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LIE-CARTAN DIFFERENTIAL INVARIANTS AND POINCARÉ-MOSER NORMAL FORMS: CONFLUENCES

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Abstract

We study 2-nondegenerate constant Levi rank 1 rigid $\mathcal{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$ with $0 \in M^5$ given in coordinates $(z, \zeta, w = u + iv)$ as $u = F\big(z, \zeta, \overline{z}, \overline{\zeta}\big)$ under *rigid* biholomorphisms:

$$(z, \zeta, w) \longmapsto \big(f(z,\zeta),\, g(z,\zeta),\, \rho\, w + h(z,\zeta)\big) =: (z', \zeta', w').$$

In a previous article, a Cartan-type reduction to an $\{e\}$-structure was done by Foo-Merker-Ta. Three relative invariants appeared: $\boldsymbol{V}_0$, $\boldsymbol{I}_0$ (primary) and $\boldsymbol{Q}_0$ (derived).

On the other hand, a Poincaré-Moser complete normal form:

$$u = \frac{z\overline{z} + \frac{1}{2}\, z^2\overline{\zeta} + \frac{1}{2}\, \overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geqslant 3}} G_{a,b,c,d}\big(F_\bullet\big)\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d,$$

with $0 = G_{a,b,0,0} = G_{a,b,1,0} = G_{a,b,2,0}$ and $0 = G_{3,0,0,1} = \operatorname{Im} G_{1,1,3,0}$, has been recently obtained by the authors.

The model $u = \frac{z\overline{z} + \frac{1}{2}\, z^2\overline{\zeta} + \frac{1}{2}\, \overline{z}^2\zeta}{1 - \zeta\overline{\zeta}}$ is equivalent to the future light cone $(\mathsf{Im}\, z_0)^2 = (\mathsf{Im}\, z_1)^2 + (\mathsf{Im}\, z_2)^2$ with $\mathsf{Im}\, z_0 > 0$ deeply investigated by Sergeev.

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In order to compare the two approaches, we compute (relative) invariants at *every* point, not only at the central point, and we 'discover' the proportionalities:

$$G_{0,1,4,0}(F_\bullet) \propto \boldsymbol{V}_0, \qquad G_{0,2,3,0}(F_\bullet) \propto \boldsymbol{I}_0, \qquad \operatorname{Re} G_{1,1,3,0}(F_\bullet) \propto \boldsymbol{Q}_0.$$

With this, a *bridge* Poincaré $\longleftrightarrow$ Cartan is constructed.

In terms of F, the numerators of $\boldsymbol{V}_0$, $\boldsymbol{I}_0$, $\boldsymbol{Q}_0$ incorporate 11, 52, 824 differential monomials.

1. Introduction

The problem of equivalence for CR manifolds was begun by Poincaré [44] in 1907, who, by a counting argument, pointed out that real hypersurfaces $M^3 \subset \mathbb{C}^2$ must *a priori* possess infinitely many *invariants* under biholomorphic transformations. In [1], Beloshapka argues that Poincaré's approach remains competitive in the study of *infinite-dimensional* geometry, and the present article will confirm this statement.

The study of real hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ is a classical subject, and there is an intensive activity since the seminal article [8] of Chern-Moser, devoted to Levi nondegenerate ones. Moser's part produces normal forms. Chern's part sets up an $\{e\}$-structure and even a Cartan connection. The link between these two parts is usually understood at the origin.

In the recent years, beyond [8], remarkable achievements appeared.

- Determination by Beloshapka [2] of universal models of CR manifolds of finite type and computation of their Lie algebras of infinitesimal CR automorphisms.
- Completion by Loboda [29, 30, 31], after a twenty-five-years study, of the full classification of locally homogeneous real hypersurfaces in $\mathbb{C}^3$ (*cf.* [9, 37, 42]).
- Cartan reduction and normal form for 6-dimensional generic submanifolds $M^6 \subset \mathbb{C}^4$ of codimension 2 and CR dimension 2 [10, 12].
- Normal forms for finite type hypersurfaces $M^3 \subset \mathbb{C}^2$ [11, 24, 25].
- Normal form for a real hypersurface in $\mathbb{C}^{2n+1}$ at a generic (not uniform) Levi-degeneracy in the sense of Webster [28].
- Survey of these and of several other results [27].
- Cartan-type reduction for constant Levi rank 1 and 2-nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ [15, 23, 32, 33, 37, 38].

- Normal forms for constant Levi rank 1 and 2-nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ which are not necessarily rigid [3, 17, 26].
- Cartan-type reduction and normal form for constant Levi rank 1 and 2-nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$ that are *rigid* [5, 16].

Most of the time, articles applying Cartan's method and articles applying Moser's method are published separately. On Cartan's side, computations are known to ⟨⟨ *explode* ⟩⟩. For instance [36], the numerator of the Cartan curvature of a Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ has $\sim 1\,500\,000$ monomials. The present article therefore focuses on certain CR structures, call *rigid* as defined below, for which actual computations remain tamed. The largest number of numerator monomials will be 824.

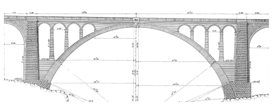
Cartan's method studies geometric structures *at every point* of the base manifold, and there is a ⟨⟨ complexity price ⟩⟩ to pay for this generality.

Moser's method is more 'simple', computationally speaking, since it usually proceeds at only *one* point, often the origin, of a manifold, by manipulating power series expanded at that point.

In comparison to Cartan's method, Moser's method seems to capture invariants only at one point.

But recently, Chen-Merker [6] found an alternative (probably known) method to capture differential invariants at *all points* while working *only at one point*. This method avoids to move the origin everywhere nearby by translations, and it works most of the times, namely when the group of transformations is (only) assumed to contain all translations, *see* especially [6, Sec. 12]. Hence this method clearly applies to the group of rigid biholomorphisms. Chen-Merker studied mainly parabolic (real) surfaces $S^2 \subset \mathbb{R}^3$ under the group of special affine transformations of $\mathbb{R}^3$, and developed an analog of Moser's method in this context, *see* also [7].

Since the technique of [6] seems not to have been well developed or understood by CR geometers up to now, we decided to write up the present memoir. Its main goal is to construct a *bridge:*

Cartan's method  Moser's method, and to exhibit how (relative) differential invariants pass from one side of the river to the other side, *computationally.* Reading the simple Section 2 below is enough to understand the key "arch-ideas" of such a bridge. We indeed first focus on the 'toy' case of rigid equivalences of rigid hypersurfaces in $\mathbb{C}^2$ (easily reached results), before passing to the not so simple case of rigid equivalences in the rigid class denoted $\mathfrak{C}_{2,1}$ by Alexander Isaev which consists, as written above, of 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$.

Such CR structures whose Levi form degenerates everywhere, but are not even locally straightenable, were deeply investigated by Sergeev and his collaborators in several memoirs — *e.g.* [45, 46, 47] — to study for instance integral representations of solutions to the $\overline{\partial}$-equation in the future tube.

Throughout all of this memoir, concentrated on CR geometry, all CR manifolds will be assumed embedded, real analytic ($\mathscr{C}^\omega$), and *rigid*.

The interest of studying *rigidly equivalent* — in Alexander Isaev's terminology — *rigid* hypersurfaces was pointed out to us during his February 2019 stay in Orsay. In recent publications [19, 20, 21, 22], Alexander Isaev *integrated* Pocchiola's zero CR curvature equations $\boldsymbol{W} = 0 = \boldsymbol{J}$ of tube and rigid 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$.

Relevant background on CR geometry may be found in [34, 35, 39].

A local hypersurface M^{2n+1} in $\mathbb{C}^{n+1}$ with coordinates $\mathrm{z} = (\mathrm{z}_1, \dots, \mathrm{z}_{n+1})$ is said to be *rigid* if there exists an infinitesimal CR automorphism, namely a vector field T tangent to M of the form $T = X + \overline{X}$ with a nonzero holomorphic vector field $X = \sum_{i=1}^{n+1} a_i(\mathrm{z})\, \partial_{\mathrm{z}_i}$, which is *transversal* to the complex tangent space T^cM in the sense that $TM = T^cM \oplus \mathbb{R}T$. After a local biholomorphic straightening, one makes $X = i\,\frac{\partial}{\partial w}$ with $w = \mathrm{z}_{n+1}$, and tangency of $X + \overline{X} = \frac{\partial}{\partial v}$ to M shows that, restricting considerations to dimensions $n + 1 = 2, 3$, writing coordinates $\mathbb{C}^2 \ni (z, w)$ and $\mathbb{C}^3 \ni (z, \zeta, w)$, the right-hand side $\mathscr{C}^\omega$ graphing functions:

$$M^3\colon \quad u = F(z, \overline{z}), \qquad\qquad M^5\colon \quad u = F(z, \zeta, \overline{z}, \overline{\zeta}),$$

are independent of v, where $w = u + i\, v$:

Alexander Isaev's concept of *rigid biholomorphic transformation* is as follows. In $\mathbb{C}^2$ and in $\mathbb{C}^3$, such are biholomorphisms of the form:

$$(z,w) \longmapsto \big(f(z),\, \rho\, w+g(z)\big), \qquad (z,\zeta,w) \longmapsto \big(f(z,\zeta),\, g(z,\zeta),\, \rho\, w+h(z,\zeta)\big),$$

where f, g, h are holomorphic of their arguments, *independently of* w, and where $\rho \in \mathbb{R}^*$. The interest is that rigid biholomorphisms send rigid hypersurfaces to rigid hypersurfaces: they respect the pre-given CR symmetry.

In $\mathbb{C}^2$, on the Cartan side of the bridge, we construct in Section 2 an absolute parallelism on $P^5 := M^3 \times \mathbb{C}$ equipped with coordinates $(z, \overline{z}, v, \mathsf{c}, \overline{\mathsf{c}})$ consisting of 5 differential 1-forms:

$$\big\{\rho,\ \zeta,\ \overline{\zeta},\ \pi,\ \overline{\pi}\big\} \qquad\qquad {\scriptstyle (\overline{\rho}=\rho)},$$

which satisfy invariant structure equations of the shape:

$$\begin{aligned} d\rho &= (\pi + \overline{\pi}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\ d\zeta &= \pi \wedge \zeta, & d\overline{\zeta} &= \overline{\pi} \wedge \overline{\zeta}, \\ d\pi &= \tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{R}\,\zeta \wedge \overline{\zeta}, & d\overline{\pi} &= -\,\tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\overline{\boldsymbol{R}}\,\zeta \wedge \overline{\zeta}, \end{aligned}$$

where there is only one (relative) invariant function:

$$\boldsymbol{R} := \frac{F_{zz\overline{z}\overline{z}}\,F_{z\overline{z}} - F_{zz\overline{z}}\,F_{z\overline{z}\overline{z}}}{(F_{z\overline{z}})^2}.$$

We show that M is rigidly equivalent to $\{u = z\overline{z}\}$ if and only if $\boldsymbol{R}(F) \equiv 0$.

On the Moser side of the bridge, starting from a given $u = \sum_{j+k\geqslant 1} F_{j,k}\, z^j \overline{z}^k$ passing by the origin, we perform as said above a few normalizing biholomorphisms in order to reach:

$$\begin{aligned} 0 &= F_{j,0} = F_{0,k} && {\scriptstyle (j\,\geqslant\,1,\ k\,\geqslant\,1)}, \\ 1 &= F_{1,1}, \\ 0 &= F_{j,1} = F_{1,k} && {\scriptstyle (j\,\geqslant\,2,\ k\,\geqslant\,2)}, \end{aligned}$$

and the key feature of the method is to *keep track* of all performed rigid biholomorphic transformations, which will give us at the end:

$$u = z\overline{z} + \Big[\frac{F_{2,2}\,F_{1,1} - F_{2,1}\,F_{1,2}}{F_{1,1}^3}\Big]\, z^2\overline{z}^2 + z^2\overline{z}^3\big(\cdots\big) + z^3\overline{z}^2\big(\cdots\big).$$

From this rational expression of the final $F'_{2,2}$ coefficient at the origin, it is easy to recognize/reconstitute/translate Cartan's invariant $\boldsymbol{R}(F)$ *at every point* (up to a nowhere vanishing factor $\mathsf{const}\cdot F_{z\overline{z}}$).

Now, pass to $\mathbb{C}^3$. The class of (local) hypersurfaces $M^5 \subset \mathbb{C}^3$ passing by the origin $0 \in M$ that are 2-nondegenerate and whose Levi form has constant rank 1 is denoted:

$$\mathfrak{C}_{2,1}.$$

The right graphed equation for the model light cone $M_{\mathsf{LC}} \subset \mathbb{C}^3$ in $\mathfrak{C}_{2,1}$ was set up by Gaussier-Merker[1] in [18]:

$$M_{\mathsf{LC}}\colon \qquad u = \frac{z\overline{z} + \frac{1}{2}\,z^2\overline{\zeta} + \frac{1}{2}\,\overline{z}^2\zeta}{1-\zeta\overline{\zeta}} =: \boldsymbol{m}\big(z,\zeta,\overline{z},\overline{\zeta}\big).$$

Here, the letter $\boldsymbol{m}$ is from $\boldsymbol{m}$odel. By luck, M_{LC} is rigid!

Start with $M^5 \subset \mathbb{C}^3$, with $0 \in M$, rigid, graphed as:

$$u = F(z,\zeta,\overline{z},\overline{\zeta}).$$

Constant Levi rank 1 means, possibly after a linear transformation in $\mathbb{C}^2_{z,\zeta}$, that:

$$F_{z\overline{z}} \neq 0 \equiv \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{vmatrix} =: \mathsf{Levi}(F), \tag{1.1}$$

while 2-nondegeneracy means that:

$$0 \neq \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{vmatrix}. \tag{1.2}$$

By direct symbolic computations, Propositions 3.1 and 3.2 in [5] establish

[1]Fels-Kaup showed in [13, 14], that the Gaussier-Merker model is locally biholomorphically equivalent to the tube over the light cone.

invariancy of these vanishing/nonvanishing properties under rigid changes of holomorphic coordinates.

Since the Gaussier-Merker function:

$$m(z,\zeta,\overline{z},\overline{\zeta}) = \frac{z\overline{z} + \frac{1}{2}\,z^2\overline{\zeta} + \overline{z}^2\zeta}{1-\zeta\overline{\zeta}}$$

is homogeneous of degree 2 in $(z,\overline{z})$, it is natural to assign the following weights to the coordinate variables:

$$[z] := 1 =: [\overline{z}], \qquad [\zeta] := 0 =: [\overline{\zeta}], \qquad [w] := 2 =: [\overline{w}].$$

In [5], the authors showed that every $\mathscr{C}^\omega$ hypersurface $M^5 \in \mathfrak{C}_{2,1}$ is equivalent, through a local rigid biholomorphism, to a rigid $\mathscr{C}^\omega$ hypersurface $M'^5 \subset \mathbb{C}'^3$ which, dropping primes for target coordinates, is a perturbation of the Gaussier-Merker model:

$$u = \frac{z\overline{z} + \frac{1}{2}\,z^2\overline{\zeta} + \frac{1}{2}\,\overline{z}^2\zeta}{1-\zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d\in\mathbb{N}\\ a+c\geqslant 3}} G_{a,b,c,d}\, z^a\zeta^b\overline{z}^c\overline{\zeta}^d,$$

with a simplified remainder G which:

(1) is normalized to be an $\mathrm{O}_{z,\overline{z}}(3)$;

(2) satisfies the prenormalization conditions $G = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = \mathrm{O}_z(3) + \mathrm{O}_\zeta(1)$:

$$\begin{aligned} G_{a,b,0,0} &= 0 = G_{0,0,c,d},\\ G_{a,b,1,0} &= 0 = G_{1,0,c,d},\\ G_{a,b,2,0} &= 0 = G_{2,0,c,d}; \end{aligned}$$

(3) satisfies in addition the sporadic normalization conditions:

$$\begin{aligned} G_{3,0,0,1} &= 0 = G_{0,1,3,0},\\ \mathsf{Im}\, G_{3,0,1,1} &= 0 = \mathsf{Im}\, G_{1,1,3,0}. \end{aligned}$$

Furthermore, two such rigid $\mathscr{C}^\omega$ hypersurfaces $M^5 \subset \mathbb{C}^3$ and $M'^5 \subset \mathbb{C}'^3$, both brought into such a normal form, are rigidly biholomorphically equivalent if and only if there exist two constants $\rho \in \mathbb{R}_+^*$, $\varphi \in \mathbb{R}$, such that for all a, b, c, d:

$$G_{a,b,c,d} = G'_{a,b,c,d}\, \rho^{\frac{a+c-2}{2}}\, e^{i\varphi(a+2b-c-2d)}.$$

When producing such a normal form, calculations are done only at one point (the origin), by manipulating only Taylor coefficients.

