to i_\nu^{a_\nu} i_{\nu-1}^{a_{\nu-1}} \dots i_1^{a_1}$$

defines an involution $A \mapsto A^*$ of $\mathcal{U}_K$.

1.2. For $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$ and $k \in [1, \nu]$, we have

$$s_{i_1} s_{i_2} \dots s_{i_{k-1}}(i'_k) = \sum_{h \in I} r_{h,k} h'$$

(in $\mathbf{Z}[I]$) where $r_{h,k} \in \mathbf{N}$. For $\mathbf{a} = (a_1, a_2, \dots, a_\nu) \in K^\nu, h \in I$ we set

$$||\mathbf{i}, \mathbf{a}||_h = \prod_{k \in [1,\nu]} a_k^{r_{h,k}} \in K.$$

We show:

(a) If $(\mathbf{i}, \mathbf{a}), (\mathbf{i}', \mathbf{a}')$ in $\mathcal{I} \times K^\nu$ are adjacent, then $||\mathbf{i}, \mathbf{a}||_h = ||\mathbf{i}', \mathbf{a}'||_h$.

Let $r'_{h,k} \in \mathbf{N}$ be defined in terms of $\mathbf{i}'$ in the same way as $r_{h,k}$ was defined in terms of $\mathbf{i}$. In case 1.1(i) we must show:

(b) $$a_l^{r_{h,l}} a_{l+1}^{r_{h,l+1}} a_{l+2}^{r_{h,l+2}} = {a'_l}^{r'_{h,l}} {a'_{l+1}}^{r'_{h,l+1}} {a'_{l+2}}^{r'_{h,l+2}}.$$

For some $w \in W$ we have

$w(i') = \sum_{h_1 \in I} r_{h_1,l} h'_1$,

$ws_i(j') = w(i') + w(j') = \sum_{h_1 \in I} r_{h_1,l+1} h'_1$,

$ws_i s_j(i') = w(j') = \sum_{h_1 \in I} r_{h_1,l+2} h'_1$,

$w(j') = \sum_{h_1 \in I} r'_{h_1,l} h'_1$,

$ws_j(i') = w(i') + w(j') = \sum_{h_1 \in I} r'_{h_1,l+1} h'_1$,

$ws_j s_i(j') = w(i') = \sum_{h_1 \in I} r'_{h_1,l+2} h'_1$.

Thus,

$r_{h,l} = r'_{h,l+2}$, $r_{h,l+1} = r'_{h,l+1} = r_{h,l} + r_{h,l+2}$, $r_{h,l+2} = r'_{h,l}$.

Thus, (b) is equivalent to

$$(a_l a_{l+1})^{r_{h,l}} (a_{l+1} a_{l+2})^{r_{h,l+2}} = (a'_l a'_{l+1})^{r_{h,l}} (a'_{l+1} a'_{l+2})^{r_{h,l+2}}.$$

This follows from

$$a_l a_{l+1} = a'_{l+1} a'_{l+2}, \quad a'_l a'_{l+1} = a_{l+1} a_{l+2}.$$

This proves (a) in case 1.1(i).

In case 1.1(ii) we must show

(c) $$a_l^{r_{h,l}} a_{l+1}^{r_{h,l+1}} = a_l'^{r'_{h,l}} {a'_{l+1}}^{r'_{h,l+1}}.$$

For some $w \in W$ we have

$w(i') = \sum_{h_1 \in I} r_{h_1,l} h'_1$,

$ws_i(j') = w(j') = \sum_{h_1 \in I} r_{h_1,l+1} h'_1$,

$w(j') = \sum_{h_1 \in I} r'_{h_1,l} h'_1$,

$ws_j(i') = w(i') = \sum_{h_1 \in I} r'_{h_1,l+1} h'_1$.

Thus, $r_{h,l} = r'_{h,l+1}$, $r_{h,l+1} = r'_{h,l}$ so that

$r_{h,l} = r'_{h,l+1}$, $r_{h,l+1} = r'_{h,l}$.

Thus, (c) is equivalent to

$$a_l^{r_{h,l}} a_{l+1}^{r_{h,l+1}} = a_l'^{r_{h,l+1}} {a'_{l+1}}^{r_{h,l}}$$

and this follows from $a'_l = a_{l+1}$, $a'_{l+1} = a_l$. This proves (a) in case 1.1(ii).

In view of (a) we can define $||A||_h \in K$ for any $A \in \mathcal{U}_K, h \in I$ to be $||\mathbf{i}, \mathbf{a}||_h$ for any $(\mathbf{i}, \mathbf{a})$ in A.

1.3. Let $\mathbf{i} \in \mathcal{I}$. We define a map $f_{\mathbf{i}} : K^\nu \to \mathcal{U}_K$ by sending $\mathbf{a} \in K^\nu$ to the equivalence class of $(\mathbf{i}, \mathbf{a})$. We show:

(a) $f_{\mathbf{i}}$ is a bijection.
Let $A \in \mathcal{U}_K$. If $(\mathbf{i}', \mathbf{a}') \in A$ then using Matsumoto's theorem [13, 1.9] for w_0, we can find a sequence $(\mathbf{i}^1, \mathbf{a}^1), \dots, (\mathbf{i}^n, \mathbf{a}^n)$ in $\mathcal{I} \times K^\nu$ in which any two consecutive terms are adjacent such that $(\mathbf{i}^n, \mathbf{a}^n) = (\mathbf{i}', \mathbf{a}')$, $\mathbf{i}^1 = \mathbf{i}$. We have $(\mathbf{i}, \mathbf{a}^1) \in A$. Thus $f_{\mathbf{i}}$ is surjective.

Next we prove that $f_{\mathbf{i}}$ is injective. Assume first that K is contained on the multiplicative group of a field $\mathbf{k}$ of characteristic zero with $+, \times$ induced from that of $\mathbf{k}$. Then $f_{\mathbf{i}}$ is injective by 4.2(a).

Next we consider a general K. Assume that $\mathbf{a} = (a_1, \dots, a_\nu) \in K^\nu, \mathbf{a}' = (a'_1, \dots, a'_\nu)$ in K^ν are such that $f_{\mathbf{i}}(\mathbf{a}) = f_{\mathbf{i}}(\mathbf{a}')$. Then we can find

$$(\mathbf{i}^1, \mathbf{a}^1), \dots, (\mathbf{i}^n, \mathbf{a}^n)$$

in $\mathcal{I} \times K^\nu$ in which any two consecutive terms are adjacent and $(\mathbf{i}^1, \mathbf{a}^1) = (\mathbf{i}, \mathbf{a})$, $(\mathbf{i}^n, \mathbf{a}^n) = (\mathbf{i}, \mathbf{a}')$. By [2, 2.1.6] we can find a semifield $\tilde{K}$ as in the first part of the proof and a homomorphism of semifields $z : \tilde{K} \to K$ such that

$$a_1 = z(\tilde{a}_1), \dots, a_\nu = z(\tilde{a}_\nu)$$

for some

$$\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_\nu) \in \tilde{K}^\nu.$$

We define

$$\tilde{\mathbf{a}}^1, \tilde{\mathbf{a}}^2, \dots, \tilde{\mathbf{a}}^\nu$$

by the condition that any two consecutive terms of

$$(\mathbf{i}^1, \tilde{\mathbf{a}}^1), (\mathbf{i}^2, \tilde{\mathbf{a}}^2), \dots, (\mathbf{i}^n, \tilde{\mathbf{a}}^n)$$

are adjacent (in $\mathcal{I} \times \tilde{K}^\nu$). Now z takes $\tilde{\mathbf{a}}^1$ to $\mathbf{a}^1$ and then it automatically takes $\tilde{\mathbf{a}}^2, \dots, \tilde{\mathbf{a}}^n$ to $\mathbf{a}^2, \dots, \mathbf{a}^n$. We have $\mathbf{i}^1 = \mathbf{i}^n = \mathbf{i}$ and the injectivity of our map (for $\tilde{K}$) implies that $\tilde{\mathbf{a}}^1 = \tilde{\mathbf{a}}^n$. Applying z we see that $\mathbf{a}^1 = \mathbf{a}^n$ that is $\mathbf{a} = \mathbf{a}'$. This proves injectivity of $f_{\mathbf{i}}$. This proves (a).

1.4. Let $i \in I, c \in K$. If $A \in \mathcal{U}_K$ we can find $(\mathbf{i}, \mathbf{a}) \in A$ such that the first term of $\mathbf{i}$ is i. (We use Matsumoto's theorem [13, 1.9] for w_0.) We set ${}_c\mathbf{a} = (a_1 c, a_2, \dots, a_\nu)$ where $\mathbf{a} = (a_1, a_2, \dots, a_\nu)$. Assume that $(\mathbf{i}', \mathbf{a}') \in A$ is also such that the first term of $\mathbf{i}'$ is i. Then ${}_c\mathbf{a}' \in K^\nu$ is defined. We show that

(a) $(\mathbf{i}, {}_c\mathbf{a}), (\mathbf{i}', {}_c\mathbf{a}')$ are equivalent.

Using Matsumoto's theorem for $s_i w_0$, we see that we can find a sequence

$$(\mathbf{i}^1, \mathbf{a}^1), \dots, (\mathbf{i}^n, \mathbf{a}^n)$$

in $\mathcal{I} \times K^\nu$ in which any two consecutive terms are adjacent as in 1.1(i),(ii) with $l \geq 2$ such that

$$(\mathbf{i}^1, \mathbf{a}^1) = (\mathbf{i}, \mathbf{a}), \mathbf{i}^n = \mathbf{i}'$$

(in particular each $\mathbf{i}^1, \mathbf{i}^2, \ldots$ starts with i). Then $(\mathbf{i}', \mathbf{a}^n), (\mathbf{i}', \mathbf{a}')$ are both in A hence by 1.3(a) we must have $\mathbf{a}^n = \mathbf{a}'$. Now any two consecutive terms of

$$(\mathbf{i}^1, {}_c\mathbf{a}^1), \ldots, (\mathbf{i}^n, {}_c\mathbf{a}^n)$$

are adjacent and we have

$$(\mathbf{i}^1, {}_c\mathbf{a}^1) = (\mathbf{i}, {}_c\mathbf{a}), (\mathbf{i}^n, {}_c\mathbf{a}^n) = (\mathbf{i}', {}_c\mathbf{a}').$$

This proves (a).

We see that $(\mathbf{i}, \mathbf{a}) \mapsto (\mathbf{i}, \mathbf{a}_c)$ defines a map (in fact a bijection) $T_{i,c} : \mathcal{U}_K \to \mathcal{U}_K$.

For c, c' in K we have $T_{i,c} T_{i,c'} = T_{i,cc'}$.

1.5. Let $i \in I, c \in K, A \in \mathcal{U}_K$.

We can find $(\mathbf{i}, \mathbf{a}) \in A$ such that the first term of $\mathbf{i}$ is i. Define $r_{h,k}$ in terms of $\mathbf{i}$ as in 1.2. We have $r_{h,1} = \delta_{h,i} 1$ (this is 1 if $h = i$ and is 0 if $h \neq i$). We have

$$||\mathbf{i}, \mathbf{a}||_h = a_1^{\delta_{i,h}} \prod_{k \in [2,\nu]} a_k^{r_{h,k}} \in K.$$
$$||\mathbf{i}, {}_c\mathbf{a}||_h = (ca_1)^{\delta_{i,h}} \prod_{k \in [2,\nu]} a_k^{r_{h,k}} = c^{\delta_{i,h}} ||\mathbf{i}, \mathbf{a}||_h.$$

Thus we have

$$||T_{i,c} A||_h = c^{\delta_{i,h}} ||A||_h.$$

1.6. We regard K^I as a group (the product of (p_i) and (p'_i) is $(p_i p'_i)$).

For $p = (p_i)_{i \in I} \in K^I$ there is well defined bijection $S_p : \mathcal{U}_K \to \mathcal{U}_K$ given by

$$i_1^{a_1} i_2^{a_2} \ldots i_\nu^{a_\nu} \mapsto i_1^{a_1 p_{i_1}} i_2^{a_2 p_{i_2}} \ldots i_\nu^{a_\nu p_{i_\nu}}$$

for $(\mathbf{i}, \mathbf{a}) \in \mathcal{I} \times K^\nu$, see [16, no.8].

