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A SIMPLE SOLUTION FORMULA FOR THE STOKES EQUATIONS IN THE HALF SPACE

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Abstract

This note studies the Stokes equations in the half space $\mathbb{R}^d_+$ with the non-slip boundary condition. We present an explicit solution formula by using the hybrid Fourier-Fourier sine transform, which is simpler than already known ones.

1. Introduction

In this note we are concerned with the initial-boundary value problem of the Stokes equations in the half space $\mathbb{R}^d_+ := \mathbb{R}_+ \times \mathbb{R}^{d-1} = \{(z_1, \dots, z_d) \in \mathbb{R}^d : z_1 > 0\}$ for $d \geq 3$ with the non-slip boundary condition:

$$u_t - \Delta u + \nabla p = 0 \qquad \text{in } \mathbb{R}^d_+ \times (0, \infty), \tag{1.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^d_+ \times (0, \infty), \tag{1.2}$$

$$u|_{x_1=0} = 0, \tag{1.3}$$

$$u|_{t=0} = a, \tag{1.4}$$

where $u = (u_i(x,t))_{1\leq i\leq d}$ is an unknown velocity field with an associated pressure $p = p(x,t)$ and $a = (a_i(x))_{1\leq i\leq d}$ is a prescribed velocity field satisfying $\nabla \cdot a = 0$ in $\mathbb{R}^d_+$.

Our main purpose of this note is to construct a solution operator $\{S[\,\cdot\,](t)\}_{t\geq 0}$ such that $u(t) = S[a](t)$ is a smooth solution to the Stokes IBVP

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(1.1)-(1.4) for $t > 0$. Such explicit solution formulae play a fundamental role in establishing various estimates of solutions and the gradient (cf. [1, 4, 5]).

Our solution formula presented in Theorem 3.1 is obtained by using the hybrid Fourier-Fourier sine transform, which is simpler than already known ones. Indeed, the first component u_1 is determined only by the initial data a_1 similarly as in the whole space case $\mathbb{R}^d$. As is well-known, each component u_i of a solution u to the corresponding Cauchy problem on $\mathbb{R}^d$ is determined only by the initial data a_i:

$$u_i = \sum_{j=1}^{d} e^{-t\Delta}(\delta_{ij} + R_i R_j) a_j = e^{-t\Delta} a_i$$

for $a = (a_i(x))_{1\le i\le d}$ satisfying $\nabla \cdot a = 0$, where $e^{-t\Delta}$ is the heat semigroup in $\mathbb{R}^d$ and R_i is the Riesz operator in $\mathbb{R}^d$ with the symbol $\sqrt{-1}\xi_i/|\xi|$. In addition, our formula is not necessary for the compatibility boundary condition $a_1|_{x_1} = 0$.

2. Preliminaries

We use the standard notation for differentiation: $\partial_t = \partial/\partial t$ and $\partial_i := \partial/\partial x_i$ for $i = 1, \ldots, d$.

Let $\mathcal{F}[\,\cdot\,]$ denote the Fourier transform in $\mathbb{R}^d$:

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} e^{-\sqrt{-1}x\cdot\xi} f(x)\,dx,$$

and let $\mathcal{F}^{-1}[\,\cdot\,]$ denote the associated inverse transform in $\mathbb{R}^d$:

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\sqrt{-1}x\cdot\xi} \hat{f}(\xi)\,d\xi.$$

Let $\mathcal{F}'[f]$ denote the x_1-tangential Fourier transform of $f = f(x)$ in $\mathbb{R}^d_+$:

$$\mathcal{F}'[f](x_1,\xi') := \int_{\mathbb{R}^{d-1}} e^{-\sqrt{-1}x'\cdot\xi'} f(x_1,x')\,dx'$$

and let $\mathcal{F}'^{-1}[\hat{f}]$ denote the associated inverse transform of $\hat{f} = \hat{f}(x_1,\xi')$:

$$\mathcal{F}'^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}} e^{\sqrt{-1}x'\cdot\xi'} \hat{f}(x_1,\xi')\,d\xi',$$

where

$$x' = (x_2, \dots, x_d), \quad \xi' = (\xi_2, \dots, \xi_d), \quad x' \cdot \xi' = \sum_{k=2}^{d} x_k \xi_k.$$