On the other hand, Cartan's method of equivalence manipulates functions defined in some neighborhood of the origin. So, Cartan's method seems to be stronger.

Our first goal in this article is to show that a suitable enhancement of Moser's method enables one to recover Cartan's curvatures *at every point near the origin while working only at the origin.*

Indeed, starting from a 2-nondegenerate constant Levi rank 1 rigid $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ with $0 \in M^5$ in coordinates $(z, \zeta, w = u + iv)$:

$$u = F\big(z, \zeta, \overline{z}, \overline{\zeta}\big) = \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+b+c+d \geqslant 1}} F_{a,b,c,d}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d,$$

we perform a reduction to normal form with conditions **(1)**, **(2)**, **(3)**, and we keep track of all intermediate changes of coordinates in order to express the final power series coefficients in terms of the initial power series coefficients:

$$u = \frac{z\overline{z} + \frac{1}{2}\, z^2\overline{\zeta} + \frac{1}{2}\,\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geqslant 3}} G_{a,b,c,d}\big(F_\bullet\big)\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d,$$

and we express the $G_{a,b,c,d}$ *explicitly* in terms of the initial $F_\bullet = \big\{F_{a,b,c,d}\big\}$.

For this, we apply the method of Chen-Merker [6], which enables us to compute (relative) invariants at *every* point, not only at the central point.

Theorem 1.3. *The three normalized Taylor coefficients:*

$$G_{0,1,4,0}\big(F_\bullet\big) \qquad\qquad G_{0,2,3,0}\big(F_\bullet\big) \qquad\qquad \operatorname{Re} G_{1,1,3,0}\big(F_\bullet\big)$$

are explicit rational expressions whose numerators have 11*,* 52*,* 824 *monomials in the initial coefficients* $F_{a,b,c,d}$*.*

More details are given in Section 4, in which the method is explained, and the first two numerators are typed. Beyond, we can create further normalization branches caused by the valued of $\boldsymbol{I}_0$ and of $\boldsymbol{V}_0$, *see* below.

We then 'discover' that the obtained coefficients:

$$G_{0,1,4,0}\big(F_\bullet\big) \propto \boldsymbol{V}_0, \qquad G_{0,2,3,0}\big(F_\bullet\big) \propto \boldsymbol{I}_0, \qquad \operatorname{Re} G_{1,1,3,0}\big(F_\bullet\big) \propto \boldsymbol{Q}_0,$$

are *proportional* — in fact *equal* after adjustment — to the (relative) invariants $\boldsymbol{V}_0$, $\boldsymbol{I}_0$, $\boldsymbol{Q}_0$ found by a completely different approach, namely Cartan's method.

Before explaining this, let us survey the results of the article [16], from Cartan's side of the river, inspired and guided by Olver's works [40, 41].

Consider as before a rigid $M^5 \subset \mathbb{C}^3$ with $0 \in M$, which is 2-nondegenerate and has Levi form of constant rank 1, *i.e.* belongs to the class $\mathfrak{C}_{2,1}$, and which is graphed as:

$$u = F\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big).$$

The letter ζ is protected, hence not used instead of z_2, since ζ will denote a 1-form. The two natural generators of $T^{1,0}M$ and $T^{0,1}M$ are:

$$\mathscr{L}_1 := \partial_{z_1} - i\, F_{z_1}\, \partial_v \qquad \text{and} \qquad \mathscr{L}_2 := \partial_{z_2} - i\, F_{z_2}\, \partial_v,$$

in the intrinsic coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v)$ on M. The Levi kernel bundle $K^{1,0}M \subset T^{1,0}M$ is generated by:

$$\mathscr{K} := \boldsymbol{k}\, \mathscr{L}_1 + \mathscr{L}_2, \qquad \text{where} \qquad \boldsymbol{k} := -\frac{F_{z_2\overline{z}_1}}{F_{z_1\overline{z}_1}},$$

is the slant function. The hypothesis of 2-nondegeneracy is equivalent to the nonvanishing:

$$0 \neq \overline{\mathscr{L}}_1(\boldsymbol{k}).$$

Also, the conjugate $\overline{\mathscr{K}}$ generates the conjugate Levi kernel bundle $K^{0,1} \subset T^{0,1}M$.

There is a second fundamental function, and no more:

$$\boldsymbol{P} := \frac{F_{z_1z_1\overline{z}_1}}{F_{z_1\overline{z}_1}}.$$

In the rigid case, it looks so simple! But in the nonrigid case, $\boldsymbol{P}$ has a numerator involving **69** differential monomials!

Foo-Merker-Ta produced in [16] reduction to an $\{e\}$-structure for the equivalence problem, under *rigid* (local) biholomorphic transformations, of such rigid $M^5 \in \mathfrak{C}_{2,1}$. They constructed an invariant 7-dimensional bundle $P^7 \longrightarrow M^5$ equipped with coordinates:

$$\big(z_1, z_2, \overline{z}_1, \overline{z}_2, v, \, \mathsf{c}, \overline{\mathsf{c}}\big),$$

with $\mathsf{c} \in \mathbb{C}$, together with a collection of seven complex-valued 1-form which make a frame for T^*P^7, denoted:

$$\big\{\rho, \, \kappa, \, \zeta, \, \overline{\kappa}, \, \overline{\zeta}, \, \alpha, \, \overline{\alpha}\big\} \qquad {\scriptstyle (\overline{\rho}=\rho)},$$

which satisfy 7 invariant structure equations of the form:

$$\begin{aligned}
d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho + i\, \kappa \wedge \overline{\kappa}, \\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \overline{\kappa}, \\
d\zeta &= \big(\alpha - \overline{\alpha}\big) \wedge \zeta + \frac{1}{\mathsf{c}} \boldsymbol{I}_0\, \kappa \wedge \zeta + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} \boldsymbol{V}_0\, \kappa \wedge \overline{\kappa}, \\
d\alpha &= \zeta \wedge \overline{\zeta} - \frac{1}{\mathsf{c}} \boldsymbol{I}_0\, \zeta \wedge \overline{\kappa} + \frac{1}{\mathsf{c}\overline{\mathsf{c}}} \boldsymbol{Q}_0\, \kappa \wedge \overline{\kappa} + \frac{1}{\overline{\mathsf{c}}} \overline{\boldsymbol{I}}_0\, \overline{\zeta} \wedge \kappa,
\end{aligned}$$

conjugate structure equations for $d\overline{\kappa}$, $d\overline{\zeta}$, $d\overline{\alpha}$ being easily deduced. Since α is not real, there is no obvious reason that $\boldsymbol{Q}_0$ should be real. But the fact that, on the other side, Re $G_{1,1,3,0}$ *is* real, led us to suspect that $\boldsymbol{Q}_0$ *is* real too.

Here, as in Pocchiola's Ph.D. [43] partly published as [38], there are exactly *two* primary Cartan-curvature invariants:

$$\begin{aligned}
\boldsymbol{V}_0 :=& -\frac{1}{3} \frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{5}{9} \left(\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right)^2 \\
& -\frac{1}{9} \frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\, \overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{1}{3} \overline{\mathscr{L}}_1(\overline{\boldsymbol{P}}) - \frac{1}{9} \overline{\boldsymbol{P}}\, \overline{\boldsymbol{P}}, \\
\boldsymbol{I}_0 :=& -\frac{1}{3} \frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})^2} + \frac{1}{3} \frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\, \overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})^3} \\
& + \frac{2}{3} \frac{\mathscr{L}_1\big(\mathscr{L}_1(\overline{\boldsymbol{k}})\big)}{\mathscr{L}_1(\overline{\boldsymbol{k}})} + \frac{2}{3} \frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}.
\end{aligned}$$

One can check that Pocchiola's relative invariant $\boldsymbol{W}_0$ (*see* [38]) which occurs under *general* biholomorphic transformations of $\mathbb{C}^3$ (not necessarily rigid!), when written for a *rigid* $M^5 \subset \mathbb{C}^3$, identifies with:

$$\boldsymbol{I}_0\big(F(z_1, z_2, \overline{z}_1, \overline{z}_2)\big) \equiv \boldsymbol{W}_0\big(F(z_1, z_2, \overline{z}_1, \overline{z}_2)\big).$$

Furthermore, there is *one* secondary invariant whose unpolished expression is:

$$\boldsymbol{Q}_0 := \frac{1}{2}\overline{\mathscr{L}}_1(\boldsymbol{I}_0) - \frac{1}{3}\bigg(\boldsymbol{P} - \frac{\mathscr{L}_1\big(\mathscr{L}_1(\overline{\boldsymbol{k}})\big)}{\mathscr{L}_1(\overline{\boldsymbol{k}})}\bigg)\overline{\boldsymbol{I}}_0 - \frac{1}{6}\bigg(\overline{\boldsymbol{P}} - \frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\bigg)\boldsymbol{I}_0 - \frac{1}{2}\frac{\mathscr{K}(\boldsymbol{V}_0)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}.$$

Visibly indeed, the vanishing of $\boldsymbol{I}_0$ and $\boldsymbol{V}_0$ implies the vanishing of $\boldsymbol{Q}_0$. In fact, a consequence of Cartan's general theory is:

$$0 \equiv \boldsymbol{V}_0 \equiv \boldsymbol{I}_0 \iff M \text{ is rigidly equivalent to the Gaussier-Merker model.}$$

However, it is not visible from its expression that $\boldsymbol{Q}_0$ is real.

In [16], by deducing new relations from the structure equations above, it was proved indirectly that $\boldsymbol{Q}_0$ is real-valued, but a finalized expression was missing there. A clean finalized expression of $\boldsymbol{Q}_0$, in terms of only the two fundamental functions $\boldsymbol{k}$, $\boldsymbol{P}$ (and their conjugates), from which one immediately sees real-valuedness, is:

$$\begin{aligned}
\boldsymbol{Q}_0 := 2\,\mathrm{Re}\bigg\{ & \frac{1}{9}\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)^2}{\overline{\mathscr{L}}_1(\boldsymbol{k})^4} - \frac{1}{9}\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\big)\,\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})^3} \\
& - \frac{1}{9}\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})^3} - \frac{1}{9}\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})^2} \\
& + \frac{1}{9}\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\big)\,\overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})^2} - \frac{2}{9}\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})} - \frac{1}{9}\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\,\boldsymbol{P}}{\overline{\mathscr{L}}_1(\boldsymbol{k})} \\
& + \frac{1}{3}\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{1}{6}\overline{\mathscr{L}}_1(\boldsymbol{P})\bigg\} - \frac{1}{9}\big|\overline{\boldsymbol{P}}\big|^2 + \frac{1}{3}\bigg|\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(\boldsymbol{k})\big)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\bigg|^2.
\end{aligned}$$

Section 6 discusses briefly the details of the necessary, nontrivial computations, *see* also [4].

Having $\boldsymbol{Q}_0$ in finalized form shows that Cartan's method and Moser's method bring complementary information. With this, a *bridge* between the two methods is constructed.

Lastly, we pursue the normalizations as follows, *see* Theorem 5.1 for a complete statement.

Theorem 1.4. *Within the branch* $\boldsymbol{I}_0 \neq 0$*, the hypersurface is, in a unique way, equivalent to:*

$$\begin{aligned} u = z\,\overline{z} &+ \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}\,z^3\,\overline{z}\,\overline{\zeta} \\ &+ \tfrac{1}{24}\,\frac{\boldsymbol{V}_0}{\overline{\boldsymbol{I}_0}^2}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,\frac{\overline{\boldsymbol{V}_0}}{\boldsymbol{I}_0^2}\,z^4\,\overline{\zeta} + \tfrac{1}{12}\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,z^3\,\overline{\zeta}^2 \\ &+ \zeta\,\overline{\zeta}\,(\cdots) + \sum_{a+b+c+d\geqslant 6,\, b\,d=0} \frac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d, \end{aligned}$$

without any harmonic monomial $z^j\,\zeta^{n-j}$, $\forall\, n \geqslant 0$, $0 \leqslant j \leqslant n$ *and any monomial* $z^a\,\zeta^b\,\overline{z}^c$, $\forall a+b \geqslant 2$, $c \in \{1,2\}$. *Collections of coefficients:* $\frac{\boldsymbol{V}_0}{\overline{\boldsymbol{I}_0}^2}$, $\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}$ *and* $\big\{F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\, b\,d=0}$, *are in one-to-one correspondence with biholomorphic equivalent classes.*

2. Rigid Equivalences of Rigid Hypersurfaces in $\mathbb{C}^2$: A Toy Study

We first consider the equivalence problem of rigid hypersurfaces in $\mathbb{C}^2$ under the action of rigid biholomorphic transformations. We will solve this problem with both Cartan's method of equivalence and Moser's method of normal forms. The calculations here are simple, and they will serve as a toy study for our more substantial problem in $\mathbb{C}^3$ later. Throughout this section, we use the complex coordinates (z, w) on $\mathbb{C}^2$ with $w = u + iv$, where $u, v \in \mathbb{R}$.

We recall that a real analytic hypersurface in $\mathbb{C}^2$ is called *rigid* if it can be written $\big\{u = F(z, \overline{z})\big\}$, where F is a converging power series in $z, \overline{z}$. A local biholomorphic map of $\mathbb{C}^2$ of the form:

$$(z, w) \longmapsto \big(f(z),\, a\,w + g(z)\big), \tag{2.1}$$

with $a \in \mathbb{R}^*$, $c \in \mathbb{R}$, will be called *rigid*. Most of the times, we will assume that the origin is fixed, whence $0 = f(0) = g(0)$.

Since rigid transformations send rigid hypersurfaces to hypersurfaces which are again rigid, it then makes sense to consider rigid equivalences of rigid hypersurfaces in $\mathbb{C}^2$, as we do here. The homogeneous model here is (still) the Heisenberg sphere $\{u = z\overline{z}\}$, whose rigid automorphisms fixing the origin can be extracted from the set of general automorphisms of the sphere (exercise).

As a starter, consider a rigid biholomorphic map $(z, w) \longmapsto \big(f(z),\, a\,w + g(z)\big) =: (z', w')$ between two hypersurfaces $\{u = F(z, \overline{z})\}$ in $\mathbb{C}^2$ and $\{u' = F'(z', \overline{z}')\}$ in $\mathbb{C}^2$ too. From:

$$F'\big(f(z), \overline{f}(\overline{z})\big) = F'\big(z', \overline{z}'\big) = u' = a\,u + \mathrm{Re}\; g(z) = a\,F(z, \overline{z}) + \tfrac{1}{2}\, g(z) + \tfrac{1}{2}\, \overline{g}(\overline{z}),$$

it comes the *fundamental equation*, identically satisfied:

$$F'\big(f(z), \overline{f}(\overline{z})\big) \equiv a\,F(z, \overline{z}) + \tfrac{1}{2}\, g(z) + \tfrac{1}{2}\, \overline{g}(\overline{z}). \tag{2.2}$$

Lemma 2.3. *Through a rigid biholomorphism between two rigid hypersurfaces $\{u = F\}$ and $\{u' = F'\}$ in $\mathbb{C}^2$, it holds:*

$$F_{z\overline{z}} = \tfrac{1}{a}\,\big|f_z\big|^2\, F'_{z'\overline{z}'}.$$

Proof. Applying $\partial_z \partial_{\overline{z}}$ eliminates g and $\overline{g}$ above and yields the result. □

Thus, $F_{z\overline{z}}$ is a *relative invariant:* it is nonvanishing in one system of coordinates if and only if it is nonvanishing in any other system of coordinates. Of course, M is *Levi nondegenerate* in the classical sense if and only if $F_{z\overline{z}} \neq 0$. We will constantly assume that this holds at *every* point.

2.4. Cartan's method of equivalence

Consider a real analytic graphed hypersurface $M^3 = \{u = F(z, \overline{z})\}$ passing through the origin in $\mathbb{C}^2$. Its holomorphic tangent space $T^{1,0}M := (\mathbb{C} \otimes TM) \cap T^{1,0}\mathbb{C}$ is a 1-dimensional complex vector bundle on M. One can check directly that the vector field $\mathscr{L} := \frac{\partial}{\partial z} - iF_z \frac{\partial}{\partial v}$ generates $T^{1,0}M$, in the intrinsic coordinates $(z, \overline{z}, v)$ on M. We abbreviate $A := -i\, F_z$ so that $\mathscr{L} = \frac{\partial}{\partial z} + A\, \frac{\partial}{\partial v}$ and $\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \overline{A} \frac{\partial}{\partial v}$.