This defines an action of the group K^I on $\mathcal{U}_K$.

For $p \in K^I, i \in I, c \in K$ we have

(a) $T_{i,c}S_p = S_pT_{i,c}$.
There is a well defined involution $\iota : \mathcal{U}_K \to \mathcal{U}_K$ such that

$$i_1^{a_1} i_2^{a_2} \dots i_\nu^{a_\nu} \mapsto (i_1^!)^{a_1}(i_2^!)^{a_2} \dots (i_\nu^!)^{a_\nu}$$

for $(\mathbf{i}, \mathbf{a}) \in \mathcal{I} \times K^\nu$. For $p \in K^I$ we have $\iota S_p = S_{p^!}\iota$ where $p^! \in K^I$ is given by $(p^!)_i = p_{i^!}$.

For $i \in I, c \in K$ we have

(b) $T_{i,c}\iota = \iota T_{i^!,c}$

We show:

(c) The action of K^I on $\mathcal{U}_K$ described above is free.
We must show that if $p = (p_i) \in K^I$ and $A \in \mathcal{U}_K$ are such that $S_pA = A$ then $p_i = 1$ for all i. Let $(\mathbf{i}, \mathbf{a}) \in A$ with $\mathbf{i} = (i_1, i_2, \dots, i_\nu), \mathbf{a} = (a_1, a_2, \dots, a_\nu)$. Let $\mathbf{a}' = (a_1p_{i_1}, a_2p_{i_2}, \dots, a_\nu p_{i_\nu}) \in K^\nu$. By assumption we have $(\mathbf{i}, \mathbf{a}') \in A$. Using 1.3(a) we deduce that $\mathbf{a}' = \mathbf{a}$. Thus $a_k = a_kp_{i_k}$ for $k = 1, \dots, \nu$ so that $p_{i_k} = 1$ for $k = 1, \dots, \nu$. For any $i \in I$ we can find k such that $i_k = i$. It follows that $p_i = 1$ for $i \in I$. This proves (c).

1.7. For a semifield K, let $\phi_K : \mathcal{U}_K \to \mathcal{U}_K$ be the bijection defined in 4.6, see also [16, no.11]. We have $\phi_K^2 = 1$.

For example, if $I = \{i\}$ we have $\phi_K(i^a) = i^{1/a}$; if $I = \{i, j\}$ with $a_{ij} = 1$, we have

$$\phi_K(i^aj^bi^c) = i^{a/c(a+c)}j^{(a+c)/ab}i^{1/(a+c)} = j^{c/ab}i^{1/c}j^{1/b}.$$

We have

(a) $\iota\phi_K = \phi_K\iota$.
For $p \in K^I$ we have

(b) $S_p\phi_K = \phi_KS_{p^{-1}}$;
hence $S_p\phi_K : \mathcal{U}_K \to \mathcal{U}_K$ has square 1.

For $i \in I, c \in K$ we have

(c) $$T_{i,c}\phi_K = \phi_K T_{i^!,c^{-1}}.$$

Now (b),(c) can be viewed as sequences of equalities between certain rational functions which are quotients of nonzero polynomials in several variables with all coefficients in $\mathbf{N}$ (after substituting elements of K for the variables). It is enough to prove these equalities in the case where $K = \mathbf{R}_{>0}$; in that case (b) is proved in [15, 4.3(d)] and (c) is proved in [16, 10(a)].

1.8. Let $\mathcal{B}_K = \{(A, A') \in \mathcal{U}_K \times \mathcal{U}_K; \phi_K(A) = A'\}$ be the graph of ϕ_K. We define an involution $\underline{\phi}_K : \mathcal{B}_K \to \mathcal{B}_K$ by $\underline{\phi}_K(A, A') = (A', A)$. (We use that $\phi_K^2 = 1$.) We define an involution $\underline{\iota} : \mathcal{B}_K \to \mathcal{B}_K$ by $\underline{\iota}(A, A') = (\iota(A), \iota(A'))$. (We use 1.7(a).) Now the group K^I acts on $\mathcal{B}_K$ by $p : B \mapsto \underline{S}_p(B)$ where $\underline{S}_p(A, A') = (S_p A, S_{p^{-1}} A')$. (We use 1.7(b)).

For $p \in K^I$ we have $\underline{S}_p \underline{\phi}_K = \underline{\phi}_K \underline{S}_{p^{-1}}$.

1.9. Let $A \in \mathcal{U}_K$ and let $(\mathbf{i}, \mathbf{a}) \in A$ with $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$, $\mathbf{a} = (a_1, a_2, \dots, a_\nu)$. Let $\mathbf{i}' = (i_\nu, \dots, i_2, i_1) \in \mathcal{I}$. Define $\mathbf{a}' = (a'_1, a'_2, \dots, a'_\nu)$ by $(\mathbf{i}', \mathbf{a}') \in \phi_K(A)$.

Lemma 1.10. *We have* $a'_\nu = (\sum_{k \in [1,\nu]; i_k = i_1} a_k)^{-1}$.

Using [2, 2.1.6] we see that we can reduce the general case to the case where K is as in 4.2. In that case the result follows from 4.5. (More precisely 4.5 gives the analogous result for ϕ'_K in 4.6 instead of 4.5, but the case of ϕ_K is then a consequence.)

2. The Subsets $\mathcal{B}^+_{\mathbf{N}}, \mathcal{B}^-_{\mathbf{N}}$ of $\mathcal{B}_{\mathbf{Z}}$

2.1. In this section we assume that $K = \mathbf{Z}$ with the usual semifield structure: the sum of a, b is taken to be $\min(a, b)$; the product of a, b is taken to be $a + b$. Then $\mathcal{U}_{\mathbf{Z}}$ is defined as in 1.1. Note that the subset $\mathcal{I} \times \mathbf{N}^\nu$ of $\mathcal{I} \times \mathbf{Z}^\nu$ is a union of equivalence classes for $\sim$. (We use that, if $a, c \in \mathbf{N}$, then $a - \min(a, c) \in \mathbf{N}$.) The set of these equivalence classes is a subset $\mathcal{U}_{\mathbf{N}}$ of $\mathcal{U}_{\mathbf{Z}}$.

For $i \in I$ there is a well defined map $g'_i : \mathcal{U}_{\mathbf{N}} \to \mathbf{N}$ such that $i_1^{a_1} i_2^{a_2} \dots i_\nu^{a_\nu} \mapsto a_\nu$ whenever $(\mathbf{i}, \mathbf{a}) \in \mathcal{I} \times \mathbf{N}^\nu$ satisfies $i_\nu = i$.

For any $p = (p_i) \in \mathbf{N}^I$ we define

$$\mathcal{U}_{\mathbf{N},p} = \{x \in \mathcal{U}_{\mathbf{N}}; g'_i(x) \le p_i \quad \forall i \in I\}.$$

Note that $\iota : \mathcal{U}_{\mathbf{Z}} \to \mathcal{U}_{\mathbf{Z}}$ restricts to a bijection

$$\mathcal{U}_{\mathbf{N},p} \xrightarrow{\sim} \mathcal{U}_{\mathbf{N},p^!}$$

where $p^! \in \mathbf{N}^I$ is as in 1.6.

2.2. For $p = (p_i) \in \mathbf{N}^I$ we set $\mathbf{h}_p = \dot{i}_1^{p_{i_1}} \dot{i}_2^{p_{i_2}} \dots \dot{i}_\nu^{p_{i_\nu}} \in \mathcal{U}_{\mathbf{N}}$ where $\mathbf{i} \in \mathcal{I}$. From the definition we see that this is independent of the choice of $\mathbf{i}$ hence is well defined.

Note that $\mathbf{h}_0 \in \mathcal{U}_{\mathbf{N},p}, \mathbf{h}_p \in \mathcal{U}_{\mathbf{N},p}$. We have

(a) $$\mathbf{h}_p = S(p)(\mathbf{h}_0),$$

(b) $$\iota(\mathbf{h}_p) = \mathbf{h}_{p^!},$$

From [12, 2.9(a), 3.9] we have

(c) $$\phi_{\mathbf{Z}}(\mathbf{h}_0) = \mathbf{h}_0.$$

In fact, by [12, 2.9], $\phi_{\mathbf{Z}}$ is the unique bijection $\mathcal{U}_{\mathbf{Z}} \to \mathcal{U}_{\mathbf{Z}}$ satisfying (c) and 1.7(c) (with $K = \mathbf{Z}$).

2.3. Let $p = (p_i) \in \mathbf{N}^I$. By definition, for $h \in I$ we have

$$||\mathbf{h}_p||_h = \sum_{k \in [1,\nu]} r_{h,k} p_{i_k} \in \mathbf{N}$$

where $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$ and for $k \in [1, \nu]$, $r_{h,k} \in \mathbf{N}$ are defined by

$$s_{i_1} s_{i_2} \dots s_{i_{k-1}}(i'_k) = \sum_{h \in I} r_{h,k} h'$$

(in $\mathbf{Z}[I]$). We state the following result.

(a) $$||\mathbf{h}_p||_h = \sum_{i\in I}(p_i + (p^!)_i)b_{ih} \in \mathbf{N}$$

where (b_{ij}) is the inverse of the Cartan matrix (a_{ij}). It is known that $b_{ij} \in \mathbf{Q}_{\geq 0}$ (see for example [17]). Now (a) can be verified case by case.

Assume for example that $I = \{i\}$. We have $||h_p||_i = p_i$, $b_{ii} = 1/2$, $(p_i + (p^!)_i)b_{ii} = p_i$ hence (a) holds.

Assume now that $I = \{i, j\}$ with $a_{ij} = -1$. Let $\mathbf{i} = (i, j, i)$. We have $\mathbf{i} \in \mathcal{I}$. The corresponding sequence $\sum_{h\in I} r_{h,k}h'$, $(k = 1, 2, 3)$ is $i', i' + j', j'$ hence

$$\sum_{h\in I} ||\mathbf{h}_p||_h h' = p_i i' + p_j(i' + j') + p_i j' = (p_i + p_j)(i' + j').$$

We have $b_{ii} = b_{jj} = 2/3, b_{ij} = b_{ji} = 1/3$. Hence

$$\begin{aligned}&\sum_{h\in I}(p_i + (p^!)_i)b_{ih}h' + \sum_{h\in I}(p_j + (p^!)_j)b_{jh}h'\\ &\quad = (p_i + p_j)((2/3)i' + (1/3)j') + (p_j + p_i)((1/3)i' + (2/3)j')\\ &\quad = (p_i + p_j)(i' + j').\end{aligned}$$

Thus, (a) holds.

Assume now that $I = \{0, c, d, e\}$ with

$$a_{0c} = a_{c0} = a_{0d} = a_{d0} = a_{0e} = a_{e0} = -1,$$

$$a_{cd} = a_{dc} = a_{ce} = a_{ec} = a_{de} = a_{ed} = 0.$$

Let $\mathbf{i} = (c, d, e, 0, c, d, e, 0, c, d, e, 0)$. We have $\mathbf{i} \in \mathcal{I}$.