We define the x_1-directional Fourier sine (resp. cosine) transform of a function $f = f(x)$ in $\mathbb{R}^d_+$ by $\mathcal{S}_1[f]$ (resp. $\mathcal{C}_1[f]$) as follows: for any $\xi_1 \in \mathbb{R}$,

$$\mathcal{S}_1[f](\xi_1, x') := 2\int_0^\infty \sin(x_1\xi_1) f(x_1, x')\, dx_1$$
$$\left(\text{resp. } \mathcal{C}_1[f](\xi_1, x') := 2\int_0^\infty \cos(x_1\xi_1) f(x_1, x')\, dx_1\right)$$

with the associated inverse transform of $\hat{f} = \hat{f}(\xi_1, x')$ in $\mathbb{R}^d_+$:

$$\mathcal{S}_1^{-1}[\hat{f}](x) := \frac{1}{\pi}\int_0^\infty \sin(x_1\xi_1)\hat{f}(\xi_1, x')\, d\xi_1.$$

Let $\mathcal{O}_1[\,\cdot\,]$ denote the odd extension operator in x_1:

$$\mathcal{O}_1[f](x) := \begin{cases} f(x_1, x') & \text{for } x_1 > 0, \\ -f(-x_1, x') & \text{for } x_1 < 0. \end{cases}$$

Note that $\mathcal{O}_1[\mathcal{S}_1[f]] = \mathcal{S}_1[f]$, $\mathcal{O}_1[\mathcal{S}_1^{-1}[\hat{f}]] = \mathcal{S}_1^{-1}[\hat{f}]$
and $\mathcal{F}[\mathcal{O}_1[f]] = \frac{1}{\sqrt{-1}}\mathcal{S}_1[\mathcal{F}'[f]]$. We have the inversion formula:

$$f = \frac{1}{\sqrt{-1}}\mathcal{F}^{-1}[\mathcal{S}_1[\mathcal{F}'[f]]]|_{\mathbb{R}^d_+} = \mathcal{F}'^{-1}[\mathcal{S}_1^{-1}[\mathcal{S}_1[\mathcal{F}'[f]]]]|_{\mathbb{R}^d_+}.$$

In addition, we have the formal identities:

$$\xi_1\mathcal{C}_1[\mathcal{F}'[f]] = -\mathcal{S}_1[\mathcal{F}'[\partial_1 f]], \qquad \xi_1^2\mathcal{S}_1[\mathcal{F}'[f]] = -\mathcal{S}_1[\mathcal{F}'[\partial_1^2 f]],$$

provided that $f = f(x)$ satisfies $f|_{x_1=0} = 0$ and $f(x_1, x') \to 0$ as $x_1 \to +\infty$.

We define the following two operators:

$$(-\Delta)^{-1} f := \frac{1}{\sqrt{-1}}\mathcal{F}^{-1}\left[-\frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[f]]\right]\Bigg|_{\mathbb{R}^d_+}$$
$$= \frac{\Gamma(\frac{d}{2}+1)}{d(d-2)\pi^{\frac{d}{2}}}\int_{\mathbb{R}^d_+}\left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|(x_1+y_1, x'-y')|^{d-2}}\right) f(y)dy$$

and

$$(-\Delta')^{-\frac{1}{2}}f := \frac{1}{\sqrt{-1}}\mathcal{F}^{-1}\left[\frac{1}{|\xi'|}\mathcal{S}_1[\mathcal{F}'[f]]\right]\bigg|_{\mathbb{R}^d_+} = \mathcal{F}'^{-1}\left[\frac{1}{|\xi'|}\mathcal{F}'[f]\right]$$
$$= \frac{1}{2\pi^{\frac{d-1}{2}}}\frac{\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})}\int_{\mathbb{R}^{d-1}}\frac{f(x_1,y')}{|x'-y'|^{d-2}}\,dy'$$

with $|\xi'| := \sqrt{\sum_{k=2}^d \xi_k^2}$. The above formulae follow from the kernels of the Newtonian and Riesz potentials respectively (cf. [3] and [2, Theorem 2.4.6] for instance).