Assume that M is everywhere Levi nondegenerate, namely $F_{z\overline{z}} \neq 0$. Next, define the real vector field $\mathscr{T}$ on M by $\mathscr{T} := -i\,[\mathscr{L}, \overline{\mathscr{L}}] = \ell\, \frac{\partial}{\partial v}$, where

$\ell := -2F_{z\overline{z}}$. As in [16], introduce also the auxiliary function on M:

$$\boldsymbol{P} := \frac{\ell_z}{\ell} = \frac{F_{zz\overline{z}}}{F_{z\overline{z}}}.$$

Lemma 2.5. *The vector fields $\mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}$ constitute a frame on $\mathbb{C} \otimes TM$, with Lie brackets:*

$$\big[\mathscr{T}, \mathscr{L}\big] = -\boldsymbol{P}\, \mathscr{T}, \qquad \big[\mathscr{T}, \overline{\mathscr{L}}\big] = -\overline{\boldsymbol{P}}\, \mathscr{T}, \qquad \big[\mathscr{L}, \overline{\mathscr{L}}\big] = -\, i\, \mathscr{T}.$$

Next, denote by $\rho_0, \zeta_0, \overline{\zeta}_0$ the (complex) 1-forms on M which are dual to the (complex) vector fields $\mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}$, respectively. More precisely, the expressions of $\rho_0, \zeta_0, \overline{\zeta}_0$ in terms of $dv, dz, d\overline{z}$ are:

$$\rho_0 := \tfrac{1}{\ell}\,\big(dv - A\, dz - \overline{A}\, d\overline{z}\big), \qquad \zeta_0 := dz, \qquad \overline{\zeta}_0 = d\overline{z}.$$

This gives us an initial coframe for $\mathbb{C} \otimes TM$ having structure equations:

$$\begin{aligned} d\rho_0 &= \boldsymbol{P}\, \rho_0 \wedge \zeta_0 + \overline{\boldsymbol{P}}\, \rho_0 \wedge \overline{\zeta}_0 + i\, \zeta_0 \wedge \overline{\zeta_0}, \\ d\zeta_0 &= d\overline{\zeta}_0 = 0. \end{aligned}$$

We now look at the action of rigid transformations on M in order to set up an initial G-structure. Observe that if a rigid biholomorphism $h\colon (z, w) \longmapsto \big(f(z),\, aw + g(z)\big) =: (z', w')$ fixing the origin maps a rigid hypersurface $M \subset \mathbb{C}^2$ to another rigid hypersurface $M' \subset \mathbb{C}'^2$, then h sends $T^{1,0}M$ to $T^{1,0}M'$, *i.e.* $h_*(T^{1,0}M) = T^{1,0}M'$. Without loss of generality, we can assume that the target $M' = \{u' = F'(z', \overline{z}')\}$ is also graphed, and is equipped with a similar frame $\{\mathscr{T}', \mathscr{L}', \overline{\mathscr{L}}'\}$. It follows that there exists a uniquely defined nowhere vanishing function $c'\colon M' \longrightarrow \mathbb{C}^*$ so that $h_*(\mathscr{L}) = c'\, \mathscr{L}'$.

Similary, $h_*(\mathscr{T}) = a'\, \mathscr{T} + b'\, \mathscr{L} + \overline{b}'\, \overline{\mathscr{L}}'$. From Definition 2.1, it is clear that $h_*(\partial_v) = a\, \partial_{v'}$. Since $\mathscr{T} = \ell\, \partial_v$ and $\mathscr{T}' = \ell'\, \partial_{v'}$, it comes $h_*(\mathscr{T}) = a\, \frac{\ell}{\ell'}\, \mathscr{T}'$. Hence $b' = 0$. Furthermore:

$$h_*(\mathscr{T}) = h_*\big(-i\,[\mathscr{L}, \overline{\mathscr{L}}]\big) = -\, i\, \big[h_*(\mathscr{L}),\, h_*(\overline{\mathscr{L}})\big] = -\, i\, \big[c'\mathscr{L}',\, \overline{c}'\overline{\mathscr{L}}'\big] = c'\overline{c}'\, \mathscr{T}',$$

with necessarily $0 \equiv \mathscr{L}'(\overline{c}')$ while expanding the bracket thanks to $b' = 0$, and we conclude that the function $a' = c'\overline{c}'$ is determined.

Consequently, under the action of h, the frame $\{\mathscr{T}, \mathscr{L}, \overline{\mathscr{L}}\}$ changes as:

$$h_* \begin{pmatrix} \mathscr{T} \\ \mathscr{L} \\ \overline{\mathscr{L}} \end{pmatrix} = \begin{pmatrix} c'\overline{c}' & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & \overline{c}' \end{pmatrix} \begin{pmatrix} \mathscr{T}' \\ \mathscr{L}' \\ \overline{\mathscr{L}}' \end{pmatrix} \qquad (c' \neq 0).$$

This gives us the transfer relation between the two *dual* coframes, in terms of a nowhere vanishing function $c\colon M \longrightarrow \mathbb{C}^*$:

$$h^* \begin{pmatrix} \rho_0' \\ \zeta_0' \\ \overline{\zeta}_0' \end{pmatrix} = \begin{pmatrix} c\overline{c} & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \overline{c} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix}.$$

The initial G-structure is now obtained as follows. Such a function c is replaced by a free variable $\mathsf{c} \in \mathbb{C}^*$, an unknown of the problem. The structure group is the 2-dimensional Lie group of matrices of the form:

$$g = \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 \\ 0 & \mathsf{c} & 0 \\ 0 & 0 & \overline{\mathsf{c}} \end{pmatrix} \qquad (\mathsf{c} \neq 0),$$

and we introduce the *lifted coframe:*

$$\begin{pmatrix} \rho \\ \zeta \\ \overline{\zeta} \end{pmatrix} := g \cdot \begin{pmatrix} \rho_0 \\ \zeta_0 \\ \overline{\zeta}_0 \end{pmatrix}.$$

We are now in the position to apply Cartan's method of equivalence to the G-structure just obtained. First, we compute the Maurer-Cartan matrix as:

$$dg \cdot g^{-1} = \begin{pmatrix} \frac{d\mathsf{c}}{\mathsf{c}} + \frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} & 0 & 0 \\ 0 & \frac{d\mathsf{c}}{\mathsf{c}} & 0 \\ 0 & 0 & \frac{d\overline{\mathsf{c}}}{\overline{\mathsf{c}}} \end{pmatrix},$$

and there is only one (complex-valued) Maurer-Cartan form $\alpha := \frac{d\mathsf{c}}{\mathsf{c}}$. The structure equations are as follows:

$$\begin{aligned} d\rho &= \big(\alpha + \overline{\alpha}\big) \wedge \rho + \frac{1}{\mathsf{c}} \boldsymbol{P}\, \rho \wedge \zeta + \frac{1}{\overline{\mathsf{c}}} \overline{\boldsymbol{P}}\, \rho \wedge \overline{\zeta} + i\, \zeta \wedge \overline{\zeta}, \\ d\zeta &= \alpha \wedge \zeta, \\ d\overline{\zeta} &= \overline{\alpha} \wedge \overline{\zeta}. \end{aligned}$$

We proceed to absorption of torsion by introducing the *modified Maurer-Cartan form*:

$$\pi := \alpha - \tfrac{1}{\mathsf{c}} \boldsymbol{P}\, \zeta,$$

in terms of which the structure equations contract as:

$$\begin{aligned} d\rho &= (\pi + \overline{\pi}) \wedge \rho + i\, \zeta \wedge \overline{\zeta}, \\ d\zeta &= \pi \wedge \zeta, \qquad\qquad d\overline{\zeta} = \overline{\pi} \wedge \overline{\zeta}. \end{aligned}$$

At this point, no more absorption can be performed, because if one modifies the 1-form π as $\tilde{\pi} := \pi - A\, \rho - B\, \zeta - C\, \overline{\zeta}$, which transforms the structure equations into:

$$\begin{aligned} d\rho &= \big(\tilde{\pi} + \overline{\tilde{\pi}}\big) \wedge \rho - (B + \overline{C})\, \rho \wedge \zeta - (\overline{B} + C)\, \rho \wedge \overline{\zeta} + i\, \zeta \wedge \overline{\zeta}, \\ d\zeta &= \tilde{\pi} \wedge \zeta + A\, \rho \wedge \zeta - C\, \zeta \wedge \overline{\zeta}, \end{aligned}$$

all the functions A, B, C must be zero to conserve the same shape. In other words, the prolongation reduces to identity, and π is uniquely defined.

Therefore, Cartan's process stops, and to finish, it remains to finalize the expression of:

$$\begin{aligned} d\pi &= \underline{d\alpha_\circ} + \tfrac{1}{\mathsf{c}} \tfrac{d\mathsf{c}}{\mathsf{c}} \boldsymbol{P} \wedge \zeta - \tfrac{1}{\mathsf{c}} d\boldsymbol{P} \wedge \zeta - \tfrac{1}{\mathsf{c}} \boldsymbol{P}\, d\zeta \\ &= 0 + \tfrac{1}{\mathsf{c}} \big(\pi + \tfrac{1}{\mathsf{c}} \boldsymbol{P}\, \zeta\big) \boldsymbol{P} \wedge \zeta - \tfrac{1}{\mathsf{c}} \big(\boldsymbol{P}_z\, dz + \boldsymbol{P}_{\overline{z}}\, d\overline{z}\big) \wedge \zeta - \tfrac{1}{\mathsf{c}} \boldsymbol{P}\, \pi \wedge \zeta \\ &= -\tfrac{1}{\mathsf{c}} \big(\boldsymbol{P}_z\, \tfrac{1}{\mathsf{c}}\, \zeta + \boldsymbol{P}_{\overline{z}}\, \tfrac{1}{\overline{\mathsf{c}}}\, \overline{\zeta}\big) \wedge \zeta, \end{aligned}$$

where we need to know/abbreviate just:

$$\boldsymbol{P}_{\overline{z}} = \tfrac{F_{zz\overline{z}\overline{z}} F_{z\overline{z}} - F_{zz\overline{z}}\, F_{z\overline{z}\overline{z}}}{(F_{z\overline{z}})^2} \;\; =: \; \boldsymbol{R},$$

whence:

$$d\pi = \tfrac{1}{\mathsf{c}\overline{\mathsf{c}}} \boldsymbol{R}\, \zeta \wedge \overline{\zeta}.$$

Visibly, $\overline{\boldsymbol{R}} = \boldsymbol{R}$ is real, because $\overline{F} = F$ is, whence $\overline{F_{z^a\overline{z}^c}} = F_{\overline{z}^a z^c}$.

Theorem 2.6. *The equivalence problem under local rigid biholomorphisms for $\mathscr{C}^\omega$ rigid real hypersurfaces $\{u = F(z,\overline{z})\}$ in $\mathbb{C}^2$ whose Levi form is everywhere nondegenerate reduces to classifying $\{e\}$-structures on the 5-dimensional bundle $M^3 \times \mathbb{C}$ equipped with coordinates $(z, \overline{z}, v, \mathsf{c}, \overline{\mathsf{c}})$ together with a coframe of 5 differential 1-forms:*

$$\big\{\rho,\ \zeta,\ \overline{\zeta},\ \pi,\ \overline{\pi}\big\} \qquad {\scriptstyle (\overline{\rho}=\rho)},$$

which satisfy invariant structure equations of the shape:

$$\begin{aligned}
d\rho &= (\pi + \overline{\pi}) \wedge \rho + i\,\zeta \wedge \overline{\zeta}, \\
d\zeta &= \pi \wedge \zeta, & d\overline{\zeta} &= \overline{\pi} \wedge \overline{\zeta}, \\
d\pi &= \tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{R}\,\zeta \wedge \overline{\zeta}, & d\overline{\pi} &= -\tfrac{1}{\mathsf{c}\overline{\mathsf{c}}}\overline{\boldsymbol{R}}\,\zeta \wedge \overline{\zeta}.
\end{aligned}$$

Another way to see that $\overline{\boldsymbol{R}} = \boldsymbol{R}$ is real from the structure equations is as follows, using Poincaré's relation:

$$\begin{aligned}
0 = d \circ d\rho &= \big(d\pi + d\overline{\pi}\big) \wedge \rho - \big(\pi + \overline{\pi}\big) \wedge d\rho + i\, d\zeta \wedge \overline{\zeta} - i\,\zeta \wedge d\overline{\zeta} \\
&= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\boldsymbol{R}\,\zeta \wedge \overline{\zeta} \wedge \rho + \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\overline{\boldsymbol{R}}\,\overline{\zeta} \wedge \zeta \wedge \rho - \big(\pi+\overline{\pi}\big)\Big[\underline{\big(\pi+\overline{\pi}\big)}_{\circ} \wedge \rho + i\,\zeta \wedge \overline{\zeta}\Big] \\
&\quad + i\,\pi \wedge \zeta \wedge \overline{\zeta} - i\,\zeta \wedge \overline{\pi} \wedge \overline{\zeta} \\
&= \frac{1}{\mathsf{c}\overline{\mathsf{c}}}\big(\boldsymbol{R} - \overline{\boldsymbol{R}}\big)\,\rho \wedge \zeta \wedge \overline{\zeta}.
\end{aligned}$$

Thus, the only invariant here is:

$$\boldsymbol{R} := \frac{F_{zz\overline{z}\overline{z}}\,F_{z\overline{z}} - F_{zz\overline{z}}\,F_{z\overline{z}\overline{z}}}{(F_{z\overline{z}})^2}. \tag{2.7}$$

When $\boldsymbol{R} \equiv 0$, the structure equations have constants coefficients, which shows, by Cartan's theory, that all rigid hypersurfaces with $\boldsymbol{R} \equiv 0$ are rigidly equivalent to each other, and equivalent to the model $\{u = z\overline{z}\}$. There also are direct arguments to get this.

Proposition 2.8. *A rigid $M = \{u = F(z,\overline{z})\}$ in $\mathbb{C}^2$ is rigidly biholomorphically equivalent to the Heisenberg sphere $\{u' = z'\overline{z}'\}$ if and only if:*

$$0 \equiv \boldsymbol{R}(F) \equiv F_{zz\overline{z}\overline{z}}\,F_{z\overline{z}} - F_{zz\overline{z}}\,F_{z\overline{z}\overline{z}}.$$

Proof. Recall that the condition $\boldsymbol{R}(F) \equiv 0$ is invariant under rigid biholomorphisms.

Trivially, $F := z\overline{z}$ implies $\boldsymbol{R}(F) \equiv 0$.

For the converse, Lemma 2.3 guarantees that M is of course Levi-nondegenerate too, and by invariancy of $\boldsymbol{R} = 0$, we can assume that $F = z\overline{z} + \mathrm{O}_{z,\overline{z}}(3)$.

Set $G := F_{z\overline{z}}$, a function which is also real-valued, with $G(0) = 1$. Thus:

$$0 \equiv G_{z\overline{z}}\, G - G_z\, G_{\overline{z}} \qquad\qquad \Longleftrightarrow \qquad\qquad \big(\log\, G\big)_{z\overline{z}} \equiv 0.$$

Consequently $\log\, G(z, \overline{z}) = \varphi(z) + \overline{\varphi}(\overline{z})$ for some holomorphic function with $\varphi(0) = 0$, whence $G(z, \overline{z}) = \psi(z) \cdot \overline{\psi}(\overline{z})$ with $\psi(0) = 1$, and

$$F(z, \overline{z}) = \int_0^z \psi(\zeta)\, d\zeta \cdot \int_0^{\overline{z}} \overline{\psi}(\overline{\zeta})\, d\overline{\zeta} \; =: \; f(z) \cdot \overline{f}(\overline{z}),$$

with $f(z) = z + \mathrm{O}_z(2)$. Thus $u = f(z)\, \overline{f}(\overline{z})$, and the rigid biholomorphism $z' := f(z)$ terminates. □

We know from Lemma 2.3 that $F_{z\overline{z}}$ is a relative invariant. What about $\boldsymbol{R}$? It suffices to examine how the numerator of $\boldsymbol{R}$ behaves under transformations.