The corresponding sequence $\sum_{h\in I} r_{h,k}h'$ $(k = 1, 2, \dots, 12)$ is

$$c', d', e', 0'c'd'e', 0'd'e', 0'c'e', 0'c'd', 0'0'c'd'e', 0'c', 0'd', 0'e', 0'$$

where we omit + signs (for example we write $0'c'$ for $0' + c'$.)

Thus we have

$$\sum_{h\in I}||\mathbf{h}_p||_h h' = p_c c' + p_d d' + p_e e' + p_0(0' + c' + d + e') + p_c(0' + d' + e')$$
$$+ p_d(0' + c' + e') + p_e(0' + c' + d') + p_0(0' + 0' + c' + d' + e')$$
$$+ p_c(0' + c') + p_d(0' + d') + p_e(0' + e') + p_0 0'$$
$$+ (2p_c + p_d + p_e + 2p_0)c' + (p_c + 2p_d + p_e + 2p_0)d'$$
$$+ (p_c + p_d + 2p_e + 2p_0)e' + (2p_c + 2p_d + 2p_e + 4p_0)0'.$$

We have

$$b_{00} = 2, b_{cc} = b_{dd} = b_{ee} = 1,$$
$$b_{0c} = b_{c0} = b_{0d} = b_{d0} = b_{0e} = b_{e0} = 1,$$
$$b_{cd} = b_{dc} = b_{ce} = b_{ec} = b_{de} = b_{ed} = 1/2.$$

Moreover, $p = p^!$. We see that (a) holds.

2.4. We define two subsets of $\mathcal{B}_{\mathbf{Z}}$ (see 1.8) by $\mathcal{B}^+_{\mathbf{N}} = \{(A, A') \in \mathcal{B}_{\mathbf{Z}}; A \in \mathcal{U}_{\mathbf{N}}\}$, $\mathcal{B}^-_{\mathbf{N}} = \{(A, A') \in \mathcal{B}_{\mathbf{Z}}; A' \in \mathcal{U}_{\mathbf{N}}\}$.

For any $p \in \mathbf{Z}^I$ we define

$$\mathcal{B}_{\mathbf{N},p} = \mathcal{B}^+_{\mathbf{N}} \cap \underline{S}_p(\mathcal{B}^-_{\mathbf{N}}) \subset \mathcal{B}_{\mathbf{Z}}.$$

Clearly, $(A, A') \mapsto A$ is a bijection

(a) $$\mathcal{B}_{\mathbf{N},p} \xrightarrow{\sim} \{A \in \mathcal{U}_{\mathbf{N}}; S_p\phi_{\mathbf{Z}}(A) \in \mathcal{U}_{\mathbf{N}}\}.$$

Note that if $p \in \mathbf{N}^I$ then $(\mathbf{h}_0, \mathbf{h}_p) \in \mathcal{B}_{\mathbf{N},p}$. (We use 2.2(a),(c).) In particular, in this case we have $\mathcal{B}_{\mathbf{N},p} \neq \emptyset$.

2.5. We show that, conversely,

(a) if $p \in \mathbf{Z}^I$ and $\mathcal{B}_{\mathbf{N},p} \neq \emptyset$, then $p \in \mathbf{N}^I$.
Assume for example that $I = \{i, j\}$ with $a_{ij} = -1$, and that $A \in \mathcal{U}_{\mathbf{N}}$ satisfies $S_p\phi_{\mathbf{Z}}(A) \in \mathcal{U}_{\mathbf{N}}$. We show that $p \in \mathbf{N}^I$. We have $i^a j^b i^c \in A$ with a, b, c in $\mathbf{N}$ and $j^{c-a-b+p_j} i^{-c+p_i} j^{-b+p_j} \in S_p\phi_{\mathbf{Z}}(A)$ with $c - a - b + p_j \in \mathbf{N}, -c + p_i \in \mathbf{N}, -b + p_j \in \mathbf{N}$. Then $p_i \geq c \geq 0$, $p_j \geq b \geq 0$, so that indeed $p \in \mathbf{N}^I$.

We now consider the general case. Assume that $A \in \mathcal{U}_{\mathbf{N}}$ satisfies $S_p\phi_{\mathbf{Z}}(A) \in \mathcal{U}_{\mathbf{N}}$. Let $i \in I$. We can find $(\mathbf{i}, \mathbf{a}) \in A$ with $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$, $\mathbf{a} = (a_1, a_2, \dots, a_\nu) \in \mathbf{N}^\nu$ such that $i_\nu = i$. Let $\mathbf{i}' = (i_\nu, \dots, i_2, i_1) \in \mathcal{I}$. Define $\mathbf{a}' = (a'_1, a'_2, \dots, a'_\nu)$ by $(\mathbf{i}', \mathbf{a}') \in \phi_K(A)$. By 1.10 we have $a'_\nu = -\min_{k\in[1,\nu]; i_k = i_1} a_k$. In particular we have $a'_\nu \le 0$. By assumption we have $p_i + a'_\nu \ge 0$ hence $p_i \ge -a'_\nu \ge 0$. This proves (a).

3. Canonical Bases

3.1. We fix a simply laced root datum; this consists of a finite set I, two finitely generated free abelian groups Y, X; a perfect pairing $(,) : Y \times X \to \mathbf{Z}$; an imbedding $I \subset X, i \mapsto i'$ and an imbedding $I \subset Y, i \mapsto i$. It is assumed that I is as in 1.1 and $(i, j') = a_{ij}$ where a_{ij} is as in 1.1. We identify $\mathbf{Z}[I]$ in 1.1 with a subgroup of X by $i' \mapsto i'$. The action of s_i on $\mathbf{Z}[I]$ extends to an action on X by $s_i : x \mapsto x - (i, x)i'$. Here $i \in I$. Thus the W-action of on $\mathbf{Z}[I]$ extends to a W-action on X. For $\lambda \in X$ we set $\lambda^! = -w_0(\lambda)$; then $\lambda \mapsto \lambda^!$ is an involution of X.

Let $X^+ = \{\lambda \in X; (i, \lambda) \in \mathbf{N} \quad \forall i \in I\}$. Note that X^+ is stable under $\lambda \mapsto \lambda^!$.

For λ, λ' in X we write $\lambda' \ge \lambda$ if $\lambda' - \lambda \in \sum_{i\in I} \mathbf{N}i'$ and $\lambda' > \lambda$ if $\lambda' \ge \lambda$ and $\lambda' \ne \lambda$.

3.2. Let v be an indeterminate. Let $\mathbf{f}$ be the associative algebra with 1 over $\mathbf{Q}(v)$ with generators $\{\theta_i; i \in I\}$ associated to the matrix (a_{ij}) in [10, 1.2.5]. This can be identified with the + part of the algebra U (see below) attached to the root datum.

There is a unique algebra antiautomorphism $\mathbf{f} \to \mathbf{f}$ $(x \mapsto x^*)$ such that $\theta_i^* = \theta_i$ for all $i \in I$. It has square 1.

Let U be the Drinfeld-Jimbo quantized enveloping algebra attached to the root datum. This is an associative algebra with 1 over $\mathbf{Q}(v)$. As a vector space, U can be identified with $\oplus_{\gamma\in Y} \mathbf{f} \otimes \mathbf{f}$ in two different ways: one by $(x \otimes x')_y \mapsto x^+ \mathcal{K}_y x'^-$ and one by $(x \otimes x')_y \mapsto x^- \mathcal{K}_y x'^+$; here $\mathcal{K}_0$ is the unit element. The map $\mathbf{f} \to U, x \mapsto x^-$ and the map $\mathbf{f} \to U, x \mapsto x^+$ are imbeddings of algebras with 1.

3.3. Let $\dot{U}$ be the modified form of U, see [9, §23]. This is an associative algebra (without 1 in general) over $\mathbf{Q}(v)$. In type A it was defined in [1]; the definition in the general case is the same. As a vector space, $\dot{U}$ can be identified with $\oplus_{\lambda\in X}\mathbf{f}\otimes\mathbf{f}$ in two different ways: one by $(x\otimes x')_\lambda \mapsto x^+1_\lambda x'^-$ and one by $(x\otimes x')_\lambda \mapsto x^-1_\lambda x'^+$.

There is a unique vector space isomorphism $\sharp : \dot{U}\to\dot{U}$ such that

$$\sharp(x^+1_\lambda x'^-) = (x'^*)^+1_\lambda(x^*)^-$$

for x, x' in $\mathbf{f}$, $\lambda\in X$; we have also

$$\sharp(x^-1_\lambda x'^+) = (x'^*)^-1_\lambda(x^*)^-$$

for x, x' in $\mathbf{f}$, $\lambda\in X$; hence $\sharp^2 = 1$. Moreover, $\sharp$ is an algebra antiautomorphism.

There is a unique vector space isomorphism $\omega : \dot{U}\to\dot{U}$ such that

$$\omega(x^+1_\lambda x'^-) = x^-1_{-\lambda}x'^+$$

for x, x' in $\mathbf{f}$, $\lambda\in X$; we have also $\omega(x^-1_\lambda x'^+) = x^+1_{-\lambda}x'^-$ for x, x' in $\mathbf{f}$, hence $\omega^2 = 1$. Moreover ω is an algebra automorphism satisfying $\omega\sharp = \sharp\omega$.

3.4. For $\lambda\in X^+$ let Λ_λ be the simple U-module defined in [9, 3.5.6]. We shall regard Λ_λ also as (unital) $\dot{U}$-module as in [9, 23.1.4]. We have $\dim\Lambda_\lambda < \infty$. Let $\eta_\lambda\in\Lambda_\lambda$ be as in [9, 3.5.7]. We have $1_\lambda\eta_\lambda = \eta_\lambda$.

Let $(,)_\lambda$ be the symmetric bilinear form $\Lambda_\lambda\times\Lambda_\lambda\to\mathbf{Q}(v)$ defined in [9, 19.1.2]. Recall that $(\eta_\lambda,\eta_\lambda)_\lambda = 1$.

Let $\dot{U}[\geq\lambda]$ (resp. $\dot{U}[>\lambda]$) be the set of all $u\in\dot{U}$ such that the following condition holds.

For any $\lambda'\in X^+$ such that u acts on $\Lambda_{\lambda'}$ by a nonzero map we have $\lambda'\geq\lambda$ (resp.$\lambda'>\lambda$).

Clearly, $\dot{U}[\geq\lambda]$ and $\dot{U}[>\lambda]$ are two-sided ideals of $\dot{U}$.

3.5. Let $\mathbf{B}$ be the canonical basis of $\mathbf{f}$ (see [6], [9]). By [6, 3.3],

for $b\in\mathbf{B}$ we have $b^*\in\mathbf{B}$ and $b\mapsto b^*$ is a bijection $\mathbf{B}\xrightarrow{\sim}\mathbf{B}$.

If $b \in \mathbf{B}$ then there is a well defined element $wt(b) \in \sum_{h\in I} \mathbf{N}h' \in X$ such that the following holds: b is $\mathbf{Q}(v)$-linear combination of elements $\theta_{i_1}\theta_{i_2}\dots\theta_{i_n}$ where $i_1, i_2, \dots, i_n$ in I satisfy $i'_1 + i'_2 + \dots + i'_n = wt(b)$.