For a given function $f : \mathbb{R}^d_+ \to \mathbb{R}$, let $\hat{v} = \hat{v}(x_1, \xi', t)$ be a solution of the IBVP of the 1-D heat equation in $\mathbb{R}_+$ with a parameter $\xi' \in \mathbb{R}^{d-1}$:

$$\partial_t\hat{v} - \partial_1^2\hat{v} + |\xi'|^2\hat{v} = 0, \qquad \hat{v}|_{x_1=0} = 0, \qquad \hat{v}|_{t=0} = \mathcal{F}'[f]. \tag{2.1}$$

Then we can observe that $\hat{w}(\xi,t) := \mathcal{S}_1[\hat{v}]$ is governed by the linear ODE:

$$\frac{d}{dt}\hat{w} + |\xi|^2\hat{w} = 0, \qquad \hat{w}|_{\xi_1=0} = 0, \qquad \hat{w}|_{t=0} = \mathcal{S}_1[\mathcal{F}'[f]] \tag{2.2}$$

and that $v(x,t) := \mathcal{F}'^{-1}[\hat{v}] = \frac{1}{\sqrt{-1}}\mathcal{F}^{-1}[\hat{w}]$ is governed by the heat equation in $\mathbb{R}^d_+$:

$$\partial_t v = \Delta v, \qquad v|_{x_1=0} = 0, \qquad v|_{t=0} = f. \tag{2.3}$$

Note that the above problems (2.1)-(2.3) are equivalent via the inversion formulae with the restriction on $\mathbb{R}^d_+$. By the reflection principle, we obtain the solution formulae for $\hat{v}$ and v respectively:

$$\hat{v}(t) = e^{-|\xi'|^2t}\int_0^\infty (G(x_1-y_1,t) - G(x_1+y_1,t))\mathcal{F}'[f](y_1,\xi')\,dy_1 \tag{2.4}$$

and

$$v(t) = \int_{\mathbb{R}^d_+}\big(G(x_1-y_1,t) - G(x_1+y_1,t)\big)\prod_{k=2}^d G(x_k-y_k,t)f(y)\,dy$$
$$=: H(t)f, \tag{2.5}$$

where the 1-D heat kernel $G(s,t) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{s^2}{4t})$. In addition, if a function

$\hat{w} = \mathcal{F}'[w](x_1, \xi', t)$ satisfies the integral form:

$$\hat{w}(t) = e^{-|\xi'|^2 t} \int_0^\infty \big(G(x_1 - y_1, t) + G(x_1 + y_1, t)\big) \mathcal{F}'[f](y_1, \xi')\, dy_1, \tag{2.6}$$

then we get the solution formula:

$$\begin{aligned} w(t) &= \int_{\mathbb{R}^d_+} \big(G(x_1 - y_1, t) + G(x_1 + y_1, t)\big) \prod_{k=2}^{d} G(x_k - y_k, t) f(y)\, dy \\ &=: K(t) f, \end{aligned} \tag{2.7}$$

which is a solution of the IBVP of the heat equation subject to the zero-Neumann boundary condition: $\partial_1 w|_{x_1=0} = 0$. Therefore we deduce the alternative formulae for (2.4) and (2.6) respectively:

$$\hat{v}(t) = \mathcal{F}'[H(t)f], \qquad \hat{w}(t) = \mathcal{F}'[K(t)f]. \tag{2.8}$$

3. A Solution Formula

In this section, we shall derive the following solution formula.

Theorem 3.1. *Let* $S[a](t) = (S_1[a_1](t), S_2[a_2\,; a_1](t), \ldots, S_d[a_d\,; a_1](t))$ *be the operator defined by*

$$\begin{aligned} S_1[a_1](t) := (-\Delta')^{-\frac{1}{2}}(1 - \partial_1)\left(1 - \partial_1^2(-\Delta)^{-1}\right) H(t) a_1 \\ - \partial_1(1 - \partial_1)(-\Delta)^{-1} K(t) a_1 \end{aligned} \tag{3.1}$$

and for $i = 2, \ldots, d$,

$$\begin{aligned} S_i[a_i\,; a_1](t) := H(t) a_i + \partial_i\{(-\Delta')^{-\frac{1}{2}} + (1 - \partial_1)(-\Delta)^{-1}\} H(t) a_1 \\ + \partial_i(-\Delta')^{-\frac{1}{2}} \partial_1(1 - \partial_1)(-\Delta)^{-1} K(t) a_1. \end{aligned} \tag{3.2}$$

Then $u(t) := S[a](t)$ *is a solution to the problem* (1.1)-(1.4).