Lemma 2.9. *Through a rigid biholomorphism* $(z, w) \longmapsto \big(f(z),\, a\, w + g(z)\big)$ $=: (z', w')$ *between two rigid hypersurfaces* $\{u = F\}$ *and* $\{u' = F'\}$ *in* $\mathbb{C}^2$, *it holds:*

$$F_{zz\overline{z}\overline{z}}\, F_{z\overline{z}} - F_{zz\overline{z}}\, F_{z\overline{z}\overline{z}} \equiv \tfrac{1}{a^2} \left(f_z\, \overline{f}_{\overline{z}}\right)^3 \Big[F'_{z'z'\overline{z}'\overline{z}'}\, F'_{z'\overline{z}'} - F'_{z'z'\overline{z}'}\, F'_{z'\overline{z}'\overline{z}'}\Big].$$

Proof. Differentiate the fundamental identity (2.2) four appropriate times:

$$\begin{aligned}
a\, F_{z\overline{z}} &\equiv f_z\, \overline{f}_{\overline{z}}\, F'_{z'\overline{z}'}, \\
a\, F_{zz\overline{z}} &\equiv f_{zz}\, \overline{f}_{\overline{z}}\, F'_{z'\overline{z}'} + f_z \overline{f}_{\overline{z}} f_z\, F'_{z'z'\overline{z}'}, \\
a\, F_{z\overline{z}\overline{z}} &\equiv f_z \overline{f}_{\overline{z}\overline{z}}\, F'_{z'\overline{z}'} + f_z \overline{f}_{\overline{z}} \overline{f}_{\overline{z}}\, F_{z'\overline{z}'\overline{z}'}, \\
a\, F_{zz\overline{z}\overline{z}} &\equiv f_{zz} \overline{f}_{\overline{z}\overline{z}}\, F'_{z'\overline{z}'} + f_{zz} \overline{f}_{\overline{z}} \overline{f}_{\overline{z}}\, F'_{z'\overline{z}'\overline{z}'} + f_z \overline{f}_{\overline{z}\overline{z}} f_z\, F'_{z'z'\overline{z}'} + f_z \overline{f}_{\overline{z}} f_z \overline{f}_{\overline{z}}\, F'_{z'z'\overline{z}'\overline{z}'},
\end{aligned}$$

perform the necessary products, substract, and get the result. □

2.10. Method of normal forms of Moser

In this subsection, following the method of Moser, we will approach the equivalence problem for rigid hypersurfaces in $\mathbb{C}^2$ under rigid biholomorphisms by constructing a normal form. Notice that although the problem is (much) simpler than that considered by Moser for general hypersurfaces in $\mathbb{C}^2$, our problem here is not a special case of what is already known.

The goal is to simplify the defining function $u = F(z, \overline{z})$ of a given hypersurface $M^3 \subset \mathbb{C}^2$ as much as possible by applying rigid holomorphic changes of variables $(z, w) \mapsto \big(f(z),\, \rho\, w + g(z)\big) =: (z', w')$, with $\rho \in \mathbb{R}^*$. We will find step by step changes, so that the transformed graphing functions F' for successive $M' = \big\{u' = F'(z', \overline{z'})\big\}$ will contain more and more zero coefficients.

Take a real analytic hypersurface $M = \{u = F(z, \overline{z})\}$ passing through the origin in $\mathbb{C}^2$, and expand:

$$u = \tfrac{1}{2}\,\big(w + \overline{w}\big) = \sum_{j+k\geqslant 1} F_{j,k}\, z^j \overline{z}^k,$$

with $F_{j,k} = \overline{F}_{k,j}$. At first, set $z' := z$ and:

$$w' := w - 2 \sum_{j\geqslant 1} F_{j,0}\, z^j,$$

in order to subtract all harmonic monomials $F_{j,0}\, z^j$ and $F_{0,k}\, \overline{z}^k$ to obtain:

$$u' = \sum_{\substack{j\geqslant 1\\ k\geqslant 1}} F_{j,k}\, z^j \overline{z}^k = F_{1,1}\, z\overline{z} + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 1 \text{ and } k\geqslant 1}} F_{j,k}\, z^j \overline{z}^k.$$

The invariant property $F_{1,1} \neq 0$ characterizes Levi nondegeneracy of M at the origin (hence in a neighborhood). Switching $u \longmapsto -u$ if necessary, we may assume $F_{1,1} > 0$.

Next, make the rigid biholomorphism $z' := \sqrt{F_{1,1}}\, z$ with $w' := w$, drop the prime, single out monomials of degree 1 in either z or $\overline{z}$, factorize, and point out remainders:

$$
\begin{aligned}
u &= z\overline{z} + \sum_{\substack{j+k\geqslant 3\\ j\geqslant 1 \text{ and } k\geqslant 1}} \frac{F_{j,k}}{\sqrt{F_{1,1}}^{\,j+k}}\, z^j\overline{z}^k \\
&= z\overline{z}+\overline{z}\bigg(\frac{F_{2,1}}{F_{1,1}^{3/2}}z^2+\sum_{j\geqslant 3}\frac{F_{j,1}}{F_{1,1}^{(j+1)/2}}z^j\bigg)+z\bigg(\frac{F_{1,2}}{F_{1,1}^{3/2}}\overline{z}^2+\sum_{k\geqslant 3}\frac{F_{1,k}}{F_{1,1}^{(1+k)/2}}\overline{z}^k\bigg) \\
&\quad+\frac{F_{2,2}}{F_{1,1}^2}z^2\overline{z}^2+\sum_{\substack{j+k\geqslant 5\\ j\geqslant 2 \text{ and } k\geqslant 2}}\frac{F_{j,k}}{F_{1,1}^{(j+k)/2}}z^j\overline{z}^k \\
&= \bigg(z+\frac{F_{2,1}}{F_{1,1}^{3/2}}z^2+\sum_{j\geqslant 3}\frac{F_{j,1}}{F_{1,1}^{(j+1)/2}}z^j\bigg)\bigg(\overline{z}+\frac{F_{1,2}}{F_{1,1}^{3/2}}\overline{z}^2+\sum_{k\geqslant 3}\frac{F_{1,k}}{F_{1,1}^{(1+k)/2}}\overline{z}^k\bigg) \\
&\quad-\frac{F_{2,1}\,F_{1,2}}{F_{1,1}^3}z^2\overline{z}^2-z^2\overline{z}^3\big(\cdots\big)-z^3\overline{z}^2\big(\cdots\big) \\
&\quad+\frac{F_{2,2}}{F_{1,1}^2}z^2\overline{z}^2+z^2\overline{z}^3\big(\cdots\big)+z^3\overline{z}^2\big(\cdots\big).
\end{aligned}
$$

Such a factorization suggests to perform the rigid biholomorphism:

$$
z' := z + \frac{F_{2,1}}{F_{1,1}^{3/2}}\, z^2 + \sum_{j\geqslant 3} \frac{F_{j,1}}{F_{1,1}^{(j+1)/2}}\, z^j,
$$

again with untouched $w' := w$. Its inverse is of the form $z = z'\big(1+z'^2(\cdots)\big)$, so $\mathrm{O}\big(z^l\overline{z}^m\big) = \mathrm{O}\big(z'^l\overline{z}'^m\big)$, and finally, dropping primes, we have proved the

Proposition 2.11. *Any rigid $M = \big\{u = \sum F_{j,k} z^j\overline{z}^k\big\}$ can be brought, by a rigid biholomorphic transformation fixing the origin, to:*

$$
u = z\overline{z} + \bigg[\frac{F_{2,2}\,F_{1,1} - F_{2,1}\,F_{1,2}}{F_{1,1}^3}\bigg]\, z^2\overline{z}^2 + z^2\overline{z}^3\big(\cdots\big) + z^3\overline{z}^2\big(\cdots\big).
$$

In other words:

$$
\begin{aligned}
0 &= F_{j,0} = F_{0,k} && (j \geqslant 1,\ k \geqslant 1), \\
1 &= F_{1,1}, \\
0 &= F_{j,1} = F_{1,k} && (j \geqslant 2,\ k \geqslant 2).
\end{aligned}
$$

Can one normalize the graphing function F further? For instance, can one annihilate some other $F_{j,k}$? Not much freedom is left, as states the next

Lemma 2.12. *If two rigid hypersurfaces in* $\mathbb{C}^2$ *having the form:*

$$u = z\overline{z} + \sum_{j,k\geqslant 2} F_{j,k}\, z^j \overline{z}^k \qquad \text{and} \qquad u' = z'\overline{z}' + \sum_{j,k\geqslant 2} F'_{j,k}\, z'^j \overline{z}'^k,$$

are equivalent through a rigid biholomorphism fixing the origin, then there exist $\rho \in \mathbb{R}_+^*$ *and* $\varphi \in \mathbb{R}$ *such that:*

$$z' = \rho^{1/2}\, e^{i\varphi}\, z, \qquad\qquad w' = \rho\, w.$$

In particular, this shows that the group of rigid transformations fixing the origin of the Heisenberg sphere $\{u = z\overline{z}\}$ is 2-dimensional, generated by these obvious rotation/dilation commuting transformations (solution of the exercise).

Proof. Write as above $(z', w') = \big(f(z),\, \rho\, w + g(z)\big)$, with $f(0) = 0 = g(0)$. The fundamental equation reads:

$$\rho\, F(z, \overline{z}) + \tfrac{1}{2}\, g(z) + \tfrac{1}{2}\, \overline{g}(\overline{z}) \;\equiv\; F'\big(f(z), \overline{f}(\overline{z})\big).$$

Put $\overline{z} := 0$, get $\overline{g}(\overline{z}) \equiv 0$. Thus:

$$\rho\,\big(z\overline{z} + z^2\overline{z}^2(\cdots)\big) \equiv f(z)\overline{f}(\overline{z}) + f(z)^2\overline{f}(\overline{z})^2\,\big(\cdots\big),$$

and using $f(z) = \mathrm{O}(z)$:

$$\rho\, z\overline{z} \equiv f(z)\overline{f}(\overline{z}) + z^2\overline{z}^2\big(\cdots\big).$$

Invertibility of the Jacobian yields $f_z(0) \neq 0$. Apply $\partial_{\overline{z}}\big|_0$ and get:

$$\rho\, z \equiv f(z)\, \overline{f}'(0),$$

so $f(z) = \lambda\, z$ for some $\lambda \in \mathbb{C}^*$. Lastly, $\rho = \lambda\overline{\lambda}$, which concludes. □

Corollary 2.13. *Two rigid hypersurfaces in* $\mathbb{C}^2$*:*

$$u = z\overline{z} + \sum_{j,k\geqslant 2} F_{j,k}\, z^j \overline{z}^k \qquad \text{and} \qquad u' = z'\overline{z}' + \sum_{j,k\geqslant 2} F'_{j,k}\, z'^j \overline{z}'^k,$$

are rigidly biholomorphically equivalent if and only if there exist $\rho \in \mathbb{R}_+^*$ *and* $\varphi \in \mathbb{R}$ *such that:*

$$F_{j,k} = \rho^{\frac{j+k-2}{2}} \, e^{i\,\varphi\,(j-k)} \, F'_{j,k} \qquad (j \geqslant 2,\ k \geqslant 2).$$

At any point $(z_0, w_0) \in M$ close to the origin, all these results are also valid, and using the recentered holomorphic coordinates $z - z_0$ and $w - w_0$, one obtains:

$$u - u_0 = (z - z_0)\,\big(\overline{z} - \overline{z}_0\big) + \frac{4\,F_{zz\overline{z}\overline{z}}(z_0)\,F_{z\overline{z}}(z_0) - 2\,F_{zz\overline{z}}(z_0) 2\,F_{z\overline{z}\overline{z}}(z_0)}{F_{z\overline{z}}(z_0)^3}(z - z_0)^2\big(\overline{z} - \overline{z}_0\big)^2 + \cdots .$$

The $(2,2)$-coefficient at various points z_0 is, up to a power of $F_{z\overline{z}}$ in the denominator, exactly equal to the relative invariant function $\boldsymbol{R}$ found in (2.7) by applying Cartan's method.

3. Caves Beneath a Waterfall

This section (whose title will be explained at its end) displays the technique of calculating differential invariants under *infinite*-dimensional Lie group actions thanks to *finite*-dimensional (power series) approximations. First, we introduce some notations.

Definition 3.1. The (local) *rigid transformation group* of $\mathbb{C}^{2+1}$ fixing the origin will be denoted:

$$G := \big\{(z, \zeta, w) \mapsto (z', \zeta', w') = \big(f(z,\zeta), g(z,\zeta), \rho\, w\big)\big\},$$

where $\rho \in \mathbb{R}^*$ and f, g are holomorphic functions near $0 \in \mathbb{C}^2$ with $f(0,0) = g(0,0) = 0$ and with *nonzero* Jacobian determinant:

$$0 \neq \begin{vmatrix} f_z & f_\zeta \\ g_z & g_\zeta \end{vmatrix}.$$

Multiplications and inversions are induced by compositions and inversions of transformations.

Proposition 3.2. *A map* $(z,\zeta) \longmapsto \big(f(z,\zeta), g(z,\zeta)\big)$ *defines a biholomorphism between two neighborhoods of* $0 \in \mathbb{C}^2$ *and* $0' \in \mathbb{C}'^2$ *if and only if its Jacobian matrix is invertible at the origin.*

We will denote the inverse map as:

$$\tilde{f}(z',\zeta') = z, \qquad\qquad \tilde{g}(z',\zeta') = \zeta.$$

The power series expansions of $\tilde{f}$ and $\tilde{g}$ can be calculated homogeneous degree by homogeneous degree from the identities:

$$f\big(\tilde{f}(z',\zeta'), \tilde{g}(z',\zeta')\big) \equiv z', \qquad g\big(\tilde{f}(z',\zeta'), \tilde{g}(z',\zeta')\big) \equiv \zeta'.$$

At each degree, a linear system has to be solved, for example at degree 1:

$$\begin{pmatrix} f_{1,0} & f_{0,1} \\ g_{1,0} & g_{0,1} \end{pmatrix} \cdot \begin{pmatrix} \tilde{f}_{1,0} & \tilde{f}_{0,1} \\ \tilde{g}_{1,0} & \tilde{g}_{0,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next, recall that $w = u + i\,v$.

Definition 3.3. The space of all Levi-rank 1 and 2-non-degenerate CR graphed hypersurfaces passing by the origin in $\mathbb{C}^3$ will be denoted:

$$\mathscr{H} := \big\{u = F(z,\zeta,\overline{z},\overline{\zeta})\big\},$$

where:

- (real-valued analytic) F is an analytic real-valued function in a neigborhood of $(0,0,0,0) \in \mathbb{C}^4$, so that $\overline{F}\big(\overline{z},\overline{\zeta},z,\zeta\big) \equiv F\big(z,\zeta,\overline{z},\overline{\zeta}\big)$;
- (passing by the origin) $F(0,0,0,0) = 0$;
- (no harmonic monomials) $\partial_z^a \partial_\zeta^b F(0,0,0,0) = 0$ for any $a,b \in \mathbb{N}$ and $\partial_{\overline{z}}^c \partial_{\overline{\zeta}}^d F(0,0,0,0) = 0$ for any $c,d \in \mathbb{N}$;
- (Levi-rank 1) the matrix
$$\begin{pmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{pmatrix}$$
has rank 1 everywhere;
- (2-nondegenerate) the matrix
$$\begin{pmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{pmatrix}$$
is invertible at the origin.

There is a natural action of the group G of local rigid transformations on the space $\mathscr{H}$. Indeed, from $w' = \rho\, w$ whence $u' = \rho\, u$, a graphed hypersurface $u = F(z, \zeta, \overline{z}, \overline{\zeta})$ is transformed into another graphed hypersurface $u' = F'(z', \zeta', \overline{z'}, \overline{\zeta'})$ when F and F' are linked by the *fundamental equation:*

$$F'\big(f(z,\zeta), g(z,\zeta), \overline{f}(\overline{z},\overline{\zeta}), \overline{g}(\overline{z},\overline{\zeta})\big) \equiv \rho\, F\big(z,\zeta,\overline{z},\overline{\zeta}\big).$$

Equivalently in terms of the inverse $(\tilde{f}, \tilde{g})$:

$$F'\big(z',\zeta',\overline{z}',\overline{\zeta}'\big) \equiv \rho\, F\big(\tilde{f}(z',\zeta'), \tilde{g}(z',\zeta'), \overline{\tilde{f}}(\overline{z}',\overline{\zeta}'), \overline{\tilde{g}}(\overline{z}',\overline{\zeta}')\big).$$

This second identity brings convenience to obtain explicit information on the action.

Both the group G and the space $\mathscr{H}$ are infinite-dimensional in the sense that they depend on infinitely many independent parameters.

Concerning G, any transformation is defined by $\rho \in \mathbb{R}^*$ and two holomorphic functions f, g with expansions:

$$\begin{aligned} f(z,\zeta) &= \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{f_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}, \\ g(z,\zeta) &= \sum_{n=1}^{\infty} \sum_{j=0}^{n} \tfrac{g_{j,n-j}}{j!\,(n-j)!}\, z^j\, \zeta^{n-j}, \end{aligned}$$

where $f_{j,k}, g_{j,k} \in \mathbb{C}$, $f_{1,0}g_{0,1} - f_{0,1}g_{1,0} \neq 0$. The group G is hence parametrized by $f_{j,k}, g_{j,k}$ and ρ.

Concerning $\mathscr{H}$, any graphed hypersurface in $\mathscr{H}$ admits an expansion:

$$u = F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\infty} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d,$$

where $F_{a,b,c,d} \in \mathbb{C}$, $F_{c,d,a,b} = \overline{F_{a,b,c,d}}$, $F_{a,b,0,0} = 0$ and conditions of constant Levi-rank 1 and of 2-nondegeneracy are satisfied. The space is hence parametrized by $F_{a,b,c,d}$.