3.6. Let $\lambda \in X^+$. By [6],

there is a unique $\mathbf{Q}(v)$-basis B_λ of Λ_λ and a unique subset $\mathbf{B}(\lambda)$ of $\mathbf{B}$ such that $b \mapsto b^-\eta_\lambda$ maps $\mathbf{B} - \mathbf{B}(\lambda)$ to 0 and restricts to a bijection $\mathbf{B}(\lambda) \xrightarrow{\sim} B_\lambda$.

Let ξ_λ be the unique element in B_λ such that $1_{-\lambda^!}\xi_\lambda = \xi_\lambda$.

By [9, §21],

(a) there is a unique vector space isomorphism $\tau : \Lambda_\lambda \to \Lambda_{\lambda^!}$ such that $\tau(ux) = \omega(u)\tau(x)$ for $u \in \dot{U}, x \in \Lambda_\lambda$ and $\tau(\eta_\lambda) = \xi_{\lambda^!}$. It satisfies $\tau(B_\lambda) = B_{\lambda^!}$ and $\tau(\xi_\lambda) = \eta_{\lambda^!}$.

We see that there is a unique bijection $\kappa : \mathbf{B}(\lambda) \xrightarrow{\sim} \mathbf{B}(\lambda^!)$ such that $\tau(b^-\eta_\lambda) = \kappa(b)^-\eta_{\lambda^!}$ for $b \in \mathbf{B}(\lambda)$. For $b \in \mathbf{B}(\lambda)$ we have

(b) $b^+\xi_{\lambda^!} = \kappa(b)^-\eta_{\lambda^!}$.

This follows from $\tau(b^-\eta_\lambda) = b^+\xi_{\lambda^!}$.

3.7. From [9, 19.1.4] for any $b \in \mathbf{B}(\lambda)$ we have

$$(b^-\eta_\lambda, b^-\eta_\lambda)_\lambda \in 1 + v^{-1}\mathbf{Q}[[v^{-1}]].$$

3.8. Let $\dot{\mathbf{B}}$ be the canonical basis of $\dot{U}$ defined in [8], see also [9, 25.2]. By [9, 26.3.2],

(a) for $\beta \in \dot{\mathbf{B}}$ we have $\omega\sharp(\beta) \in \pm\dot{\mathbf{B}}$ and $\omega(\beta) \in \pm\dot{\mathbf{B}}$.

In *loc.cit.* it was conjectured that the signs in (a) are +. The fact that the sign is + for $\omega\sharp$ is proved in [5, 4.3.2]. I thank H.Nakajima for pointing out to me that [5, 4.3.2] together with [11, 4.14] imply that the sign for ω is +. See also 3.16(a) for a more precise statement.

3.9. For $\lambda \in X^+$ let $\dot{\mathbf{B}}[\lambda]$ be the set of all $\beta \in \dot{\mathbf{B}} \cap \dot{U}[\geq \lambda]$ such that β acts on Λ_λ by a nonzero map. By [9, 29.1.2, 29.1.3, 29.1.4] we have a partition

$\dot{\mathbf{B}} = \sqcup_{\lambda \in X^+} \dot{\mathbf{B}}[\lambda]$. Note that for $\lambda \in X^+$, $\dot{U}[\geq \lambda]$ (resp. $\dot{U}[> \lambda]$) is the subspace of $\dot{U}$ with basis $\sqcup_{\lambda' \in X^+; \lambda' \geq \lambda} \dot{\mathbf{B}}[\lambda']$ (resp. $\sqcup_{\lambda' \in X^+; \lambda' > \lambda} \dot{\mathbf{B}}[\lambda']$).

3.10. Let $\lambda \in X^+$.

By [11, 4.4(a)], for $b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda)$, there exists a unique element $\beta \in \dot{\mathbf{B}}[\lambda]$ such that $b_1 1_\lambda b_2^{*+} - \beta \in \dot{U}[> \lambda]$. We set $\beta = \beta_\lambda(b_1, b_2)$.

By [11, 4.4(b)],

(a) *the map* $f : \sqcup_{\lambda \in X^+} \boldsymbol{B}(\lambda) \times \boldsymbol{B}(\lambda) \to \dot{\boldsymbol{B}}[\lambda]$ *given by* $(\lambda, (b_1, b_2)) \mapsto \beta_\lambda(b_1, b_2)$ *is bijective.*

Lemma 3.11. *Let* $\lambda \in X^+$, $b_1 \in \boldsymbol{B}(\lambda), b_2 \in \boldsymbol{B}(\lambda)$. *For any* $r \in \boldsymbol{Z}$ *we set*

$$u_r := 1_\lambda b_2^{*+} b_1^- 1_\lambda - v^r (b_1^- \eta_\lambda, b_2^- \eta_\lambda)_\lambda 1_\lambda.$$

Then for some $r = r_{b_2,\lambda} \in \boldsymbol{Z}$ *we have* $u_r \in \dot{U}[> \lambda]$.

Here we write $r = r_{b_2,\lambda}$ instead of: r depending on b_2, λ but not on b_1.

The following proof is almost copied from [11, 4.7].

Since $1_\lambda \in \dot{U}[\geq \lambda]$ and $\dot{U}[\geq \lambda]$ is a two-sided ideal of $\dot{U}$, we have $u_r \in \dot{U}[\geq \lambda]$ and it is enough to show that for some $r = r_{b_2,\lambda} \in \mathbf{Z}$, u_r acts as 0 on Λ_λ. Since $u_r = u_r 1_\lambda$ and $1_\lambda \Lambda_\lambda$ is the line spanned by η_λ, it is enough to show that $u_r \eta_\lambda = 0$ for some $r = r_{b_2,\lambda} \in \mathbf{Z}$. Since $u_r = 1_\lambda u_r$, we have $u_r \eta_\lambda = z_r \eta_\lambda$ for some $z_r \in \mathbf{Q}(v)$. We have

$$z_r (\eta_\lambda, \eta_\lambda)_\lambda = (u_r \eta_\lambda, \eta_\lambda)_\lambda.$$

By the definition of $(,)_\lambda$, for some $r_0 = (r_0)_{b_2,\lambda} \in \mathbf{Z}$ we have

$$(1_\lambda b_2^{*+} b_1^- 1_\lambda \eta_\lambda, \eta_\lambda)_\lambda = (b_1^- \eta_\lambda, v^{r_0} \sharp (1_\lambda b_2^{*+}) \eta_\lambda)_\lambda = (b_1^- \eta_\lambda, v^{r_0} b_2^- \eta_\lambda)_\lambda,$$

so that

$$z_{r_0} (\eta_\lambda, \eta_\lambda)_\lambda = (b_1^- \eta_\lambda, b_2^- \eta_\lambda)_\lambda)(v^{r_0} - v^{r_0} (\eta_\lambda, \eta_\lambda)_\lambda).$$

Since $(\eta_\lambda, \eta_\lambda)_\lambda = 1$ we see that $z_{r_0} = 0$ so that $u_{r_0} \eta_\lambda = 0$. The lemma is proved.

3.12. In the setup of Lemma 3.11 let $b_0 \in \mathbf{B}(\lambda)$. By Lemma 3.11 and its proof, we can find $r_0 = (r_0)_{b_2,\lambda} \in \mathbf{Z}$ such that $u_{r_0}\eta_\lambda = 0$; we then have $b_0^- u_{r_0}\eta_\lambda = 0$. We see that

$$b_0^- 1_\lambda b_2^{*+} b_1^- 1_\lambda \eta_\lambda = v^{r_0}(b_1^-\eta_\lambda, b_2^-\eta_\lambda)_\lambda b_0^-\eta_\lambda.$$

We now replace λ, b_0, b_1, b_2 by $\lambda^!, \kappa(b_0), \kappa(b_1), \kappa(b_2)$. (Recall that $\kappa(b_0)$, $\kappa(b_1)$, $\kappa(b_2)$ are in $\mathbf{B}(\lambda^!)$). We see that for any $\lambda \in X^+$ and any b_0, b_1, b_2 in $\mathbf{B}(\lambda)$ we can find $\tilde{r}_0 = (\tilde{r}_0)_{b_2,\lambda} \in \mathbf{Z}$ such that

$$\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+}\kappa(b_1)^- 1_{\lambda^!}\eta_{\lambda^!} = v^{\tilde{r}_0}(\kappa(b_1)^-\eta_{\lambda^!}, \kappa(b_2)^-\eta_{\lambda^!})_{\lambda^!}\kappa(b_0)^-\eta_{\lambda^!}.$$

Using 3.6(b), we deduce

(a) $$\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+}\kappa(b_1)^- 1_{\lambda^!}\eta_{\lambda^!} = v^{\tilde{r}_0}(b_1^+\xi_{\lambda^!}, b_2^+\xi_{\lambda^!})_{\lambda^!} b_0^+\xi_{\lambda^!}.$$

Lemma 3.13. *Let $\lambda \in X^+$, $b_1 \in \boldsymbol{B}(\lambda), b_2 \in \boldsymbol{B}(\lambda)$. Let $\delta = (\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!}$. For any $r \in \boldsymbol{Z}$ we set*

$$u'_r := 1_{-\lambda} b_2^{*-} b_1^+ 1_{-\lambda} - v^r \delta^{-1}(b_1^+\xi_{\lambda^!}, b_2^+\xi_{\lambda^!})_{\lambda^!} 1_{-\lambda}.$$

Then for some $r = r_{b_2,\lambda} \in \boldsymbol{Z}$ we have $u'_r \in \dot{U}[> \lambda^!]$.

The proof is similar to that of Lemma 3.11.

Since $1_{-\lambda} \in \dot{U}[\ge \lambda^!]$ and $\dot{U}[\ge \lambda^!]$ is a two-sided ideal of $\dot{U}$ we have $u'_r \in \dot{U}[\ge \lambda^!]$ and it is enough to show that for some $r = r_{b_2,\lambda} \in \mathbf{Z}$, u'_r acts as 0 on $\Lambda_{\lambda^!}$. Since $u'_r = u'_r 1_{-\lambda}$ and $1_{-\lambda}\Lambda_{\lambda^!}$ is the line spanned by $\xi_{\lambda^!}$, it is enough to show that $u'_r\xi_{\lambda^!} = 0$ for some $r = r_{b_2,\lambda} \in \mathbf{Z}$. Since $u'_r = 1_{-\lambda}u'_r$, we have $u'_r\xi_{\lambda^!} = z'_r\xi_{\lambda^!}$ for some $z'_r \in \mathbf{Q}(v)$. We have $z'_r(\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!} = (u'_r\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!})$. By the definition of $(,)_{\lambda^!}$, for some $r'_0 = (r'_0)_{b_2,\lambda} \in \mathbf{Z}$ we have

$$(1_{-\lambda} b_2^{*-} b_1^+ 1_{-\lambda}\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!} = (b_1^+\xi_{\lambda^!}, v^{r'_0}\sharp(1_{-\lambda}b_2^{*-})\xi_{\lambda^!})_{\lambda^!} = (b_1^+\xi_{\lambda^!}, v^{r'_0} b_2^+\xi_{\lambda^!})_{\lambda^!},$$

so that

$$z'_{r'_0}(\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!} = (b_1^+\xi_{\lambda^!}, b_2^+\xi_{\lambda^!})_{\lambda^!})(v^{r'_0} - v^{r'_0}\delta^{-1}(\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!}).$$

Since $(\xi_{\lambda^!}, \xi_{\lambda^!})_{\lambda^!} = \delta \ne 0$ (see 3.7), we see that $z'_{r'_0} = 0$ so that $u'_{r'_0}\xi_{\lambda^!} = 0$. The lemma is proved.