Proof. Suppose that $\{u, p\}$ is a sufficiently regular solution to (1.1)-(1.4) on $\overline{\mathbb{R}^d_+} \times [0, \infty)$. Let us set

$$\hat{u}_i := \mathcal{F}'[u_i](x_1, \xi', t), \quad \hat{p} := \mathcal{F}'[p](x_1, \xi', t), \quad \hat{a}_i := \mathcal{F}'[a_i](x_1, \xi') \tag{3.3}$$

for $i = 1, \dots, d$. Applying $\nabla\cdot$ to the first equation (1.1), we have that $\Delta p = 0$, which yields the following ODE:

$$(\partial_1^2 - |\xi'|^2)\hat{p} = 0.$$

We deduce that

$$\hat{p} = \hat{p}(x_1, \xi', t) = e^{-x_1|\xi'|}\hat{p}(0, \xi', t).$$

Note that $\hat{p} \to 0$ as $|\xi'| \to \infty$ or $x_1 \to \infty$ and

$$(\partial_1 + |\xi'|)\hat{p} = 0. \tag{3.4}$$

From the first equation (1.1) for $i = 1$, we get

$$\partial_t \hat{u}_1 - \partial_1^2 \hat{u}_1 + |\xi'|^2 \hat{u}_1 + \partial_1 \hat{p} = 0. \tag{3.5}$$

Let

$$\hat{v} := |\xi'|\hat{u}_1 + \partial_1 \hat{u}_1. \tag{3.6}$$

From (3.4)-(3.5), we obtain the 1-D heat equation in $\mathbb{R}^+$:

$$\partial_t \hat{v} - \partial_1^2 \hat{v} + |\xi'|^2 \hat{v} = 0. \tag{3.7}$$

On the other hand, we can rewrite by using the second equation (1.2),

$$\begin{aligned}\hat{v} &= |\xi'|\hat{u}_1 + \mathcal{F}'[\partial_1 u_1] = |\xi'|\hat{u}_1 + \mathcal{F}'\left[-\sum_{j=2}^{d} \partial_j u_j\right] \\ &= |\xi'|\hat{u}_1 - \sqrt{-1}\sum_{j=2}^{d} \xi_j \int_{\mathbb{R}^{d-1}} e^{-\sqrt{-1}x'\cdot\xi'} u_j(x_1, x', t)\, dx',\end{aligned}$$

which implies the boundary condition

$$\hat{v}|_{x_1=0} = 0. \tag{3.8}$$

We also have the initial condition

$$\hat{v}|_{t=0} = |\xi'|\hat{a}_1 + \partial_1 \hat{a}_1. \tag{3.9}$$

In view of (2.1)-(2.8), we observe that the solution $\hat{v}$ to the IBVP (3.7)-(3.9) satisfies

$$\begin{aligned}\hat{v} &= |\xi'|\mathcal{F}'[H(t)a_1] + e^{-|\xi'|^2 t}\int_0^\infty (G(x_1-y_1,t)-G(x_1+y_1,t))\frac{\partial}{\partial y_1}\hat{a}_1(y_1,\xi')dy_1 \\ &= |\xi'|\mathcal{F}'[H(t)a_1] - e^{-|\xi'|^2 t}\int_0^\infty \frac{\partial}{\partial y_1}(G(x_1-y_1,t)-G(x_1+y_1,t))\hat{a}_1(y_1,\xi')dy_1 \\ &= |\xi'|\mathcal{F}'[H(t)a_1] + e^{-|\xi'|^2 t}\int_0^\infty \frac{\partial}{\partial x_1}(G(x_1-y_1,t)+G(x_1+y_1,t))\hat{a}_1(y_1,\xi')dy_1 \\ &= |\xi'|\mathcal{F}'[H(t)a_1] + \mathcal{F}'[\partial_1 K(t)a_1].\end{aligned}$$

Here we solve the ODE (3.6) with $\hat{u}_1|_{x_1=0} = 0$ to get

$$\hat{u}_1(x_1,\xi',t) = \int_0^{x_1} e^{(s-x_1)|\xi'|}\hat{v}(s,\xi',t)ds. \tag{3.10}$$

Therefore we deduce that $\mathcal{S}_1[\hat{u}_1] = \mathcal{S}_1[\mathcal{F}'[u_1]]$ satisfies