Fortunately these infinite-dimensional objects have finite-dimensional approximations. They can be truncated by degrees in expansions. Then they can be viewed as inverse or projective limits of those finite-dimensional truncations.

Definition 3.4. The δ^{th} residue group H_δ is the subgroup of G with

$$f(z,\zeta) = z + \mathrm{O}(\delta), \qquad g(z,\zeta) = \zeta + \mathrm{O}(\delta), \qquad \rho = 1.$$

One can verify

Proposition 3.5. *The group H_δ is a normal subgroup of G.*

Definition 3.6. The δ^{th} *approximation group* G_δ is the quotient group $G/H_{\delta+1}$. Each element has a *polynomial* representative:

$$\begin{aligned} f(z,\zeta) &= \sum_{n=1}^{\delta}\sum_{j=0}^{n} \tfrac{f_{j,n-j}}{j!\,(n-j)!}\, z^j\,\zeta^{n-j}, \\ g(z,\zeta) &= \sum_{n=1}^{\delta}\sum_{j=0}^{n} \tfrac{g_{j,n-j}}{j!\,(n-j)!}\, z^j\,\zeta^{n-j}. \end{aligned}$$

The group G_δ is a *finite*-dimensional Lie group parametrized by ρ and $f_{j,n-j}, g_{j,n-j}$ with $1 \leqslant n \leqslant \delta$, $0 \leqslant j \leqslant n$. Here is a table.

δ	1	2	3	4	5	6	7	δ
$\dim_{\mathbb{R}} G_\delta$	9	21	37	57	81	109	141	$2\,\delta^2 + 6\,\delta + 1$

Multiplication and inversion in G_δ are obtained by dropping terms of degree $\geqslant \delta + 1$ in the multiplication and inversion of G.

Proposition 3.7. *For any $\delta, \delta' \in \mathbb{Z}_+$ with $\delta > \delta'$ there is a projection $G_\delta \longrightarrow G_{\delta'}$ induced by the injection $H_\delta \longrightarrow H_{\delta'}$. For any $\delta, \delta', \delta'' \in \mathbb{Z}_+$ with $\delta > \delta' > \delta''$ the following diagram commutes:*

$$\begin{array}{ccc} G_\delta & \longrightarrow & G_{\delta'} \\ & \searrow & \downarrow \\ & & G_{\delta''}. \end{array}$$

These projections define a projective system $\{G_\delta\}_{\delta\in\mathbb{Z}_+}$. Projections $\pi_\delta : G \longrightarrow G_\delta$ are compatible with this system. By the universal property of the *projective limit, there is an injective morphism:*

$$G \longrightarrow \lim_{\overleftarrow{\delta}} G_\delta.$$

Now, pass to truncations of hypersurfaces.

Definition 3.8. For any $\delta \geqslant 2$, the δ^{th} *approximation of* $\mathscr{H}$ is the algebraic polynomial hypersurface:

$$\mathscr{H}_\delta := \Big\{ u = F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d \Big\},$$

where:

- (real-valued) $F_{a,b,c,d} = \overline{F_{c,d,a,b}}$ for any $a,b,c,d \geqslant 0$;
- (passing by the origin) $F_{0,0,0,0} = 0$;
- (no harmonic monomials) $F_{a,b,0,0} = F_{0,0,c,d} = 0$ for any $a,b,c,d \geqslant 0$;
- (2-non-degenerate) the matrix:

$$\begin{pmatrix} F_{1,0,1,0} & F_{1,0,0,1} \\ F_{2,0,1,0} & F_{2,0,0,1} \end{pmatrix}$$

  is invertible.
- (Levi-rank 1 until degree δ) $F_{1,0,1,0}$, $F_{1,0,0,1} = \overline{F_{0,1,1,0}}$ and $F_{0,1,0,1}$ are not all 0 and the complex Hessian of $F(z,\zeta,\overline{z},\overline{\zeta})$ vanishes up to order $\delta - 2$, *i.e.*:

$$\begin{vmatrix} F_{z\overline{z}} & F_{\zeta\overline{z}} \\ F_{z\overline{\zeta}} & F_{\zeta\overline{\zeta}} \end{vmatrix} = F_{z\,\overline{z}}\, F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\, F_{\zeta\,\overline{z}} = \mathrm{O}(\delta - 1).$$

The last condition may look questionable, but it is reasonable, as Proposition 3.10 will show in a while.

But before and as a preparation we must introduce *dependent* and *independent* power series coefficients. The manifolds $\mathscr{H}$ and $\mathscr{H}_\delta$ are covered by 3 open subsets: $\{F_{1,0,1,0} \neq 0\}$, $\{F_{1,0,0,1} = \overline{F_{0,1,1,0}} \neq 0\}$ and $\{F_{0,1,0,1} \neq 0\}$. We only treat the case $F_{1,0,1,0} \neq 0$ because the other two cases can be transformed into this one by changes of coordinates $(z',\zeta') = (z+\zeta, z-\zeta)$ or $(z',\zeta') = (z,\zeta)$ preserving the Levi-rank.

When $F_{1,0,1,0} \neq 0$ we have $F_{z,\overline{z}} \neq 0$ in a neighborhood of the origin. The Levi-rank 1 condition is now equivalent to:

$$F_{\zeta\,\overline{\zeta}} \equiv \frac{F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}}.$$

By differentiating both sides, all terms $F_{z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d}$ with $b \geqslant 1$ and $d \geqslant 1$ can be uniquely expressed as rational functions of $F_{z^{a'}\,\zeta^{b'}\,\overline{z}^{c'}}$ with $a'+b'+c' \leqslant a+b+c+d$ and $F_{z^{a''}\,\overline{z}^{c''}\,\overline{\zeta}^{d''}}$ with $a''+b''+c'' \leqslant a+b+c+d$. Moreover, only powers of $F_{z\,\overline{z}}$ appears in the denominators. For example:

$$F_{z\,\zeta,\overline{\zeta}} \equiv \frac{F_{z\,\zeta\,\overline{z}}\,F_{z\,\overline{\zeta}}}{F_{z\,\overline{z}}} + \frac{F_{z^2\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}} - \frac{F_{z^2\,\overline{z}}\,F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}}}{F_{z\,\overline{z}}^2}.$$

Taking values at the origin, the coefficients $F_{a,b,c,d}$ with $b \geqslant 1$ and $d \geqslant 1$ can be uniquely expressed as rational functions of $F_{a',b',c',0}$ with $a'+b'+c' \leqslant a+b+c+d$ and $F_{a'',0,c'',d''}$ with $a''+b''+c'' \leqslant a+b+c+d$. Moreover, only powers of $F_{1,0,1,0}$ appear in the denominators. For example:

$$F_{1,1,0,1} = \frac{F_{1,1,1,0}\,F_{1,0,0,1}}{F_{1,0,1,0}} + \frac{F_{2,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}} - \frac{F_{2,0,1,0}\,F_{1,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}^2}.$$

Definition 3.9. A coefficient $F_{a,b,c,d}$ will be called *dependent* if $b \geqslant 1$ and $d \geqslant 1$. Otherwise, it will be called *independent*.

Elements in the open subset $\{F_{1,0,1,0} \neq 0\}$ of $\mathscr{H}$ and $\mathscr{H}_\delta$ are uniquely determined by the independent coefficients $F_{a,b,c,d}$ with $b\,d = 0$. Since F is real-valued, *i.e.* $F_{c,d,a,b} = \overline{F_{a,b,c,d}}$, one has:

$$\dim_{\mathbb{R}} \mathscr{H}_\delta = \#\big\{(a,b,c,d)\colon\; a+b \geqslant 1,\; c+d \geqslant 1,\; a+b+c+d \leqslant \delta,\; b\,d = 0\big\},$$

with values given by:

δ	2	3	4	5	6	7	8	δ
$\dim_{\mathbb{R}} \mathscr{H}_\delta$	3	11	26	50	85	133	196	$\frac{1}{6}(2\,\delta^3 + 3\,\delta^2 - 5\,\delta)$

We can now state the promised

Proposition 3.10. *A polynomial*

$$F(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d$$

is a degree δ truncation of a formal power series $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta})$ with $\tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}} = 0$ if and only if $F_{z\,\overline{z}}\,F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}} = \mathrm{O}(\delta-1)$.

Proof. (only if) When calculating the complex Hessian of a power series:

$$\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{n=2}^{\infty}\,\sum_{a+b+c+d=n} \tfrac{\tilde{F}_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d,$$

the $\delta-2$ degree terms of $\tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}}$ involve only coefficients $\tilde{F}_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$.

Let $F(z,\zeta,\overline{z},\overline{\zeta})$ be its degree δ truncation:

$$F(z,\zeta,\overline{z},\overline{\zeta}) := \sum_{n=2}^{\delta}\,\sum_{a+b+c+d=n} \tfrac{\tilde{F}_{a,b,c,d}}{a!\,b!\,c!\,d!}\, z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d.$$

Then $F_{z\,\overline{z}}\,F_{\zeta\,\overline{\zeta}} - F_{z\,\overline{\zeta}}\,F_{\zeta\,\overline{z}} = \tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}} + \mathrm{O}(\delta-1) = \mathrm{O}(\delta-1)$.

To prove the 'if' part, one shall construct a power series:

$$\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) = F(z,\zeta,\overline{z},\overline{\zeta}) + \sum_{n=\delta+1}^{\infty}\,\sum_{a+b+c+d=n} \tfrac{\tilde{F}_{a,b,c,d}}{a!b!c!d!}\, z^a\zeta^b\overline{z}^c\overline{\zeta}^d$$

with $\tilde{F}_{z\,\overline{z}}\,\tilde{F}_{\zeta\,\overline{\zeta}} - \tilde{F}_{z\,\overline{\zeta}}\,\tilde{F}_{\zeta\,\overline{z}} = 0$. This can be achieved by taking all the independent coefficients $\tilde{F}_{a,b,c,d} = 0$ with $a+b+c+d \geqslant n+1$ and $b\,d = 0$ and by calculating all the dependent coefficients $\tilde{F}_{a,b,c,d}$ with $b \geqslant 1$ and $d \geqslant 1$ by their rational expressions in terms of the independent ones. □

Proposition 3.11. *For any $\delta, \delta' \in \mathbb{Z}_+$ with $\delta > \delta'$ there is a projection $\mathscr{H}_\delta \longrightarrow \mathscr{H}_{\delta'}$ by dropping terms of degree $\geqslant \delta'+1$. For any $\delta, \delta', \delta'' \in \mathbb{Z}_+$ with $\delta > \delta' > \delta''$ the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{H}_\delta & \longrightarrow & \mathscr{H}_{\delta'} \\ & \searrow & \downarrow \\ & & \mathscr{H}_{\delta''}. \end{array}$$

These projections define a projective system $\{\mathscr{H}_\delta\}_{\delta\in\mathbb{Z}_+}$. Projections $\pi_\delta \colon \mathscr{H} \longrightarrow \mathscr{H}_\delta$ are compatible with this system. By the universal property *of the projective limit, there is an injective morphism:*

$$\mathscr{H} \longrightarrow \varprojlim_{\delta} \mathscr{H}_\delta.$$

The manifold $\mathscr{H}_\delta$ is a finite-dimensional manifold parametrized by the independent coefficients $F_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$ and $b\,d = 0$. The action of the group G on $\mathscr{H}$ induces an action on each manifold $\mathscr{H}_\delta$:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\pi_\delta} & \mathscr{H}_\delta \\ {\scriptstyle (f,g,\rho)}\downarrow & & \vdots \\ \mathscr{H} & \xrightarrow{\pi_\delta} & \mathscr{H}_\delta. \end{array}$$

More precisely, a polynomial $F(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}_\delta$ is a degree δ truncation of a (not unique) power series $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}$, which is then transformed to another convergent power series $\tilde{F}'(z,\zeta,\overline{z},\overline{\zeta})$ by the fundamental equation:

$$\begin{aligned} &\tilde{F}'\big(z',\zeta',\overline{z}',\overline{\zeta}'\big) \\ &\quad = \rho\,\tilde{F}\big(\tilde{f}(z',\zeta'), g'(z',\zeta'), \overline{\tilde{f}}(\overline{z}',\overline{\zeta}'), \overline{g}(\overline{z}',\overline{\zeta}')\big) \\ &\quad = \rho\sum_{n=2}^{\delta}\sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!b!c!d!}\big(\tilde{f}(z',\zeta')\big)^a\big(\tilde{g}(z',\zeta')\big)^b\big(\overline{\tilde{f}}(\overline{z}',\overline{\zeta}')\big)^c\big(\overline{\tilde{g}}(\overline{z}',\overline{\zeta}')\big)^d + \mathrm{O}(\delta+1). \end{aligned}$$

The degree δ truncation of $\tilde{F}'\big(z',\zeta',\overline{z}',\overline{\zeta}'\big)$, denoted $F'(z',\zeta',\overline{z}',\overline{\zeta}') \in \mathscr{H}_\delta$, is then defined as being the *image* of $F(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}_\delta$ after the group action.

To ensure that this group action is well defined, let us verify that it is independent of the choice of a representative $\tilde{F}(z,\zeta,\overline{z},\overline{\zeta})$ and that it depends only on the coefficients $F_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$.

Proposition 3.12. *There is a group action of $G_{\delta-1}$ on $\mathscr{H}_\delta$. The group action of G on $\mathscr{H}_\delta$ factors through the projection $G \longrightarrow G_{\delta-1}$,* i.e. *the following diagram commutes:*

$$\begin{array}{ccc} G \times \mathscr{H}_\delta & \longrightarrow & \mathscr{H}_\delta \\ {\scriptstyle \pi_{\delta-1}}\downarrow & \nearrow & \\ G_{\delta-1} \times \mathscr{H}_\delta. & & \end{array}$$

Proof. When calculating the Taylor coefficients $F'_{a,b,c,d}$ in:

$$\tilde{F}'\big(z',\zeta',\overline{z}',\overline{\zeta}'\big) = \sum_{n=2}^{\delta} \tfrac{F'_{a,b,c,d}}{a!b!c!d!} z'^a\,\zeta'^b\,\overline{z}'^c\,\overline{\zeta}'^d + \mathrm{O}(\delta+1),$$

we are calculating the coefficients of $z'^a\,\zeta'^b\,\overline{z}'^c\,\overline{\zeta}'^d$ with $a+b+c+d \leqslant \delta$ from:

$$\rho \sum_{n=2}^{\delta} \sum_{a+b+c+d=n} \tfrac{F_{a,b,c,d}}{a!b!c!d!} \big(\tilde{f}(z',\zeta')\big)^a \,\big(g'(z',\zeta')\big)^b \,\big(\bar{\tilde{f}}(\overline{z}',\overline{\zeta}')\big)^c \,\big(\overline{\tilde{g}}(\overline{z}',\overline{\zeta}')\big)^d.$$

Each monomial is a product of at least 2 terms among $\tilde{f}(z',\zeta')$, $\tilde{g}(z',\zeta')$, $\bar{\tilde{f}}(\overline{z}',\overline{\zeta}')$, $\overline{\tilde{g}}(\overline{z}',\overline{\zeta}')\big\}$. The two power series in z', ζ':

$$\tilde{f}\big(z',\zeta'\big) = \sum_{n=1}^{\infty} \tfrac{\tilde{f}_{j,n-j}}{j!(n-j)!} z'^j\,\zeta'^{n-j},$$
$$\tilde{g}\big(z',\zeta'\big) = \sum_{n=1}^{\infty} \tfrac{\tilde{g}_{j,n-j}}{j!(n-j)!} z'^j\,\zeta'^{n-j},$$

start from degree 1. So only $\tilde{f}_{j,n-j}$, $\tilde{g}_{j,n-j}$ and their conjugates $\bar{\tilde{f}}_{j,n-j}$, $\overline{\tilde{g}}_{j,n-j}$ with $n \leqslant \delta - 1$ and $0 \leqslant j \leqslant n$ do contribute to $F'_{a,b,c,d}$ with $a+b+c+d \leqslant \delta$.