3.14. In the setup of Lemma 3.13, let $b_0 \in \mathbf{B}(\lambda)$. By Lemma 3.13 and its proof we can find $r'_0 = (r'_0)_{b_2,\lambda} \in \mathbf{Z}$ such that $u'_{r'_0}\xi_{\lambda^!} = 0$; we then have $(b_0^+ 1_{-\lambda})u'_{r'_0}\xi_{\lambda^!} = 0$. We see that

$$b_0^+ 1_{-\lambda} b_2^{*-} b_1^+ 1_{-\lambda}\xi_{\lambda^!} = v^{r'_0}\delta^{-1}(b_1^+\xi_{\lambda^!}, b_2^+\xi_{\lambda^!})_{\lambda^!} b_0^+\xi_{\lambda^!}.$$

Comparing with 3.12(a), we deduce

(a) $$v^{-r'_0}\delta b_0^+ 1_{-\lambda} b_2^{*-} b_1^+ 1_{-\lambda}\xi_{\lambda^!} = v^{-\tilde{r}_0}\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+}\kappa(b_1)^- 1_{\lambda^!}\eta_{\lambda^!}.$$

3.15. Let $\lambda \in X^+$, $b_0 \in \mathbf{B}(\lambda)$, $b_2 \in \mathbf{B}(\lambda)$. Since $1_{-\lambda} \in \dot{U}[\geq \lambda^!]$, $1_{\lambda^!} \in \dot{U}[\geq \lambda^!]$ and $\dot{U}[\geq \lambda^!]$ is a two-sided ideal of $\dot{U}$ we have

$$b_0^+ 1_{-\lambda} b_2^{*-} \in \dot{U}[\geq \lambda^!],$$

$$\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+} \in \dot{U}[\geq \lambda^!].$$

Let $\mu = b_0^+ 1_{-\lambda} b_2^{*-} - C\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+}$ where $C = v^{r'_0-\tilde{r}_0}\delta^{-1}$ with $\tilde{r}_0, r'_0$ as in 3.12, 3.13 and δ is as in 3.13. We show:

(a) $$\mu \in \dot{U}[> \lambda^!].$$

It is enough to show that μ acts as zero on $\Lambda_{\lambda^!}$ or that $\mu s = 0$ for any $s \in B_{\lambda^!}$ that is, for any s of the form $s = b_1^+\xi_{\lambda^!} = \kappa(b_1)^-\eta_{\lambda^!}$ with $b_1 \in \mathbf{B}(\lambda)$. Thus, it is enough to show that

$$b_0^+ 1_{-\lambda} b_2^{*-} b_1^+\xi_{\lambda^!} - C\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+}\kappa(b_1)^-\eta_{\lambda^!} = 0$$

for any $b_1 \in \mathbf{B}(\lambda)$. This clearly follows from 3.14(a).

We have $b_0^+ 1_{-\lambda} b_2^{*-} = \omega(b_0^- 1_\lambda b_2^{*+})$, $b_0^- 1_\lambda b_2^{*+} = \beta_\lambda(b_0, b_2) + \gamma$ and

$$\kappa(b_0)^- 1_{\lambda^!}\kappa(b_2)^{*+} = \beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) + \gamma'$$

where $\gamma \in \dot{U}[> \lambda], \gamma' \in \dot{U}[> \lambda^!]$ so that (a) implies

$$\omega(\beta_\lambda(b_0, b_2) + \gamma) - C(\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) + \gamma') \in \dot{U}_{>\lambda^!}.$$

From [9, 29.3.1] we see that $\omega(\dot{U}[\geq \lambda]) \subset \dot{U}[\geq \lambda^!]$ and $\omega(\dot{U}[> \lambda]) \subset \dot{U}[> \lambda^!]$. Thus $\omega(\gamma) \in \dot{U}[> \lambda^!]$. We see that

$$\omega(\beta_\lambda(b_0, b_2)) - C\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) \in \dot{U}[> \lambda^!].$$

By 3.8(a) we have $\omega(\beta_\lambda(b_0, b_2)) = \epsilon\beta'$ where $\epsilon \in \{1, -1\}$, $\beta' \in \dot{\mathbf{B}}$ is necessarily in $\dot{\mathbf{B}}[\geq \lambda^!]$ and we have

(b) $$\epsilon\beta' - C\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) \in \dot{U}[> \lambda^!].$$

If $\beta' \in \dot{\mathbf{B}}[> \lambda^!]$, then we have

$$C\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) \in \dot{U}[> \lambda^!],$$

contradicting

$$\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)) \in \dot{\mathbf{B}}[\lambda^!], C \neq 0$$

Thus, we have $\beta' \in \dot{\mathbf{B}}[\lambda^!]$ so that (b) implies

$$\beta' = \epsilon C\beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)).$$

It follows that $\epsilon C = 1$ that is, $\delta = \epsilon v^{r'_0 - \tilde{r}_0}$. Since $\delta \in 1 + v^{-1}\mathbf{Q}[[v^{-1}]]$ (see 3.7), we see that

(c) $$\delta = 1, \epsilon = 1, \tilde{r}_0 = r'_0,$$

so that $C = 1$. Thus we have the following result.

Proposition 3.16.

(a) *Let $\lambda \in X^+, b_0 \in \boldsymbol{B}(\lambda), b_2 \in \boldsymbol{B}(\lambda)$. We have*

$$\omega(\beta_\lambda(b_0, b_2)) = \beta_{\lambda^!}(\kappa(b_0), \kappa(b_2)).$$

In particular, $\omega(\dot{\boldsymbol{B}}(\lambda)) = \dot{\boldsymbol{B}}(\lambda^!)$.

(b) *We have $(\xi_\lambda, \xi_\lambda)_\lambda = 1$.*

3.17. Let $\mathbf{A} = \mathbf{Q}[[v^{-1}]] \cap \mathbf{Q}(v)$. Let $\mathbf{f}_\mathbf{A}$ be the $\mathbf{A}$-submodule of $\mathbf{f}$ with basis $\mathbf{B}$.

For $i \in I$ let $\tilde{f}_i : \mathbf{f} \to \mathbf{f}$, $\tilde{e}_i : \mathbf{f} \to \mathbf{f}$ be the linear maps defined in [4]. From [4, 7] we see that there are well defined maps $\phi_i : \mathbf{B} \to \mathbf{B}$, $\epsilon_i : \mathbf{B} \to \mathbf{B} \cup \{0\}$ such that for any $b \in \mathbf{B}$ we have $\tilde{f}_i(b) = \phi_i(b) \bmod v^{-1}\mathbf{f}_{\mathbf{A}}$, $\tilde{e}_i(b) = \epsilon_i(b) \bmod v^{-1}\mathbf{f}_{\mathbf{A}}$. Recall that for b, b' in $\mathbf{B}$ we have $\phi_i(b) = b'$ if and only if $\epsilon_i(b') = b$.

For $\lambda \in \Lambda^+$ let $\Lambda_{\lambda,\mathbf{A}}$ be the $\mathbf{A}$-submodule of Λ_λ with basis B_λ. For $i \in I$ let $\tilde{F}_i : \Lambda_\lambda \to \Lambda_\lambda$, $\tilde{E}_i : \Lambda_\lambda \to \Lambda_\lambda$ be the linear maps denoted by $\tilde{f}_i$, $\tilde{e}_i$ in [4]. From [4, 7] we see that there are well defined maps $\mathcal{F}_i : B_\lambda \to B_\lambda \cup \{0\}$, $\mathcal{E}_i : B_\lambda \to B_\lambda \cup \{0\}$ such that for any $d \in B_\lambda$ we have $\tilde{F}_i(d) = \mathcal{F}_i(d) \bmod v^{-1}\Lambda_{\lambda,A}$ $\tilde{E}_i(d) = \mathcal{E}_i(d) \bmod v^{-1}\Lambda_{\lambda,A}$. Recall that for d, d' in B_λ we have $\mathcal{F}_i(d) = d'$ if and only if $\mathcal{E}_i(d') = d$. From the definitions we have:

(a) If b, b' in $\mathbf{B}(\lambda)$, $i \in I$ satisfy $b^-\eta_l = \mathcal{F}_i(b'^-\eta_l)$ then $b = \phi_i(b')$.

Lemma 3.18. *If b, b' in $\boldsymbol{B}(\lambda^!)$, $i \in I$ satisfy $b^+\xi_l = \mathcal{F}_i(b'^+\xi_l)$, then $b = \epsilon_i(b')$.*

Consider the vector space isomorphism $\tau^{-1} : \Lambda_{\lambda^!} \to \Lambda_\lambda$, see 3.6(a). It induces an A-module isomorphism $\Lambda_{\lambda^!,A} \to L_{\lambda,A}$ (since $\tau^{-1}(B_{\lambda^!}) = B_\lambda$); moreover, we have $\tau^{-1}\tilde{F}_i = \tilde{E}_i\tau^{-1} : \Lambda_{\lambda^!} \to \Lambda_\lambda$.

Let $d \in B_{\lambda^!}$; we have $\tau^{-1}(d) \in B_\lambda$. Applying τ^{-1} to

$$\tilde{F}_i(d) = \mathcal{F}_i(d) \bmod v^{-1}\Lambda_{\lambda^!,A}$$

we obtain

$$\tilde{E}_i(\tau^{-1}(d)) = \tau^{-1}(\mathcal{F}_i(d)) \bmod v^{-1}\Lambda_{\lambda,A}.$$

We have also

$$\tilde{E}_i(\tau^{-1}(d)) = \mathcal{E}_i(\tau^{-1}(d)) \bmod v^{-1}\Lambda_{\lambda,A}.$$

Thus

$$\tau^{-1}(\mathcal{F}_i(d)) = \mathcal{E}_i(\tau^{-1}(d)) \bmod v^{-1}\Lambda_{\lambda,A}.$$

Since $\tau^{-1}(\mathcal{F}_i(d)), \mathcal{E}_i(\tau^{-1}(d))$ are in $B_\lambda \cup \{0\}$, it follows that

(a) $\tau^{-1}(\mathcal{F}_i(d)) = \mathcal{E}_i(\tau^{-1}(d))$.

Now let $d' = b^+\xi_\lambda$, $d = b'^+\xi_\lambda$. By assumption we have $d' = \mathcal{F}_i(d)$. We have $\tau^{-1}(\mathcal{F}_i(d)) = \tau^{-1}(d') = b^-\eta_\lambda$, $\tau^{-1}(d) = b'^-\eta_\lambda$. Using now (a), we deduce $b^-\eta_\lambda = \mathcal{E}_i(b'^-\eta_\lambda)$, so that $b = \epsilon_i(b')$. The lemma is proved.

4. The Involution ϕ_K

4.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let G be a connected reductive group over $\mathbf{k}$ with a fixed pinning corresponding to the root datum dual to that in 3.1. Thus, G has a given maximal torus T, given Borel subgroups B^+, B^- with intersection T (with unipotent radicals U^+, U^-) and given imbeddings of algebraic groups $x_i : \mathbf{k} \to U^+$, $y_i : \mathbf{k} \to U^-$ $(i \in I)$ with the usual properties (see for example [10, 1.1]). Note that X (resp. Y) is now the group of homomorphisms of algebraic groups $\mathrm{Hom}(\mathbf{k}^*, T)$ (resp. $\mathrm{Hom}(T, \mathbf{k}^*)$).

Let $\Omega : G \to G$ be the (involutive) automorphism of G such that $\Omega(x_i(a)) = y_i(a)$, $\Omega(y_i(a)) = x_i(a)$ for $i \in I, a \in \mathbf{k}$, $\Omega(t) = t^{-1}$ for $t \in T$.