$$\begin{aligned}\mathcal{S}_1[\hat{u}_1] =& \int_0^\infty \sin(x_1\xi_1)\int_0^{x_1} e^{(s-x_1)|\xi'|}\hat{v}(s,\xi',t)dsdx_1 \\ =& \int_0^\infty e^{s|\xi'|}\hat{v}(s,\xi',t)\left(\int_s^\infty e^{-x_1|\xi'|}\sin(x_1\xi_1)dx_1\right)ds \\ =& \frac{1}{|\xi|^2}\int_0^\infty (\sin(s\xi_1)+\xi_1\cos(s\xi_1))\,\hat{v}(s,\xi',t)ds \\ =& \frac{1}{|\xi|^2}\int_0^\infty (\sin(x_1\xi_1)+\xi_1\cos(x_1\xi_1))\big(|\xi'|\mathcal{F}'[H(t)a_1]+\mathcal{F}'[\partial_1 K(t)a_1]\big)\,dx_1 \\ =& \frac{|\xi'|}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[H(t)a_1]] + \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1 K(t)a_1]] + \frac{|\xi'|\xi_1}{|\xi|^2}\mathcal{C}_1\left[\mathcal{F}'[H(t)a_1]\right] \\ &+ \frac{\xi_1}{|\xi|^2}\mathcal{C}_1[\mathcal{F}'\left[\partial_1 K(t)a_1\right]] \\ =& \frac{|\xi'|}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[H(t)a_1]] + \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1 K(t)a_1]] - \frac{|\xi'|}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1 H(t)a_1]] \\ &- \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1^2 K(t)a_1]]] \\ =& \frac{|\xi'|}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[(1-\partial_1)H(t)a_1]] + \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1(1-\partial_1)K(t)a_1]].\end{aligned} \tag{3.11}$$

In the above computation, the elementary identity:

$$e^{-x_1|\xi'|}\sin(x_1\xi_1) = -\frac{1}{|\xi|^2}\partial_1\{e^{-x_1|\xi'|}(\sin(x_1\xi_1)+\xi_1\cos(x_1\xi_1))\}$$

is used and the condition $a_1|_{x_1=0} = 0$ is *not* used. Hence we have obtained

$$\begin{aligned} u_1(t) =& \mathcal{F}^{-1}\Big[\frac{1}{|\xi'|}\left(1-\frac{\xi_1^2}{|\xi|^2}\right)\mathcal{S}_1[\mathcal{F}'[(1-\partial_1)H(t)a_1]] \\ &+ \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_1(1-\partial_1)K(t)a_1]]\Big]\Big|_{\mathbb{R}^d_+} \\ =& S_1[a_1](t). \end{aligned}$$

Next, we get from the first equation (1.1) for $i = 2, \ldots, d$,

$$\partial_t\hat{u}_i - \partial_1^2\hat{u}_i + |\xi'|^2\hat{u}_i + \sqrt{-1}\xi_i\hat{p} = 0. \tag{3.12}$$

Since (3.4)-(3.5), we have

$$\hat{p} = \frac{1}{|\xi'|}(\partial_t\hat{u}_1 - \partial_1^2\hat{u}_1 + |\xi'|^2\hat{u}_1).$$

Thus we see that

$$\hat{w}_i := \hat{u}_i + \frac{\sqrt{-1}\xi_i\hat{u}_1}{|\xi'|} \qquad (i = 2, \ldots, d)$$

satisfies the 1-D heat equation in $\mathbb{R}^+$:

$$\partial_t\hat{w}_i - \partial_1^2\hat{w}_i + |\xi'|^2\hat{w}_i = 0$$

subject to

$$\hat{w}_i|_{x_1=0}, \qquad \hat{w}_i|_{t=0} = \hat{a}_i + \frac{\sqrt{-1}\xi_i\hat{a}_1}{|\xi'|}.$$

That is,

$$\hat{w}_i = \mathcal{F}'[H(t)a_i] + \frac{1}{|\xi'|}\mathcal{F}'[\partial_i H(t)a_1].$$

Therefore we deduce from (3.11) that

$$\begin{aligned}\mathcal{S}_1[\hat{u}_i] &= \mathcal{S}_1[\mathcal{F}'[H(t)a_i]] + \frac{1}{|\xi'|}\mathcal{S}_1[\mathcal{F}'[\partial_i H(t)a_1]] - \frac{\sqrt{-1}\xi_i}{|\xi'|}\mathcal{S}_1[\hat{u}_1] \\ &= \mathcal{S}_1[\mathcal{F}'[H(t)a_i]] + \frac{1}{|\xi'|}\mathcal{S}_1[\mathcal{F}'[\partial_i H(t)a_1]] - \frac{1}{|\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_i(1-\partial_1)H(t)a_1]] \\ &\quad - \frac{1}{|\xi'||\xi|^2}\mathcal{S}_1[\mathcal{F}'[\partial_i\partial_1(1-\partial_1)K(t)a_1]],\end{aligned}$$

which yields $u_i(t) = S_i[a_i\,; a_1](t)$. □

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