Thus the group action of $G_{\delta-1}$ on $\mathscr{H}_\delta$ is well defined and the diagram is indeed commutative. □

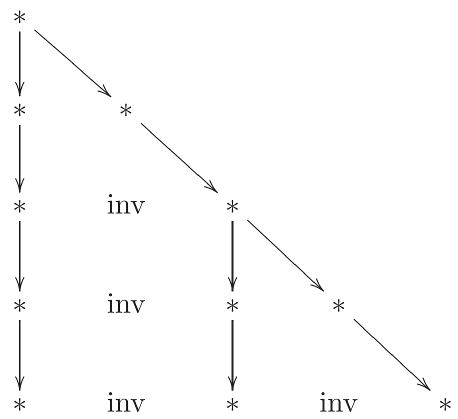
Now, compare the two tables of dimensions:

δ	2	3	4	5	6	7	8
$\dim_{\mathbb{R}} G_{\delta-1}$	9	21	37	57	81	109	141
$\dim_{\mathbb{R}} \mathscr{H}_\delta$	3	11	26	50	85	133	196

Therefore, the theory of differential invariants of finite-dimensional Lie group actions applies: the orbit dimension of $G_{\delta-1}$ on $\mathscr{H}_d$ is at most equal to $\dim_{\mathbb{R}} G_{\delta-1}$ and the equality is achieved only when the action is locally free. We see immediately that the dimension of (local) transversals to the orbits, which is equal to the number of linearly independent differential invariants up to order δ, is positive when $\delta \geqslant 6$.

Now, let us explain the *title* of this Section 3. The infinite-dimensional Lie group G can be interpreted as an infinitely long flow of water. The space

$\mathscr{H}$ can be interpreted as an infinitely high valley. At the beginning, water fills the space up. But later on as the waterfall grows wider, water cannot fill the space. Some *caves*, corresponding to the transversal dimension, or *differential invariants*, show up.



4. Invariants I_0, V_0, Q_0 at Every Point

Since the G action on $\mathscr{H}_\delta$ factors through $\pi_{\delta-1} : G \longrightarrow G_{\delta-1}$, we have the

Proposition 4.1. *A rational function on $\mathscr{H}_\delta$ is invariant under the G action if and only if it is invariant under the G_δ action.*

Thus, to calculate differential invariants of order δ under G is equivalent to calculate those under the finite-dimensional Lie group $G_{\delta-1}$. The algorithm goes as follows.

(1) Write down how $(f, g, \rho) \in G_{\delta-1}$ acts on some independent parameters $F_{a,b,c,d}$.

(2) Choose certain $(f, g, \rho) \in G_{\delta-1}$ to normalize as many independent parameters $F_{a,b,c,d}$ to 0 or 1 as possible, *i.e.* (f, g, ρ) send $F_{a,b,c,d}$ to $F^{(1)}_{a,b,c,d}$ and some $F^{(1)}_{a,b,c,d} = 0$ or 1.

(3) Calculate how the other independent parameters $F^{(1)}_{a,b,c,d}$ are changed under this special (f, g, ρ) action, *i.e.* express them as rational functions of $F_{a,b,c,d}$, $f_{j,n-j}$, $g_{j,n-j}$ and ρ.

(4) Calculate the "*stabilizer*", *i.e.* the subgroup $G^{(1)}_{\delta-1}$ of $G_{\delta-1}$ which preserves the current normalizations.

(5) Repeat steps (2), (3), (4) by studying the $G^{(1)}_{\delta-1}$ action on $F^{(1)}_{a,b,c,d}$, the $G^{(2)}_{\delta-1}$ action on $F^{(2)}_{a,b,c,d}$, and so on, until no more terms can be normalized, *i.e.* $G^{(k)}_{\delta-1}$ fixes all $F^{(k)}_{a,b,c,d}$.

(6) Express those non-constant $F^{(k)}_{a,b,c,d}$ in terms of the initial $F_{a,b,c,d}$. Obtain rational functions fixed by $G_{\delta-1}$, *i.e.* differential invariants of order $\leqslant \delta$.

We fix $\delta = 5$ in this section. The goal is to show the existence of order 5 invariants and to compute their explicit expressions. Lastly, we compare the results with similar invariant obtained in [16] through the (completely) different Cartan method of equivalence.

4.2. First normalization: degree 2 terms $= z\,\overline{z}$

We may assume that $F_{1,0,1,0} \neq 0$. In this case:

$$\begin{aligned}
F(z,\zeta,\overline{z},\overline{\zeta}) &= F_{1,0,1,0}\, z\,\overline{z} + F_{1,0,0,1}\, z\overline{\zeta} + F_{0,1,1,0}\,\zeta\,\overline{z} + \tfrac{F_{1,0,0,1}\,F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta\,\overline{\zeta} + \mathrm{O}(3)\\
&= F_{1,0,1,0}\left(z + \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta\right)\left(\overline{z} + \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\,\overline{\zeta}\right) + \mathrm{O}(3)\\
&= \underbrace{\left(F^{1/2}_{1,0,1,0}\, z + \tfrac{F_{0,1,1,0}}{F^{1/2}_{1,0,1,0}}\,\zeta\right)}_{=:\,z'}\underbrace{\left(F^{1/2}_{1,0,1,0}\,\overline{z} + \tfrac{F_{1,0,0,1}}{F^{1/2}_{1,0,1,0}}\,\overline{\zeta}\right)}_{=:\,\overline{z'}} + \mathrm{O}(3).
\end{aligned}$$

After the rigid transformation:

$$z' = F^{1/2}_{1,0,1,0}\, z + \tfrac{F_{0,1,1,0}}{F^{1/2}_{1,0,1,0}}\,\zeta, \qquad \zeta' = \zeta, \qquad w' = w,$$

the polynomial $F(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(1)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + \mathrm{O}(3)$. The other independent parameters $F^{(1)}_{a,b,c,d}$ with $a+b \geqslant 1, c+d \geqslant 1, b\,d = 0$ can also be uniquely expressed as rational functions of $F_{a,b,c,d}$ through the fundamental equation.

Since all the independent parameters $F^{(1)}_{a,b,c,d}$ have $b\,d = 0$ and $F^{(1)}_{c,d,a,b} = \overline{F^{(1)}_{a,b,c,d}}$, it suffices to calculate $F^{(1)}_{a,b,c,0}$ in terms of $F_{a,b,c,d}$. The inverse transformation is:

$$z = \tfrac{1}{F^{1/2}_{1,0,1,0}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\,\zeta', \qquad \zeta = \zeta', \qquad w = w'.$$

In the fundamental equality

$$\sum_{a,b,c,d} \tfrac{F^{(1)}_{a,b,c,d}}{a!b!c!d!} z'^a\, \zeta'^b\, \overline{z}'^c\, \overline{\zeta}'^d = \sum_{a,b,c,d} \tfrac{F_{a,b,c,d}}{a!b!c!d!} z^a\, \zeta^b\, \overline{z}^c\, \overline{\zeta}^d$$
$$= \sum_{a,b,c,d} \tfrac{F_{a,b,c,d}}{a!b!c!d!} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\, \zeta'\big)^a\, \zeta'^b\, \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, \overline{z'} - \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\, \overline{\zeta'}\big)^c\, \overline{\zeta'}^d,$$

we calculate the coefficient of $z'^a\, \zeta'^b\, \overline{z'}^c$. On the left hand side, it is $F^{(1)}_{a,b,c,0}$. On the right hand side only $F_{j,a+b-j,c,0}$ with $a \leqslant j \leqslant a+b$ contribute. Since:

$$\tfrac{F_{j,a+b-j,c,0}}{j!(a+b-j)!c!} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, z' - \tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\, \zeta'\big)^a\, \zeta'^{a+b-j}\, \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, \overline{z'} - \tfrac{F_{1,0,0,1}}{F_{1,0,1,0}}\, \overline{\zeta'}\big)^c$$
$$= \tfrac{F_{j,a+b-j,c,0}}{j!(a+b-j)!c!}\, \tfrac{j!}{a!(j-a)!} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, z'\big)^a \big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\, \zeta'\big)^{j-a}\, \zeta'^{a+b-j}\, \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\, \overline{z'}\big)^c$$
$$+ \text{ irrelevant monomials},$$

we get

$$F^{(1)}_{a,b,c,0} = \sum_{j=a}^{a+b} \tfrac{F_{j,a+b-j,c,0}}{a!(j-a)!(a+b-j)!c!} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\big)^a \big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\big)^{j-a} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\big)^c$$
$$= \sum_{j=0}^{b} \tfrac{F_{a+j,b-j,c,0}}{a!j!(b-j)!c!} \big(\tfrac{1}{F^{1/2}_{1,0,1,0}}\big)^{a+c} \big(-\tfrac{F_{0,1,1,0}}{F_{1,0,1,0}}\big)^j.$$

We define:

$$\mathscr{H}_5^{(1)} := \big\{u = F^{(1)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \mathrm{O}(3)\big\},$$

a codimension 3 submanifold of $\mathscr{H}_5$ since we have normalized $F^{(1)}_{1,0,1,0} = 1$ and $F^{(1)}_{1,0,0,1} = \overline{F^{(1)}_{0,1,1,0}} = 0$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(1)} = 50 - 3 = 47$.

Its stabilizer group $G_4^{(1)}$ consists of (f, g, ρ) such that

$$f(z,\zeta) = r\, e^{i\theta}\, z + \mathrm{O}(2), \qquad g(z,\zeta) = \mathrm{O}(1), \qquad \rho = r^2,$$

where $r \in \mathbb{R}_+$, $\theta \in [0, 2\pi)$. It is a codimension 3 subgroup of G_4, hence $\dim_{\mathbb{R}} G_4^{(1)} = 57 - 3 = 54$.

4.3. Second normalization: $F^{(2)}_{a,b,1,0} = 0$ for $(a,b) \neq (1,0)$

Now, we study the group action of $G_4^{(1)}$ on $\mathscr{H}_5^{(1)}$. Any element in $\mathscr{H}_5^{(1)}$ has expansion:

$$\begin{aligned} &F^{(1)}(z,\zeta,\overline{z},\overline{\zeta}) \\ &= z\,\overline{z} + \overline{z}\Big(\sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b\Big) + z\Big(\sum_{2\leqslant a+b\leqslant 4} \overline{\tfrac{F^{(1)}_{a,b,1,0}}{a!b!}}\,\overline{z}^c\,\overline{\zeta}^d\Big) + R(z,\zeta,\overline{z},\overline{\zeta}) \\ &= \underbrace{\Big(z + \sum_{2\leqslant a+b\leqslant 4} \tfrac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b\Big)}_{=:\,z'} \underbrace{\Big(\overline{z} + \sum_{2\leqslant a+b\leqslant 4} \overline{\tfrac{F^{(1)}_{a,b,1,0}}{a!b!}}\,\overline{z}^a\,\overline{\zeta}^b\Big)}_{=:\,\overline{z'}} + R(z,\zeta,\overline{z},\overline{\zeta}) \end{aligned}$$

whose the remainder $R(z,\zeta,\overline{z},\overline{\zeta})$ contains only terms $z^a\,\zeta^b\,\overline{z}^c\,\overline{z'}^d$ with either (a,b) or $(c,d) \notin \{(0,0),(1,0)\}$. After the rigid transformation in $G_4^{(1)}$:

$$z' = z + \sum_{2\leqslant a+b\leqslant 4} \frac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b, \qquad \zeta' = \zeta, \qquad w' = w, \tag{4.4}$$

the polynomial $F^{(1)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(2)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. It remains to show that the remainder $R'(z',\zeta',\overline{z'},\overline{\zeta'})$ contains only terms $z^a\,\zeta^b\,\overline{z}^c\,\overline{z}^d$ with either (a,b) or $(c,d) \notin \{(0,0),(1,0)\}$.

Lemma 4.5. *The inverse of the transformation* (4.4) *in* $G_4^{(1)}$ *is of the form:*

$$z = z' + \sum_{n=2}^{4}\sum_{j=0}^{n} \frac{\tilde{f}_{j,n-j}}{j!(n-j)!}\,z^j\,\zeta^{n-j}, \qquad \zeta = \zeta', \qquad w = w'.$$

Proof. It suffices to show that $z := \tilde{f}(z',\zeta') = z' + \mathrm{O}_{z',\zeta'}(2)$. From (4.4):

$$z = z' - \sum_{2\leqslant a+b\leqslant 4} \frac{F^{(1)}_{a,b,1,0}}{a!b!}\,z^a\,\zeta^b = z' - \sum_{2\leqslant a+b\leqslant 4} \frac{F^{(1)}_{a,b,1,0}}{a!b!}\,\tilde{f}(z',\zeta')^a\,\zeta'^b = z' + \mathrm{O}_{z',\zeta'}(2).$$

□

In the remainder $R(z,\zeta,\overline{z},\overline{\zeta})$, each term $z^a\,\zeta^b\,\overline{z}^c\,\overline{z}^d$ is transformed to $\big(z' + \mathrm{O}_{z',\zeta'}(2)\big)^a\,\zeta'^b\,\big(\overline{z'} + \mathrm{O}_{\overline{z'},\overline{\zeta'}}(2)\big)\,\overline{\zeta'}^d$, whose expansion still contains only terms $z'^a\,\zeta'^b\,\overline{z'}^c\,\overline{\zeta'}^d$ with either (a,b) or $(c,d) \notin \{(0,0),(1,0)\}$.

The terms $F^{(2)}_{a,b,c,0}$ such that $2 \leqslant a+b+c \leqslant 5$, $(a,b),(c,0) \notin \{(0,0),(1,0)\}$

can be solved in terms of $F^{(1)}_{a,b,c,d}$:

$$
\begin{aligned}
F^{(2)}_{0,1,2,0} &= F^{(1)}_{0,1,2,0},\\
F^{(2)}_{0,1,3,0} &= -3F^{(1)}_{0,1,2,0}F^{(1)}_{1,0,2,0}+F^{(1)}_{0,1,3,0},\\
F^{(2)}_{0,1,4,0} &= 15F^{(1)}_{0,1,2,0}(F^{(1)}_{1,0,2,0})^2-4F^{(1)}_{0,1,2,0}F^{(1)}_{1,0,3,0}-6F^{(1)}_{0,1,3,0}F^{(1)}_{1,0,2,0}+F^{(1)}_{0,1,4,0},\\
F^{(2)}_{0,2,2,0} &= -F^{(1)}_{0,2,1,0}F^{(1)}_{1,0,2,0}+F^{(1)}_{0,2,2,0},\\
F^{(2)}_{0,2,3,0} &= 3F^{(1)}_{0,2,1,0}(F^{(1)}_{1,0,2,0})^2-F^{(1)}_{0,2,1,0}F^{(1)}_{1,0,3,0}-3F^{(1)}_{0,2,2,0}F^{(1)}_{1,0,2,0}+F^{(1)}_{0,2,3,0},\\
F^{(2)}_{0,3,2,0} &= 3F^{(1)}_{0,2,1,0}F^{(1)}_{1,0,2,0}F^{(1)}_{1,1,1,0}-3F^{(1)}_{0,2,1,0}F^{(1)}_{1,1,2,0}-F^{(1)}_{0,3,1,0}F^{(1)}_{1,0,2,0}+F^{(1)}_{0,3,2,0},\\
F^{(2)}_{1,1,2,0} &= -F^{(1)}_{1,0,2,0}F^{(1)}_{1,1,1,0}+F^{(1)}_{1,1,2,0},\\
F^{(2)}_{1,1,3,0} &= 3(F^{(1)}_{1,0,2,0})^2F^{(1)}_{1,1,1,0}-3F^{(1)}_{1,0,2,0}F^{(1)}_{1,1,2,0}-F^{(1)}_{1,0,3,0}F^{(1)}_{1,1,1,0}+F^{(1)}_{1,1,3,0},\\
F^{(2)}_{1,2,2,0} &= F^{(1)}_{0,2,1,0}F^{(1)}_{1,0,2,0}F^{(1)}_{2,0,1,0}+2F^{(1)}_{1,0,2,0}(F^{(1)}_{1,1,1,0})^2-F^{(1)}_{0,2,1,0}F^{(1)}_{2,0,2,0}\\
&\quad -F^{(1)}_{1,0,2,0}F^{(1)}_{1,2,1,0}-2F^{(1)}_{1,1,1,0}F^{(1)}_{1,1,2,0}+F^{(1)}_{1,2,2,0},\\
F^{(2)}_{2,0,2,0} &= -F^{(1)}_{1,0,2,0}F^{(1)}_{2,0,1,0}+F^{(1)}_{2,0,2,0},\\
F^{(2)}_{2,0,3,0} &= 3(F^{(1)}_{1,0,2,0})^2F^{(1)}_{2,0,1,0}-3F^{(1)}_{1,0,2,0}F^{(1)}_{2,0,2,0}-F^{(1)}_{1,0,3,0}F^{(1)}_{2,0,1,0}+F^{(1)}_{2,0,3,0},\\
F^{(2)}_{2,1,2,0} &= 3F^{(1)}_{1,0,2,0}F^{(1)}_{1,1,1,0}F^{(1)}_{2,0,1,0}-F^{(1)}_{1,0,2,0}F^{(1)}_{2,1,1,0}-2F^{(1)}_{1,1,1,0}F^{(1)}_{2,0,2,0}\\
&\quad -F^{(1)}_{1,1,2,0}F^{(1)}_{2,0,1,0}+F^{(1)}_{2,1,2,0},\\
F^{(2)}_{3,0,2,0} &= 3F^{(1)}_{1,0,2,0}(F^{(1)}_{2,0,1,0})^2-F^{(1)}_{1,0,2,0}F^{(1)}_{3,0,1,0}-3F^{(1)}_{2,0,1,0}F^{(1)}_{2,0,2,0}+F^{(1)}_{3,0,2,0}.
\end{aligned}
$$

Next, define:

$$\mathscr{H}_5^{(2)} := \big\{u = F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \mathrm{O}(3)\colon\ F^{(2)}_{a,b,1,0} = 0\ \forall\,(a,b)\neq(1,0)\big\},$$

a codimension 24 submanifold of $\mathscr{H}_5^{(1)}$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(2)} = 47 - 24 = 23$.