Let $g \mapsto \Theta(g)$ be the (involutive) antiautomorphism of G such that $\Theta(x_i(a)) = x_i(a)$, $\Theta(y_i(a)) = y_i(a)$ for $i \in I, a \in \mathbf{k}$, $\Theta(t) = t^{-1}$ for $t \in T$. We have $\Theta\Omega = \Omega\Theta$.

For $i \in I$ we set $\dot{s}_i = y_i(-1)x_i(1)y_i(-1)$ (an element in the normalizer of T).

Let $\mathcal{B}$ be the variety of Borel subgroups of G. Now Ω (resp. Θ) induces an involution $\mathcal{B} \to \mathcal{B}$ denoted again by Ω (resp. Θ); now Ω interchanges B^+, B^-, while Θ preserves B^+, B^-.

4.2. We now fix a semifield K contained in $\mathbf{k}^*$ with $+, \times$ induced from $\mathbf{k}$. Let $U^+_{>0}$ (resp. $U^-_{>0}$) be the totally positive part of U^+ (resp. U^-) defined in terms of K in [10, 2.12]. In [10, 2.7] a family of bijections $g^+_{\mathbf{i}} : K^\nu \to U^+_{>0}$ indexed by the various $\mathbf{i} \in \mathcal{I}$ is considered; it is also shown, using Bruhat decomposition, that each of these maps in injective. Using [10, 2.5], we see that these maps define a (surjective) map $g^+ : \mathcal{U}_K \to U^+_{>0}$. Note that for any $\mathbf{i}$ we have $g^+_{\mathbf{i}} = g^+ f_{\mathbf{i}}$ where $f_{\mathbf{i}} : K^\nu \to \mathcal{U}_K$ is as in 1.3. Since $g^+_{\mathbf{i}}$ is injective, it follows that

(a) $f_{\mathbf{i}}$ is injective.

As shown in 1.3, $f_{\mathbf{i}}$ is surjective hence bijective so that $g^+ = g^+_{\mathbf{i}} f^{-1}_{\mathbf{i}}$. Since $g^+_{\mathbf{i}}$ is injective it follows that g^+ is injective. But it is also surjective so that it is bijective. Thus,

(b) $g^+ : \mathcal{U}_K \to U^+_{>0}$ is a bijection.

Similarly, in [10, 2.9] a family of bijections $g^-_{\mathbf{i}} : K^\nu \to U^-_{>0}$ indexed by the various $\mathbf{i} \in \mathcal{I}$ is considered. Now these maps define a map $g^- : \mathcal{U}_K \to U^-_{>0}$. As above we see that

(c) $g^- : \mathcal{U}_K \to U^-_{>0}$ is a bijection.

For $A \in \mathcal{U}_K$ we write $A^+ = g^+(A) \in U^+_{>0}$, $A^- = g^-(A) \in U^-_{>0}$; we have $A^+ = \Omega(A^-)$.

Let $T_{>0}$ be the subgroup of T generated by $\{\lambda(a); \lambda \in X = \mathrm{Hom}(\mathbf{k}^*, T), a \in K\}$.

Lemma 4.3. *Let $\boldsymbol{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$, $\boldsymbol{a} = (a_1, a_2, \dots, a_\nu) \in K^\nu$. We set $i = i_\nu$. Let $k \in [1, \nu]$. For any $b_* = (b_{k+1}, b_{k+2}, \dots, b_\nu) \in K^{\nu-k}$ there exists a unique $b'_* = (b'_k, b'_{k+1}, \dots, b'_\nu) \in K^{\nu-k+1}$ such that*

(a)
$$\begin{aligned}
&x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})x_{i_k}(a_k)y_{i_\nu}(b_\nu)y_{i_{\nu-1}}(b_{\nu-1})\dots\\
&y_{i_{k+1}}(b_{k+1})\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-\\
&= x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})y_{i_\nu}(b'_\nu)\\
&y_{i_{\nu-1}}(b'_{\nu-1})\dots y_{i_k}(b'_k)\dot{s}_{i_k}\dot{s}_{i_{k+1}}\dots\dot{s}_{i_\nu}B^-.
\end{aligned}$$

Moreover, we have

(b) $b'_\nu = b_\nu(1 + a_k b_\nu)^{-1}$ *if* $i_k = i$, $b'_\nu = b_\nu$ *if* $i_k \ne i$.

The proof is essentially a repetition of arguments in the proof in [12, 3.2] (except for (b)); with this occasion we correct some typos in that proof. We only have to prove existence; the uniqueness is immediate. Now $x_{i_k}(a_k)y_{i_\nu}(b_\nu)$ is equal to $y_{i_\nu}(b'_\nu)x_{i_k}(c)t$ where b'_ν is as in (b), $c \in K$ and $t \in T_{>0}$. Hence the left hand side of (a) is of the form

$$\begin{aligned}
&x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})y_{i_\nu}(b'_\nu)x_{i_k}(c)y_{i_{\nu-1}}(b''_{\nu-1})\dots\\
&y_{i_{k+1}}(b''_{k+1})\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-
\end{aligned}$$

for some $(b''_{k+1}, \dots, b''_\nu) \in K^{\nu-k}$. Using the usual commutation relations between $x_{i_k}(?)$ and $y_j(?)$ we see that the left hand side of (a) is of the form

$$\begin{aligned}
&x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})y_{i_\nu}(b'_\nu)y_{i_{\nu-1}}(b''_{\nu-1})\dots\\
&y_{i_{k+1}}(b''_{k+1})x_{i_k}(c')\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-
\end{aligned}$$

where $c' \in K$. We now use that $x_{i_k}(c') = y_{i_k}(1/c')\dot{s}_{i_k} y_{i_k}(1/c')t'$ where $t' \in T_{>0}$. we see that the left hand side of (a) is of the form

$$x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})y_{i_\nu}(b'_\nu)y_{i_{\nu-1}}(b''_{\nu-1})\dots$$
$$y_{i_{k+1}}(b''_{k+1})y_{i_k}(1/c')\dot{s}_{i_k} y_{i_k}(1/c')\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-$$

It remains to observe that

$$y_{i_k}(1/c')\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^- = \dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-$$

since $s_{i_k}s_{i_{k+1}}s_{i_{k+2}}\dots s_{i_\nu}$ is a reduced expression in W. This proves the lemma.

Lemma 4.4. *Let* $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$, $\boldsymbol{a} = (a_1, a_2, \dots, a_\nu) \in K^\nu$. *We set* $i = i_\nu$. *Let* $k \in [1, \nu]$ *and let* $(b_{k+1}, b_{k+2}, \dots, b_\nu) \in K^{\nu-k}$. *Assume that* $b_\nu = (\sum_{l\in[k+1,\nu];i_l=i} a_l)^{-1}$. *Then there exists a unique*

$$(b'_k, b'_{k+1}, \dots, b'_\nu) \in K^{\nu-k+1}$$

such that

$$x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_k}(a_k)y_{i_\nu}(b_\nu)y_{i_{\nu-1}}(b_{\nu-1})$$
$$\dots y_{i_{k+1}}(b_{k+1})\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-$$
$$= x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_{k-1}}(a_{k-1})y_{i_\nu}(b'_\nu)y_{i_{\nu-1}}(b'_{\nu-1})$$
$$\dots y_{i_k}(b_k)\dot{s}_{i_k}\dot{s}_{i_{k+1}}\dot{s}_{i_{k+2}}\dots\dot{s}_{i_\nu}B^-.$$

Moreover, we have $b'_\nu = (\sum_{l\in[k,\nu];i_l=i} a_l)^{-1}$.

Except for the last sentence, this is a special case of Lemma 4.3. It remains to prove the formula for b'_ν. If $i_k \neq i$, then from 4.3 we have

$$b'_\nu = \left(\sum_{l\in[k+1,\nu];i_l=i} a_l\right)^{-1} = \left(\sum_{l\in[k,\nu];i_l=i} a_l\right)^{-1}.$$

If $i_k = i$, then from 4.3 we have

$$b'_\nu = \frac{(\sum_{l\in[k+1,\nu];i_l=i} a_l)^{-1}}{1 + a_k(\sum_{l\in[k,\nu];i_l=i} a_l)^{-1}} = \left(\sum_{l\in[k,\nu];i_l=i} a_l\right)^{-1}.$$

The lemma is proved.

Proposition 4.5. *Let* $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$, $\boldsymbol{a} = (a_1, a_2, \dots, a_\nu) \in K^\nu$. *We set* $i = i_\nu$. *There exists a unique* $(b_1, b_2, \dots, b_\nu) \in K^\nu$ *such that*

$$x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_\nu}(a_\nu)B^- = y_{i_\nu}(b_\nu)y_{i_{\nu-1}}(b_{\nu-1})\dots y_{i_1}(b_1)\dot{s}_{i_1}\dot{s}_{i_2}\dots\dot{s}_{i_\nu}B^-.$$

Moreover, we have $b_\nu = (\sum_{l\in[1,\nu];i_l=i} a_l)^{-1}$.

Again this is contained in [12, 3.2] except for the last sentence. It follows by applying 4.4 repeatedly with $k = \nu, \nu-1, \dots, 1$.

4.6. From 4.5 we see that the image of the (injective) map

(a) $\mathcal{U}_K \xrightarrow{c^+} \mathcal{B}$, $A \mapsto A^+B^-(A^+)^{-1}$
is contained in the image of the (injective) map

(b) $\mathcal{U}_K \xrightarrow{c^-} \mathcal{B}$, $A \mapsto A^-B^+(A^-)^{-1}$.

Applying Ω we see that the image of c^- is contained in the image of c^+ hence

(c) the image of c^+ is equal to the image of c^-.

We denote this image by $\mathcal{B}_{K,>0}$ (a subset of $\mathcal{B}$). Applying Θ to (c) we see that

(d) the image of $\mathcal{U}_K \to \mathcal{B}$, $A \mapsto (A^+)^{-1}B^-A^+$ is equal to the image of $\mathcal{U}_K \to \mathcal{B}$, $A \mapsto (A^-)^{-1}B^+A^-$ and that these two maps are injective.

We denote this image by $\mathcal{B}_{K,<0}$ (a subset of $\mathcal{B}$ equal to $\Theta(\mathcal{B}_{K,>0})$). We see that $\Omega : \mathcal{B} \to \mathcal{B}$ restricts to an involution $\Omega'_K : \mathcal{B}_{K,>0} \to \mathcal{B}_{K,>0}$ and to an involution $\Omega_K : \mathcal{B}_{K,<0} \to \mathcal{B}_{K,<0}$.

We also see that there is a unique bijection $\phi'_K : \mathcal{U}_K \to \mathcal{U}_K$ such that

$$A^+B^-(A^+)^{-1} = \phi'_K(A)^-B^+(\phi'_K(A)^-)^{-1}$$

for any $A \in \mathcal{U}_K$, that is, $\Omega'_Kc^- = c^-\phi'_K$. Since ${\Omega'_K}^2 = 1$, it follows that ${\phi'_K}^2 = 1$. Moreover we see that there is a unique bijection $\phi_K : \mathcal{U}_K \to \mathcal{U}_K$ such that

$$(A^+)^{-1}B^-A^+ = (\phi_K(A)^-)^{-1}B^+\phi_K(A)^-$$

for any $A \in \mathcal{U}_K$; we have $\phi_K = *\phi'_K*$ ($*$ as in 1.1) hence $\phi_K^2 = 1$.