It will be a bit strange to talk about stabilizer group in this step. We in fact need to introduce a new definition of stabilizer. But after the final step, we will recover the stabilizer in the standard sense.

Definition 4.6. For any fixed element $F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) \in \mathscr{H}_5^{(2)}$, the subset of $G^{(1)}_{0,4}$ consisting of elements f, g, ρ which send $F^{(2)}(z,\zeta,\overline{z},\overline{\zeta})$ to another element in $\mathscr{H}_5^{(2)}$, is defined as $G^{(2)}_{0,4}(F^{(2)})$. It depends on the choice of the original element $F^{(2)}$.

The stabilizer $G_4^{(2)}(F^{(2)})$ is a codimension 24 subgroup of $G_4^{(1)}$ hence $\dim_{\mathbb{R}} G_4^{(2)}(F^{(2)}) = 54 - 24 = 30$. It contains elements $(f,g,\rho) = \big(r\,e^{i\,\theta}\,z + \mathrm{O}(2), g, r^2\big) \in G_4^{(1)}$ such that:

$$
\begin{aligned}
&f_{2,0} = -r\,e^{i\,\theta}\,F^{(2)}_{2,0,0,1}\,\overline{g}_{1,0}\,\overline{g}_{0,1}^{-1},\\
&f_{3,0} = -r\,e^{i\,\theta}\,F^{(2)}_{3,0,0,1}\,\overline{g}_{1,0}\,\overline{g}_{0,1}^{-1},\\
&f_{4,0} = -r\,e^{i\,\theta}\,F^{(2)}_{4,0,0,1}\,\overline{g}_{1,0}\,\overline{g}_{0,1}^{-1},\\
&f_{0,2} = 0, f_{1,1}=0, f_{0,3}=0, f_{1,2}=0, f_{2,1}=0, f_{0,4}=0, f_{1,3}=0, f_{2,2}=0, f_{3,1} = 0,
\end{aligned}
$$

which are in total 12 conditions on complex coefficients.

4.7. Third normalization: $F^{(3)}_{2,0,0,1} = \overline{F^{(3)}_{0,1,2,0}} = 1$

Any element in $\mathscr{H}_5^{(2)}$ has expansion:

$$F^{(2)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \frac{F^{(2)}_{2,0,0,1}}{2}\,z^2\,\overline{\zeta} + \frac{\overline{F^{(2)}_{2,0,0,1}}}{2}\,\overline{z}^2\,\zeta + \mathrm{O}(4).$$

By 2-non-degeneracy $F^{(2)}_{2,0,0,1} \neq 0$. So after the rigid transformation:

$$z' = z, \qquad \zeta' = \overline{F^{(2)}_{2,0,0,1}}\,\zeta = F^{(2)}_{0,1,2,0}\,\zeta, \qquad w' = w,$$

this element becomes a graph $u = F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \frac{1}{2}z^2\,\overline{\zeta} + \frac{1}{2}\overline{z}^2\,\zeta + \mathrm{O}(4)$. The relations are $F^{(3)}_{a,b,c,0} = F^{(2)}_{a,b,c,0}\,(F^{(2)}_{0,1,2,0})^{-b}$.

Next, define:

$$\mathscr{H}_5^{(3)} := \big\{u = F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} + \tfrac{1}{2}z^2\,\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\,\zeta + \mathrm{O}(4)\colon \\ F^{(3)}_{a,b,1,0} = 0\ \forall\,(a,b) \neq (1,0)\big\},$$

a codimension 2 submanifold of $\mathscr{H}_5^{(2)}$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(3)} = 23 - 2 = 21$.

For any fixed element $F^{(3)} \in \mathscr{H}_5^{(3)}$, there exists some $F^{(2)} \in \mathscr{H}_5^{(2)}$ whose third normalization is equal to $F^{(3)}$. For example, we can take $F^{(2)} = F^{(3)}$. The stabilizer $G_4^{(3)}(F^{(3)})$ is a codimension 2 subgroup of $G_4^{(2)}(F^{(3)})$. Hence $\dim_{\mathbb{R}} G_4^{(3)}(F^{(3)}) = 30 - 2 = 28$. It contains elements $(f,g,\rho) \in G_4^{(2)}(F^{(3)})$

satisfying $g_{0,1} = e^{2\,i\,\theta}$, *i.e.*:

$$\begin{aligned} f(z,\zeta) &= r\,e^{i\theta}\,z - \tfrac{1}{2}\,r\,e^{3\,i\,\theta}\,\overline{g_{1,0}}\,z^2 - \tfrac{1}{6}\,r\,e^{3\,i\,\theta}\,F^{(3)}_{3,0,0,1}\,\overline{g_{1,0}}\,z^3 \\ &\qquad - \tfrac{1}{24}\,r\,e^{3\,i\,\theta}\,F^{(3)}_{4,0,0,1}\,\overline{g_{1,0}}\,z^4 \\ g(z,\zeta) &= g_{1,0}\,z + e^{2\,i\,\theta}\,\zeta + \mathrm{O}(2), \qquad \rho = r^2. \end{aligned}$$

4.8. Fourth normalization: $F^{(4)}_{2,0,2,0} = 0$

Any element in $\mathscr{H}^{(3)}_5$ has expansion:

$$\begin{aligned} F^{(3)}(z,\zeta,\overline{z},\overline{\zeta}) &= z\,\overline{z} + \tfrac{1}{2}z^2\,\overline{\zeta} + \tfrac{1}{2}\overline{z}^2\,\zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}z^2\,\overline{z}^2 + R(z,\zeta,\overline{z},\overline{\zeta}) \\ &= z\,\overline{z} + \tfrac{1}{2}z^2\,\underbrace{\big(\overline{\zeta} + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,\overline{z}^2\big)}_{=:\overline{\zeta}'} + \tfrac{1}{2}\overline{z}^2\,\underbrace{\big(\zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z^2\big)}_{=:\zeta'} + R(z,\zeta,\overline{z},\overline{\zeta}), \end{aligned}$$

whose remainder $R(z,\zeta,\overline{z},\overline{\zeta}) = \mathrm{O}(4)$ contains no $z^2\,\overline{z}^2$ term. After the rigid transformation in $G^{(3)}_4$:

$$z' = z, \qquad \zeta' = \zeta + \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z^2, \qquad w' = w, \tag{4.9}$$

the polynomial $F^{(3)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes $F^{(4)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} + \frac{1}{2}z'^2\,\overline{\zeta'} + \frac{1}{2}\overline{z'}^2\,\zeta' + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. The inverse of (4.9) is:

$$z = z', \qquad \zeta = \zeta' - \tfrac{1}{4}F^{(3)}_{2,0,2,0}\,z'^2, \qquad w = w'.$$

So $R'(z',\zeta',\overline{z'},\overline{\zeta'}) = R\big(z',\zeta' - \frac{1}{4}F^{(3)}_{2,0,2,0}\,z'^2, \overline{z'}, \overline{\zeta'} - \frac{1}{4}F^{(3)}_{2,0,2,0}\,\overline{z'}^2\big) = \mathrm{O}(4)$ without $z'^2\,\overline{z'}^2$ term.

The relations are:

$$\begin{aligned} &F^{(4)}_{0,1,3,0} = F^{(3)}_{0,1,3,0}, \quad F^{(4)}_{0,2,2,0} = F^{(3)}_{0,2,2,0}, \quad F^{(4)}_{1,1,2,0} = F^{(3)}_{1,1,2,0}, \quad F^{(4)}_{0,1,4,0} = F^{(3)}_{0,1,4,0}, \\ &F^{(4)}_{0,2,3,0} = F^{(3)}_{0,2,3,0}, \quad F^{(4)}_{0,3,2,0} = F^{(3)}_{0,3,2,0}, \quad F^{(4)}_{1,2,2,0} = F^{(3)}_{1,2,2,0}, \\ &F^{(4)}_{2,1,2,0} = -\tfrac{1}{2}F^{(3)}_{0,2,2,0}\,F^{(3)}_{2,0,2,0} + F^{(3)}_{2,1,2,0}, \\ &F^{(4)}_{3,0,2,0} = -\tfrac{3}{2}F^{(3)}_{1,1,2,0}\,F^{(3)}_{2,0,2,0} - \tfrac{1}{2}F^{(3)}_{3,0,0,1}\,F^{(3)}_{2,0,2,0} + F^{(3)}_{3,0,2,0}, \\ &F^{(4)}_{1,1,3,0} = -\tfrac{3}{2}F^{(3)}_{2,0,2,0} + F^{(3)}_{1,1,3,0}, \\ &F^{(4)}_{2,0,3,0} = -\tfrac{1}{2}F^{(3)}_{0,1,3,0}\,F^{(3)}_{2,0,2,0} - \tfrac{3}{2}F^{(3)}_{2,0,1,1}\,F^{(3)}_{2,0,2,0} + F^{(3)}_{2,0,3,0}. \end{aligned}$$

We define $\mathscr{H}_5^{(4)}$, a codimension 1 submanifold of $\mathscr{H}_5^{(3)}$ by requiring $F_{2,0,2,0}^{(4)} = 0$. So $\dim_{\mathbb{R}} \mathscr{H}_5^{(2)} = 21 - 1 = 20$.

For any fixed element $F^{(4)} \in \mathscr{H}_5^{(4)}$, the stabilizer $G_4^{(4)}(F^{(4)})$ is a codimension 1 subgroup of some $G_4^{(3)}(F^{(4)})$. Hence $\dim_{\mathbb{R}} G_4^{(4)}(F^{(3)}) = 28 - 1 = 27$. It contains elements $(f, g, \rho) \in G_4^{(3)}(F^{(4)})$ satisfying:

$$
\begin{aligned}
g_{2,0} = e^{-2i\theta} F_{0,2,2,0}^{(4)} g_{1,0}^2 + e^{6i\theta} F_{2,0,0,2}^{(4)} \overline{g_{1,0}}^2 - e^{-4i\theta} g_{0,2}\, g_{1,0}^2 - e^{8i\theta} \overline{g_{0,2}}\, \overline{g_{1,0}}^2 \\
- 2 F_{1,1,2,0}^{(4)} g_{1,0} - 2 e^{4i\theta} F_{2,0,1,1}^{(4)} \overline{g_{1,0}} + 2 e^{-2i\theta} g_{1,0}\, g_{1,1} + 2 e^{6i\theta} \overline{g_{1,0}}\, \overline{g_{1,1}} \\
+ 3 e^{2i\theta} g_{1,0} \overline{g_{1,0}} - e^{4i\theta} \overline{g_{2,0}}.
\end{aligned}
$$

In other words

$$
\begin{aligned}
\mathrm{Re}\,\big(e^{-2i\theta} g_{2,0}\big) = \mathrm{Re}\, \big\{ - e^{-4i\theta} F_{0,2,2,0}^{(4)} g_{1,0}^2 - e^{-6i\theta} g_{0,2}\, g_{1,0}^2 \\
- 2 e^{-2i\theta} F_{1,1,2,0}^{(4)} g_{1,0} + 2 e^{-4i\theta} g_{1,0}\, g_{1,1} + \tfrac{3}{2} g_{1,0} \overline{g_{1,0}} \big\}.
\end{aligned}
$$

4.10. Fifth normalization: $F_{a,b,2,0}^{(5)} = 0$ for $2 \leqslant a + b \leqslant 3$ and $(a, b) \neq (2, 0)$

Any element in $\mathscr{H}_5^{(4)}$ has expansion:

$$
\begin{aligned}
F^{(4)}(z, \zeta, \overline{z}, \overline{\zeta}) = z\,\overline{z} + \tfrac{1}{2} z^2 \underbrace{\Big(\overline{\zeta} + \sum_{2 \leqslant a+b \leqslant 3} \tfrac{F_{2,0,a,b}^{(4)}}{a!\,b!}\, \overline{z}^a\, \overline{\zeta}^b\Big)}_{=:\, \overline{\zeta'}} \\
+ \tfrac{1}{2} \overline{z}^2 \underbrace{\Big(\zeta + \sum_{2 \leqslant a+b \leqslant 3} \tfrac{F_{a,b,2,0}^{(4)}}{a!\,b!}\, z^a\, \zeta^b\Big)}_{=:\, \zeta'} + R\big(z, \zeta, \overline{z}, \overline{\zeta}\big),
\end{aligned}
$$

whose remainder $R(z, \zeta, \overline{z}, \overline{\zeta}) = \mathrm{O}(4)$ contains no $z^a \zeta^b \overline{z}^2$ term for any $2 \leqslant a + b \leqslant 3$. After the rigid transformation in $G_4^{(4)}(F^{(4)})$:

$$
z' = z, \qquad \zeta' = \zeta + \sum_{2 \leqslant a+b \leqslant 4} \frac{F_{a,b,2,0}^{(4)}}{a!\,b!}\, z^a\, \zeta^b, \qquad w' = w, \tag{4.11}
$$

the polynomial $F^{(4)}(z, \zeta, \overline{z}, \overline{\zeta})$ becomes $F^{(5)}(z', \zeta', \overline{z'}, \overline{\zeta'}) = z'\, \overline{z'} + \frac{1}{2} z'^2\, \overline{\zeta'} +$

$\frac{1}{2}\overline{z'}^2\,\zeta' + R'(z',\zeta',\overline{z'},\overline{\zeta'})$. The inverse of (4.11) is:

$$z = z', \qquad \zeta = \zeta' + \mathrm{O}_{z',\zeta'}(2), \qquad w = w'.$$

So $R'(z',\zeta',\overline{z'},\overline{\zeta'}) = R\big(z',\zeta' + \mathrm{O}_{z',\zeta'}(2),\overline{z'},\overline{\zeta'} + \mathrm{O}_{\overline{z'},\overline{\zeta'}}(2)\big) = \mathrm{O}(4)$ without $z'^a\,\zeta'^b\,\overline{z'}^2$ terms for any $2 \leqslant a+b \leqslant 3$.

The relations are:

$$F^{(5)}_{0,1,3,0} = F^{(4)}_{0,1,3,0}, \qquad F^{(5)}_{0,1,4,0} = F^{(4)}_{0,1,4,0},$$
$$F^{(5)}_{0,2,3,0} = -2F^{(4)}_{0,1,3,0}F^{(4)}_{0,2,2,0} + F^{(4)}_{0,2,3,0}, \quad F^{(5)}_{1,1,3,0} = -2F^{(4)}_{0,1,3,0}F^{(4)}_{1,1,2,0} + F^{(4)}_{1,1,3,0}.$$

We define $\mathscr{H}_5^{(5)}$ a codimension 12 submanifold of $\mathscr{H}_5^{(4)}$ where $F^{(5)}_{a,b,2,0} = 0$ for:

$$(a,b) \in \big\{(1,1),(0,2),(3,0),(2,1),(1,2),(0,3)\big\}.$$

So $\dim_{\mathbb{R}} \mathscr{H}_5^{(5)} = 20 - 12 = 8$.