4.7. From the proof of 4.5 we see that ϕ_K (and also ϕ'_K), when expressed as a map $K^\nu \to K^\nu$, is given by rational functions which are quotients of nonzero polynomials in several variables with all coefficients in $\mathbf{N}$ (after substituting elements of K for the variables); moreover these polynomials are independent of K. Replacing the variables in these polynomials by elements of an arbitrary semifield K' we see that $\phi_{K'} : \mathcal{U}_{K'} \to \mathcal{U}_{K'}$ is defined for any semifield K' (not necessarily contained in a field). Then various properties of this map can be deduced from the analogous properties in the case where K is as in 4.2.

4.8. We can identify $\mathcal{B}_K$ (see 1.8) with $\mathcal{B}_{K,<0}$ by $(A, A') \mapsto (A^+)^{-1}B^-A^+$.

5. Connecting Bases with Objects over the Semifield Z

5.1. Let $\mathbf{u} : \mathcal{U}_\mathbf{N} \xrightarrow{\sim} \mathbf{B}$ be the bijection defined in [6]. It satisfies $\mathbf{u}(\mathbf{h}_0) = 1$. More generally, if $A \in \mathcal{U}_\mathbf{N}$ and $b = \mathbf{u}(A)$ then

(a)
$$wt(b) = \sum_{h \in I} ||A||_h h'.$$

See [6, 2.9]. From [6, 2.11] we see that under $\mathbf{u}$, the restriction to $\mathcal{U}_\mathbf{N}$ of the involution $*\iota = \iota*$ (see 1.1, 1.6, with $K = \mathbf{Z}$) corresponds to the involution $*$ of $\mathbf{B}$ in 3.2.

5.2. In [7] it is shown that for $i \in I$ we have

$$\phi_i \mathbf{u}(A) = \mathbf{u}(T_{i,1}A) \text{ for all } A \in \mathcal{U}_\mathbf{N}.$$

From (a) we can deduce:

(b) if $A \in \mathcal{U}_\mathbf{N}$ is such that $\epsilon_i \mathbf{u}(A) \in \mathbf{B}$, then $T_{i,-1}A \in \mathcal{U}_\mathbf{N}$ and $\epsilon_i \mathbf{u}(A) = \mathbf{u}(T_{i,-1}A)$.

Indeed, we have $\epsilon_i \mathbf{u}(A) = \mathbf{u}(A_1)$ for some $A_1 \in \mathcal{U}_\mathbf{N}$. Then $\mathbf{u}(A) = \phi_i \mathbf{u}(A_1)$ so that by (a) we have $A = T_{i,1}A_1$ hence $A_1 = T_{i,-1}A$.

5.5. Until the end of 5.6 we fix $\lambda \in X^+$. We define $p^\lambda = (p_i^\lambda) \in \mathbf{N}^I$ by $p_i^\lambda = (i, \lambda)$ for $i \in I$. We show:

(a)
$$\sum_{h \in I} ||\mathbf{h}_{p^\lambda}||_h h' = \lambda + \lambda^!.$$

Using 2.3(a) we see that this is equivalent to the equality

$$\sum_{h \in I} \sum_{i \in I} ((i, \lambda) + (i, \lambda^!)) b_{ih} h' = \lambda + \lambda^!.$$

that is

(b)
$$\sum_{h \in I} \sum_{i \in I} (i, \zeta) b_{ih} h' = \zeta$$

where $\zeta = \lambda + \lambda^! = \lambda - w_0(\lambda) \in \sum_{h \in I} h' \subset X$. It is enough to show that the two sides of (b) have the same $(,)$ with any $j \in I$ (viewed as an element of Y) that is,

$$\sum_{h \in I} \sum_{i \in I} (i, \zeta) b_{ih} a_{hj} = (j, \zeta)$$

for $j \in I$. This follows immediately from the definition of b_{ih}.

5.4. According to [6],

(a) $\mathbf{u}$ restricts to a bijection $\mathbf{u}^\lambda : \mathcal{U}_{\mathbf{N}, p^\lambda} \xrightarrow{\sim} \mathbf{B}(\lambda)$.

We have $\mathbf{u}^\lambda(\mathbf{h}_0)^- \eta_\lambda = \eta_\lambda$.

We show:

(b)
$$\mathbf{u}^\lambda(\mathbf{h}_{p^\lambda})^- \eta_\lambda = \xi_\lambda.$$

Let $d = \mathbf{u}^\lambda(\mathbf{h}_{p^\lambda})^- \eta_\lambda$. Since $d \in B_\lambda$ and ξ_λ is the unique element $b \in B_\lambda$ such that $1_{w_0(\lambda)} b = b$, it is enough to show that $1_{w_0(\lambda)} d = d$. Since $1_\lambda \eta_\lambda = \eta_\lambda$, we see from the definition of d that $1_{\lambda'} d = d$ where $\lambda' = \lambda - wt(\mathbf{u}^\lambda(\mathbf{h}_{p^\lambda})$. Using 5.1(a), 5.3(a) this can be rewritten as $\lambda' = \lambda - (\lambda + \lambda^!) = -\lambda^! = w_0(\lambda)$. This proves (b).

Theorem 5.5.

(i) *For any* $A \in \mathcal{U}_{\boldsymbol{N},p^\lambda}$ *we have*

$$\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A) \in \mathcal{U}_{\mathbf{N},p^{\lambda^!}}.$$

(ii) *Define a bijection* $\tilde{\kappa} : \mathcal{U}_{\boldsymbol{N},p^\lambda} \to \mathcal{U}_{\boldsymbol{N},p^{\lambda^!}}$ *by* $\boldsymbol{u}_{\lambda^!}(\tilde{\kappa}(A)) = \kappa(\boldsymbol{u}_\lambda(A))$ *for any* $A \in \mathcal{U}_{\boldsymbol{N},p^\lambda}$. *For any* $A \in \mathcal{U}_{\boldsymbol{N},p^\lambda}$ *we have*

$$\tilde{\kappa}(A) = \iota S_{p^\lambda}\phi_{\mathbf{Z}}(A).$$

Something close to this is proved in [12, 4.9], but that proof contains some misprints. For this reason we reprove it without referring to [12, 4.9].

Let $A \in \mathcal{U}_{\mathbf{N},p^\lambda}$. Let $d = \mathbf{u}^\lambda(A)^-\eta_\lambda \in B_\lambda$. From [4] it is known that there exists a sequence $i_1, i_2, \dots, i_k$ in I such that

$$d = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\dots\mathcal{F}_{i_k}\eta_\lambda.$$

The smallest such k is a number $f(A) \in \mathbf{N}$. We argue by induction on $f(A)$. If $f(A) = 0$ we have $\mathbf{u}^\lambda(A)^-\eta_\lambda = \eta_\lambda$ hence $\mathbf{u}^\lambda(A) = 1$ and $A = \mathbf{h}_0$. Using 2.2(c),(b),(a), we have

$$\iota S_{p^\lambda}\phi_{\mathbf{Z}}(\mathbf{h}_0) = \iota S_{p^\lambda}(\mathbf{h}_0) = \iota\mathbf{h}_{p^\lambda} = \mathbf{h}_{p^{\lambda^!}}$$

which belongs to $\mathcal{U}_{\mathbf{N},p^{\lambda^!}}$; thus (i) holds in this case. By the proof of (i), proving (ii) for $A = \mathbf{h}_0$ it is the same as proving that

$$\kappa(\mathbf{u}^\lambda(\mathbf{h}_0)) = \mathbf{u}^{\lambda^!}(\mathbf{h}_{\pi^{\lambda^!}})$$

or, using 5.4(b) for $\lambda^!$ instead of λ, that $\kappa(1)^-\eta_{\lambda^!} = \xi_{\lambda^!}$. From the definition of κ this is the same as $1^+\xi_{\lambda^!} = \xi_{\lambda^!}$ which is obvious. Thus (ii) is proved in our case.

We can now assume that A is such that $f(A) \geq 1$ and that the result is known when A is replaced by any $A' \in \mathcal{U}_{\mathbf{N},p^\lambda}$ such that $f(A') < f(A)$. By the definition of $f(A)$ we can find $A' \in \mathcal{U}_{\mathbf{N},p^\lambda}$ such that $f(A') = f(A) - 1$ and

$$\mathbf{u}^\lambda(A)^-\eta_\lambda = \mathcal{F}_i(\mathbf{u}^\lambda(A')^-\eta_\lambda)$$

for some $i \in I$. From 3.17(a) we then have $\mathbf{u}(A) = \phi_i(\mathbf{u}(A'))$ hence, by 5.2, we have $\mathbf{u}(A) = \mathbf{u}(T_{i,1}A')$ so that $A = T_{i,1}A'$.

Let $b = \kappa(\mathbf{u}_\lambda(A)) = \kappa(\phi_i(\mathbf{u}(A')), b' = \kappa(\mathbf{u}(A'))$. We have

$$b^+\xi_\lambda = \mathbf{u}_\lambda(A)^-\eta_\lambda = (\phi_i\mathbf{u}(A'))^-\eta_\lambda = \mathcal{F}_i(\mathbf{u}(A')^-\eta_\lambda) = \mathcal{F}_i(b'^+\xi_\lambda).$$

Using 3.18 we see that $b = \epsilon_i(b')$, that is

(a) $\kappa(\mathbf{u}_\lambda(A)) = \epsilon_i\kappa(\mathbf{u}_\lambda(A'))$.

By the induction hypothesis we have

(b) $\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A') \in \mathcal{U}_{\mathbf{N},p^\lambda!}$ and $\tilde{\kappa}(A') = \iota S_{p^\lambda}\phi_{\mathbf{Z}}(A')$.

Using (a) we have

$$\mathbf{u}(\tilde{k}(A)) = \kappa(\mathbf{u}_\lambda(A)) = \epsilon_i\kappa(\mathbf{u}_\lambda(A')) = \epsilon_i\mathbf{u}(\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A')).$$

(The third equality follows from (b).) Using 5.2(b) we see that

$$T_{i,-1}\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A') \in \mathcal{U}_{\mathbf{N}}$$

and

$$\epsilon_i\mathbf{u}(\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A')) = \mathbf{u}(T_{i,-1}\iota S_{p^\lambda}\phi_{\mathbf{Z}}(A')).$$

Here the right hand side is equal to

$$\mathbf{u}(\iota S_{p^\lambda}\phi_{\mathbf{Z}}T_{i,1}(A')) = \mathbf{u}(\iota S_{p^\lambda}\phi_{\mathbf{Z}}A).$$

(We have used 1.6(a), (b), 1.7(c).)

Thus we have $\mathbf{u}(\tilde{k}(A)) = \mathbf{u}(\iota S_{p^\lambda}\phi_{\mathbf{Z}}A)$, so that $\tilde{k}(A) = \iota S_{p^\lambda}\phi_{\mathbf{Z}}A$. We see that (i) and (ii) hold for A. The theorem is proved.

Corollary 5.6.

(i) *For any* $A \in \mathcal{U}_{\boldsymbol{N},p^\lambda}$ *we have* $S_{p^\lambda}\phi_{\mathbf{Z}}(A) \in \mathcal{U}_{\mathbf{N},p^\lambda}$.

(ii) *The map* $\mathcal{U}_{\boldsymbol{N},p^\lambda} \to \mathcal{U}_{\boldsymbol{N},p^\lambda}$ *given by* $A \mapsto S_{p^\lambda}\phi_{\boldsymbol{Z}}(A)$ *is a bijection.*

(i) follows from 5.5(i); (ii) follows from 5.5(ii).

Corollary 5.7. *Let* $p \in \boldsymbol{N}^I$, $A \in \mathcal{U}_{\boldsymbol{N},p}$. *Then* $S_p\phi_{\boldsymbol{Z}}(A) \in \mathcal{U}_{\boldsymbol{N},p}$. *Moreover, the map* $\mathcal{U}_{\boldsymbol{N},p} \to \mathcal{U}_{\boldsymbol{N},p}$ *given by* $A \mapsto S_p\phi_{\boldsymbol{Z}}(A)$ *is a bijection.*

We replace $Y, X, (,)$ by $Y', X' = \mathrm{Hom}(Y', \mathbf{Z}), (,)'$ where $Y' = \sum_i \mathbf{Z}i \subset Y$ and $(,)'$ is the obvious pairing. Define $i' \in X'$ by $(j, i')' = a_{ij}$. We apply 5.6 to this new root datum and to λ replaced by $\lambda' \in X'$ given by $(i, \lambda') = p_i$. Then 5.7 follows.

Corollary 5.8. *For any $p \in \boldsymbol{N}^I$ we have*

$$\mathcal{U}_{\boldsymbol{N},p} = \{A \in \mathcal{U}_{\boldsymbol{N}}; S_p \phi_{\boldsymbol{Z}}(A) \in \mathcal{U}_{\boldsymbol{N}}\}.$$

Hence $(A, A') \mapsto A$ is a bijection $\mathcal{B}_{\boldsymbol{N},p} \xrightarrow{\sim} \mathcal{U}_{\boldsymbol{N},p}$.

The first sentence follows immediately from 5.7, as shown in [16, no.12]. The second sentence follows from the first sentence using 2.4(a).

Corollary 5.9. *Let $\lambda \in X^+$. The map $\mathcal{B}_{\boldsymbol{N},p^\lambda} \to \boldsymbol{B}(\lambda)$ given by $(A, A') \mapsto \boldsymbol{u}(A)$ is a bijection.*

This follows from 5.4(a) and 5.8.

5.10. Let

$$\begin{aligned}\Xi &= \sqcup_{\lambda \in X^+} \mathcal{B}_{\mathbf{N},p^\lambda} \times \mathcal{B}_{\mathbf{N},p^\lambda}\\ &= \sqcup_{\lambda \in X^+} \{(B_1, B_2) \in \mathcal{B}_{\mathbf{Z}} \times \mathcal{B}_{\mathbf{Z}}; B_1 \in \mathcal{B}^+_{\mathbf{N}}, B_2 \in \mathcal{B}^+_{\mathbf{N}}, \underline{S}_{p^{-\lambda}} B_1 \in \mathcal{B}^-_{\mathbf{N}},\\ &\quad \underline{S}_{p^{-\lambda}} B_2 \in \mathcal{B}^-_{\mathbf{N}}\}.\end{aligned}$$

We define $\dot{\mathbf{u}} : \Xi \to \dot{\mathbf{B}}$ by $\dot{\mathbf{u}}(B_1, B_2, \lambda) = b_\lambda(\mathbf{u}(A_1), \mathbf{u}(A_2))$ for any $\lambda \in X^+$ and any $B_1 = (A_1, A'_1) \in \mathcal{B}_{\mathbf{N},p^\lambda}$, $B_2 = (A_2, A'_2) \in \mathcal{B}_{\mathbf{N},p^\lambda}$.

We define $\tilde{\omega} : \Xi \to \Xi$ by $\tilde{\omega}(B_1, B_2, \lambda) = (\underline{\iota\phi_{\mathbf{Z}}}(B_1), \underline{\iota\phi_{\mathbf{Z}}}(B_2), \lambda^!)$.

We define $\tilde{\sharp} : \Xi \to \Xi$ by $\tilde{\sharp}(B_1, B_2, \lambda) = (B_2, B_1, \lambda)$.

Theorem 5.11.

(a) *$\dot{\boldsymbol{u}}$ is a bijection.*

(b) *We have $\omega \dot{\boldsymbol{u}} = \dot{\boldsymbol{u}} \tilde{\omega}$.*

(c) *We have $\sharp \dot{\boldsymbol{u}} = \dot{\boldsymbol{u}} \tilde{\sharp}$.*

(a) follows from 3.10(a) and 5.9; (b) follows from 3.16, 5.5 and 5.8; (c) follows from [11, 4.14(a)].

5.12. The group X acts on $\mathcal{B}_{\mathbf{Z}} \times \mathcal{B}_{\mathbf{Z}}$ by $\lambda : (B, \tilde{B}) \mapsto (\underline{S}_{p^\lambda} B, \underline{S}_{p^\lambda} \tilde{B})$.

We now assume that the root datum in 3.1 is of simply connected type, that is, the map $X \to \mathbf{Z}^I$, $\lambda \mapsto p^\lambda$ is a bijection. Then the X-action above is free. Let

$$\begin{aligned}\Xi' = \{&(B_1, B_2, \tilde{B}_1, \tilde{B}_2) \in \mathcal{B}^+_{\mathbf{N}} \times \mathcal{B}^+_{\mathbf{N}} \times \mathcal{B}^-_{\mathbf{N}} \times \mathcal{B}^-_{\mathbf{N}};\\ &(B_1, B_2) \text{ is in the } X\text{-orbit of } (\tilde{B}_1, \tilde{B}_2)\}.\end{aligned}$$

We define $\Xi \to \Xi'$ by

(a) $$(B_1, B_2, \lambda) \mapsto (B_1, B_2, \underline{S}_{p^{-\lambda}} B_1, \underline{S}_{p^{-\lambda}} B_2).$$

We show:

(b) The map (a) is a bijection.

The injectivity follows from the freeness of the X-action. We show that (a) is surjective. It is enough to show that:

(c) if $B_1 \in \mathcal{B}^+_{\mathbf{N}}$, $\tilde{B}_1 \in \mathcal{B}^-_{\mathbf{N}}$, $\lambda \in X$ are such that $\tilde{B}_1 = \underline{S}_{p^{-\lambda}} B_1$, then $\lambda \in X^+$.

This follows from 2.5(a).

Using (b), we see that in our case, Ξ' can be viewed as an indexing set for $\dot{\mathbf{B}}$.

Appendix

A.1. Let $p = (p_i) \in \mathbf{N}^I$. Then the set $\mathcal{U}_{\mathbf{N},p}$ is finite since it is in bijection with the finite set B_λ for some $\lambda \in X^+$ (assuming that the root datum is of simply connected type). But one would like to have a proof of the finiteness of $\mathcal{U}_{\mathbf{N},p}$ independent of the theory of canonical bases. Such a proof will be given in this appendix.

Let $n \in \mathbf{N}$. Let $\mathcal{U}^n_{\mathbf{N}}$ be the set of all $A \in \mathcal{U}_{\mathbf{N}}$ such that for any $(\mathbf{i}, \mathbf{a}) \in A$ we have $a_1 \le n$.

Lemma A.2. *Let $A \in \mathcal{U}^n_{\mathbf{N}}$. For any $(\boldsymbol{i}, \boldsymbol{a}) \in A$ we have $a_k \le 2^{k-1}n$ for $k \in [1, \nu]$.*

We argue by induction on k. For $k = 1$ the result is clear. Now assume that $k \ge 2$. Let $(\mathbf{i}, \mathbf{a}) \in A$. We set $a_{k-1} = a, a_k = b$, $i_{k-1} = i, i_k = j$. If $a_{ij} = 0$ then

$$((i_1, \dots, i_{k-2}, j, i, \dots), (a_1, \dots, a_{k-2}, b, a, \dots)) \in A$$

and by the induction hypothesis we have $b \le 2^{k-2}n$ hence $b \le 2^{k-1}n$ as desired. Thus we can assume that $a_{ij} = -1$.

Case 1. Assume that $s_i s_{i_{k+1}} \dots s_{i_\nu}$ is not a reduced expression; then

$$s_{i_{k+1}} \dots s_{i_\nu} = s_i s_{j_{k+2}} \dots s_{j_\nu}$$

(reduced expression) for some $j_{k+2}, \dots, j_\nu$ in I so that for some $b_{k+1}, \dots, b_\nu$ we have

$$((i_1, \dots, i_{k-2}, i, j, i, j_{k+2}, \dots, j_\nu), (a_1, \dots, a_{k-2}, a, b, b_{k+1}, \dots, b_\nu)) \in A.$$

Here we can replace i, j, i by j, i, j and a, b, b_{k+1} by

$$b + b_{k+1} - \min(a, b_{k+1}), \min(a, b_{k+1}), a + b - \min(a, b_{k+1}).$$

From the induction hypothesis we have $b + b_{k+1} - \min(a, b_{k+1}) \le 2^{k-2}n$. If $a \le b_{k+1}$, then $b + b_{k+1} - a \le 2^{k-2}n$ hence

$$b \le b + b_{k+1} \le a + 2^{k-2}n \le 2^{k-2}n + 2^{k-2}n = 2^{k-1}n.$$

If $b_{k+1} \leq a$ then $b \leq 2^{k-2}n$ hence $b \leq 2^{k-1}n$ as desired.

Case 2. We can now assume that $s_i s_{i_{k+1}} \dots s_{i_\nu}$ is a reduced expression. Then setting $y = s_{i_{k+1}} \dots s_{i_\nu}$ we have that $|s_i y| > |y|$, $|s_j y| > |y|$, hence $|s_j s_i s_j y| = |y|+3$, hence can find $u \in W$ such that $us_j s_i s_j y = w_0$, $|u|+|s_j s_i s_j|+|y| = \nu$. Hence we can find $(\mathbf{i}', \mathbf{a}') \in A$ with

$$(i'_{k-2}, i'_{k-1}, i'_k) = (j, i, j),$$
$$(a'_{k-2}, a'_{k-1}, a'_k) = (a'_{k-2}, a, b).$$

Here we can replace j, i, j by i, j, i and a'_{k-2}, a, b by

$$a + b - \min(a'_{k-2}, b), \min(a'_{k-2}, b), a'_{k-2} + a - \min(a'_{k-2}, b).$$

By the induction hypothesis we have $a'_{k-2} \leq 2^{k-3}n$, $a+b-\min(a'_{k-2}, b) \leq 2^{k-3}n$. If $b \leq a'_{k-2}$ then $b \leq 2^{k-3}n$ hence $b \leq 2^{k-1}n$. If $a'_{k-2} \leq b$ then $a + b - a'_{k-2} \leq 2^{k-3}n$ hence

$$b \leq a + b \leq a'_{k-2} + 2^{k-3}n \leq 2^{k-3}n + 2^{k-3}n = 2^{k-2}n$$

hence $b \leq 2^{k-1}n$. This completes the induction step. The lemma is proved.

Proposition A.3. *$\mathcal{U}^n_{\mathbf{N}}$ is a finite set.*

Let $\mathbf{i} \in \mathcal{I}$. Using 1.3(a) it is enough to show the finiteness of the set of all $\mathbf{a} \in \mathbf{N}^\nu$ such that the equivalence class of $(\mathbf{i}, \mathbf{a})$ is in $\mathcal{U}^n_{\mathbf{N}}$. By the lemma, the number of elements in this set is $\leq (N+1)(2N+1)(4N+1)\cdots(2^{\nu-1}N+1)$. The proposition is proved.

A.4. We now choose n such that $n \geq p_i$ for all $i \in I$. Clearly, $\mathcal{U}_{\mathbf{N},p}$ is contained in the image of $\mathcal{U}^n_{\mathbf{N}}$ under $A \mapsto A^*$. Using A.3, we deduce that $\mathcal{U}_{\mathbf{N},p}$ is finite.

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