For any fixed element $F^{(5)} \in \mathscr{H}_5^{(5)}$, the stabilizer $G_4^{(5)}(F^{(5)})$ is a codimension 12 subgroup of some $G_4^{(4)}(F^{(5)})$. Hence $\dim_{\mathbb{R}} G_4^{(5)}(F^{(5)}) = 27-12 = 15$. It contains element $(f,g,\rho) \in G_4^{(4)}(F^{(5)})$ satisfying:

$$\begin{aligned}
g_{0,2} &= 0, \qquad g_{1,1} = -2\,e^{4\,i\,\theta}\,\overline{g_{1,0}}, \qquad g_{0,3} = 0, \qquad g_{1,2} = 0,\\
g_{2,1} &= 2\,e^{6\,i\,\theta}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}},\\
g_{3,0} &= -\,5\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}}\,g_{1,0} + e^{6\,i\,\theta}\,F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,1,1}\,\overline{g_{1,0}}\\
&\quad -\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{2,0}}.
\end{aligned}$$

Since $(f,g,\rho) \in G_4^{(4)}(F^{(5)})$ we have

$$\mathrm{Re}\,\big(e^{-2\,i\,\theta}\,g_{2,0}\big) = \mathrm{Re}\,\big(-\tfrac{5}{2}\,g_{1,0}\,\overline{g_{1,0}}\big).$$

Thus $e^{-2\,i\,\theta}\,g_{2,0} = -\frac{5}{2}\,g_{1,0}\,\overline{g_{1,0}} + i\,b_{2,0}$ for some $b_{2,0} \in \mathbb{R}$. So the last equation becomes

$$\begin{aligned}
g_{3,0} &= -\,\tfrac{5}{2}\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,\overline{g_{1,0}}\,g_{1,0} + e^{6\,i\,\theta}\,F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}}^2 - 2\,e^{4\,i\,\theta}\,F^{(5)}_{3,0,1,1}\,\overline{g_{1,0}}\\
&\quad -\,i\,e^{2\,i\,\theta}\,F^{(5)}_{3,0,0,1}\,b_{2,0}.
\end{aligned}$$

The stabilizer $G_4^{(5)}(F^{(5)})$ is parametrized by 3 real variables $b_{2,0}, r, \theta$ and 6 complex variables $g_{1,0}, g_{j,4-j}$ for $0 \leqslant j \leqslant 4$.

4.12. Final normalization: $F^{(6)}_{0,1,3,0} = 0$ and $\mathsf{Im}\, F^{(6)}_{1,1,3,0} = 0$

Any element in $\mathscr{H}_5^{(5)}$ has expansion:

$$F^{(5)}(z,\zeta,\overline{z},\overline{\zeta}) = z\overline{z} + \tfrac{1}{2}z^2\overline{\zeta} + \tfrac{1}{2}\zeta\overline{z}^2 + \tfrac{1}{6}F^{(5)}_{0,1,3,0}\zeta\overline{z}^3 + \tfrac{1}{6}F^{(5)}_{3,0,0,1}z^3\overline{\zeta}$$
$$+ \tfrac{1}{6}F^{(5)}_{1,1,3,0}z\zeta\overline{z}^3 + \tfrac{1}{6}F^{(5)}_{3,0,1,1}z^3\overline{z}\overline{\zeta} + \tfrac{1}{24}F^{(5)}_{0,1,4,0}\zeta\overline{z}^4$$
$$+ \tfrac{1}{24}F^{(5)}_{4,0,0,1}z^4\overline{\zeta} + \tfrac{1}{12}F^{(5)}_{0,2,3,0}\zeta^2\overline{z}^3 + \tfrac{1}{12}F^{(5)}_{3,0,0,2}z^3\overline{\zeta}^2 + \zeta\overline{\zeta}(\cdots).$$

We study how $g_{1,0}$ and $b_{2,0}$ act on this object, *i.e.* we consider an arbitrary $(f, g, \rho) \in G_4^{(5)}(F^{(5)})$ with $r = 1$ and $\theta = g_{j,4-j} = 0$. They have the form:

$$\begin{aligned}
f(z,\zeta) &= z - \tfrac{1}{2}\,\overline{g_{1,0}}\,z^2 + \mathrm{O}(3),\\
g(z,\zeta) &= g_{1,0}\,z + \zeta + \tfrac{1}{2}\,(-\tfrac{5}{2}\,g_{1,0}\,\overline{g_{1,0}} + i\,b_{2,0})\,z^2 + \mathrm{O}(3),\\
\rho &= 1.
\end{aligned}$$

This transformation sends $F^{(5)}$ to $F'^{(5)} \in \mathscr{H}_5^{(5)}$ such that:

$$\begin{aligned}
F'^{(5)}_{3,0,0,1} &= F^{(5)}_{3,0,0,1} + 3\,\overline{g_{1,0}},\\
F'^{(5)}_{3,0,1,1} &= F^{(5)}_{3,0,1,1} - 3\,F^{(5)}_{3,0,0,1}\,g_{1,0} - F^{(5)}_{3,0,0,2}\,\overline{g_{1,0}} + \tfrac{15}{2}\,g_{1,0}\,\overline{g_{1,0}} - 3\,i\,b_{2,0}.
\end{aligned}$$

So by a unique choice of $g_{1,0}$ and $b_{2,0}$, namely:

$$g_{1,0} = -\tfrac{1}{3}F^{(5)}_{0,1,3,0},\ b_{2,0} = \tfrac{i}{18}\big(F^{(5)}_{0,2,3,0}\,F^{(5)}_{0,1,3,0} - F^{(5)}_{3,0,0,2}F^{(5)}_{3,0,0,1} + 3F^{(5)}_{1,1,3,0} - 3F^{(5)}_{3,0,1,1}\big),$$

we can normalize $F'^{(5)}_{3,0,0,1}$ to 0 and $F'^{(5)}_{3,0,1,1}$ to a real number. The polynomial $F^{(5)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes:

$$\begin{aligned}
&F^{(6)}(z',\zeta',\overline{z'},\overline{\zeta'})\\
&= z'\overline{z'} + \tfrac{1}{2}z'^2\overline{\zeta'} + \tfrac{1}{2}\zeta'\overline{z'}^2 + \tfrac{1}{6}F^{(6)}_{1,1,3,0}z'\zeta'\overline{z'}^3 + \tfrac{1}{6}F^{(6)}_{3,0,1,1}z'^3\overline{z'}\overline{\zeta'} + \tfrac{1}{24}F^{(6)}_{0,1,4,0}\zeta'\overline{z'}^4\\
&\quad + \tfrac{1}{24}F^{(6)}_{4,0,0,1}z'^4\overline{\zeta'} + \tfrac{1}{12}F^{(6)}_{0,2,3,0}\zeta'^2\overline{z'}^3 + \tfrac{1}{12}F^{(6)}_{3,0,0,2}z'^3\overline{\zeta'}^2 + \zeta'\overline{\zeta'}(\cdots)\\
&= z'\overline{z'} + \tfrac{1}{2}z'^2\overline{\zeta'} + \tfrac{1}{2}\zeta'\overline{z'}^2 + \tfrac{1}{6}\boldsymbol{Q}_0 z'\zeta'\overline{z'}^3 + \tfrac{1}{6}\boldsymbol{Q}_0 z'^3\overline{z'}\overline{\zeta'}\\
&\quad + \tfrac{1}{24}\boldsymbol{V}_0\zeta'\overline{z'}^4 + \tfrac{1}{24}\overline{\boldsymbol{V}}_0 z'^4\overline{\zeta'} + \tfrac{1}{12}\boldsymbol{I}_0\zeta'^2\overline{z'}^3 + \tfrac{1}{12}\overline{\boldsymbol{I}}_0 z'^3\overline{\zeta'}^2 + \zeta'\overline{\zeta'}(\cdots),
\end{aligned}$$

where $\boldsymbol{I}_0 := F^{(6)}_{0,2,3,0} \in \mathbb{C}$, $\boldsymbol{V}_0 := F^{(6)}_{0,1,4,0} \in \mathbb{C}$, $\boldsymbol{Q}_0 := F^{(6)}_{1,1,3,0} \in \mathbb{R}$.

The relations are:

$$\begin{aligned}
\boldsymbol{I}_0 &= F^{(5)}_{0,2,3,0} + 2\,F^{(5)}_{3,0,0,1},\\
\boldsymbol{V}_0 &= -\tfrac{5}{3}\,(F^{(5)}_{0,1,3,0})^2 + F^{(5)}_{0,1,4,0},\\
\boldsymbol{Q}_0 &= \tfrac{1}{6}F^{(5)}_{0,2,3,0}F^{(5)}_{0,1,3,0}+\tfrac{1}{2}F^{(5)}_{3,0,0,1}F^{(5)}_{0,1,3,0}+\tfrac{1}{6}F^{(5)}_{3,0,0,2}F^{(5)}_{3,0,0,1}+\tfrac{1}{2}F^{(5)}_{1,1,3,0}+\tfrac{1}{2}F^{(5)}_{3,0,1,1}.
\end{aligned}$$

We define $\mathcal{N} = \mathscr{H}^{(6)}_5$ a codimension 3 submanifold of $\mathscr{H}^{(5)}_5$ by requiring $F^{(6)}_{0,3,1,0} = 0$ and $\mathsf{Im}\,(F^{(6)}_{1,1,3,0}) = 0$.

For any fixed element $F^{(6)} \in \mathcal{N}$, the stabilizer $G^{(6)}_4(F^{(6)})$ is a codimension 3 subgroup of some $G^{(5)}_4(F^{(6)})$. Hence $\dim_{\mathbb{R}} G^{(6)}_4(F^{(6)}) = 15 - 3 = 12$. It contains elements $(f, g, \rho) \in G^{(5)}_4(F^{(6)})$ of the form:

$$f(z,\zeta) = r\,e^{i\,\theta}\,z, \qquad g(z,\zeta) = e^{2\,i\,\theta}\,s + \mathrm{O}(4), \qquad \rho = r^2.$$

This group sends $\boldsymbol{I}_0$, $\boldsymbol{V}_0$, $\boldsymbol{Q}_0$ to $\boldsymbol{I}_0'$, $\boldsymbol{V}_0'$, $\boldsymbol{Q}_0'$ with relations:

$$\boldsymbol{I}_0' = r^{-1}\,e^{-i\,\theta}\,\boldsymbol{I}_0, \qquad \boldsymbol{V}_0' = r^{-2}\,e^{2\,i\,\theta}\,\boldsymbol{V}_0, \qquad \boldsymbol{Q}_0' = r^{-2}\,\boldsymbol{Q}_0$$

So if we ignore dilations and rotations $(z', \zeta', w') = (r\,e^{i\,\theta}\,z, e^{2\,i\,\theta}\,\zeta, r^2\,w)$, then $\boldsymbol{I}_0$, $\boldsymbol{V}_0$, $\boldsymbol{Q}_0$ are invariants.

Each $F^{(t)}_{a,b,c,d}$ is a rational function of $F^{(t-1)}_{a',b',c',d'}$ for $t = 5, 4, 3, 2$ and each $F^{(1)}_{a,b,c,d}$ is a rational function of $F_{a',b',c',d'}$. By composing these rational functions, one can express $\boldsymbol{I}_0$, $\boldsymbol{V}_0$, $\boldsymbol{Q}_0$ in terms of original coordinates $F_{a,b,c,d}$:

$$\begin{aligned}
\boldsymbol{I}_0 &= \frac{\text{52 terms in degree 9}}{F^{3/2}_{1,0,1,0}(F_{0,1,1,0}F_{1,0,2,0}-F_{0,1,2,0}F_{1,0,1,0})^3(F_{1,0,0,1}F_{2,0,1,0}-F_{1,0,1,0}F_{2,0,0,1})},\\
\boldsymbol{V}_0 &= \frac{\text{11 terms in degree 4}}{3F_{1,0,1,0}(F_{0,1,1,0}F_{1,0,2,0} - F_{0,1,2,0}F_{1,0,1,0})^2},\\
\boldsymbol{Q}_0 &= \frac{\text{824 terms in degree 18}}{6F^3_{1,0,1,0}(F_{0,1,1,0}F_{1,0,2,0}-F_{0,1,2,0}F_{1,0,1,0})^4(F_{1,0,0,1}F_{2,0,1,0}-F_{1,0,1,0}F_{2,0,0,1})^4}.
\end{aligned}$$

The numerator of $\boldsymbol{I}_0$ is shown in [4], and the numerator of $\boldsymbol{V}_0$ is:

$$\begin{aligned}
&3F^2_{0,1,1,0}F_{1,0,2,0}F_{1,0,4,0} - 5F^2_{0,1,1,0}F^2_{1,0,3,0} - 3F_{0,1,1,0}F_{0,1,2,0}F_{1,0,1,0}F_{1,0,4,0}\\
&\quad + 12F_{0,1,1,0}F_{0,1,2,0}F_{1,0,2,0}F_{1,0,3,0} + 10F_{0,1,1,0}F_{0,1,3,0}F_{1,0,1,0}F_{1,0,3,0}
\end{aligned}$$

$$- 12F_{0,1,1,0}F_{0,1,3,0}F^2_{1,0,2,0} - 3F_{0,1,1,0}F_{0,1,4,0}F_{1,0,1,0}F_{1,0,2,0}$$
$$- 12F^2_{0,1,2,0}F_{1,0,1,0}F_{1,0,3,0} + 12F_{0,1,2,0}F_{0,1,3,0}F_{1,0,1,0}F_{1,0,2,0}$$
$$+ 3F_{0,1,2,0}F_{0,1,4,0}F^2_{1,0,1,0} - 5F^2_{0,1,3,0}F^2_{1,0,1,0}$$

We define $\mathscr{H}_5^{(6)}$ a codimension 3 submanifold of $\mathscr{H}_5^{(5)}$ by requiring $F^{(6)}_{0,3,1,0} = 0$ and $Im\, F^{(6)}_{1,1,3,0} = 0$.

For any fixed element $F^{(6)} \in \mathscr{H}_5^{(6)}$, the stabilizer $G_4^{(6)}(F^{(6)})$ is a codimension 3 subgroup of some $G_4^{(5)}(F^{(6)})$. Hence $\dim_{\mathbb{R}} G_4^{(6)}(F^{(6)}) = 15 - 3 = 12$. It contains elements $(f, g, \rho) \in G_4^{(5)}(F^{(6)})$ of the form:

$$f(z,\zeta) = r\, e^{i\,\theta}\, z, \qquad g(z,\zeta) = e^{2\,i\,\theta}\,\zeta + \mathrm{O}(4), \qquad \rho = r^2.$$

Note that this stabilizer group no longer depends on the choice of $F^{(6)} \in \mathscr{H}_5^{(6)}$. We simply write it as $G_4^{(6)}$.

4.13. Passing to the infinite dimension

After these six normalizations, we have killed $f_{0,1}$ and $g_{1,0}$. It is a miracle that now we can work directly on the infinite-dimensional objects. We define $\mathscr{H}^{(7)}$ to be the subspace of $\mathscr{H}$ consisting of all power series $u = F^{(7)}(z,\zeta,\overline{z},\overline{\zeta}) = \frac{F^{(7)}_{a,b,c,d}}{a!b!c!d!} z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d$ such that:

- $F^{(7)}_{a,b,1,0} = 0$, $\forall (a,b) \neq (1,0)$; $F^{(7)}_{1,0,1,0} = 1$;
- $F^{(7)}_{a,b,2,0} = 0$, $\forall (a,b) \neq (0,1)$; $F^{(7)}_{0,1,2,0} = 1$;
- $F^{(7)}_{3,0,0,1} = 0$, $F^{(7)}_{3,0,1,1} = F^{(7)}_{1,1,3,0}$.

It is both infinite-dimensional and infinite-codimensional in $\mathscr{H}$, but it has a finite-dimensional stabilizer.

By definition, any element in $\mathscr{H}^{(7)}$ has its degree 5 truncation in $\mathscr{H}_5^{(6)}$.

Theorem 4.14. *Any element $u = F(z,\zeta,\overline{z},\overline{\zeta})$ in $\mathscr{H}$ can be sent to some element $u = F^{(7)}(z,\zeta,\overline{z},\overline{\zeta})$ in $\mathscr{H}^{(7)}$ by some (but not unique) element in G. The ambiguity can be controlled in the following sense: any element $(f,g,\rho) \in G$ sending one element $F^{(7)} \in \mathscr{H}^{(7)}$ to another $F'^{(7)} \in \mathscr{H}^{(7)}$ has the form $f(z,\zeta) = r\, e^{i\,\theta}\, z$, $g(z,\zeta) = e^{2\,i\,\theta}\,\zeta$, $\rho = r^2$.*

Proof. One shall simply use the six normalizations above with a slight modification: in the second (killing $F_{a,b,1,0}$) and the fifth (killing $F_{a,b,2,0}$) normalization, we normalize for infinitely many (a,b). More precisely, we start from $u = F(z,\zeta,\overline{z},\overline{\zeta})$ in $\mathscr{H}$. After the six normalizations above we get $u = F^{(6)}(z,\zeta,\overline{z},\overline{\zeta})$ whose degree 5 truncation $\pi_5\big(F^{(6)}(z,\zeta,\overline{z},\overline{\zeta})\big)$ is in $\mathscr{H}_5^{(6)}$, *i.e.*:

- $F^{(6)}_{a,b,1,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F^{(6)}_{1,0,1,0} = 1$;
- $F^{(6)}_{a,b,2,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F^{(6)}_{0,1,2,0} = 1$;
- $F^{(6)}_{3,0,0,1} = 0$, $F^{(6)}_{3,0,1,1} = F^{(6)}_{1,1,3,0}$.

Then we do 2 more normalizations. First:

$$z' = z + \sum_{a+b\geqslant 5} \tfrac{F^{(6)}_{a,b,1,0}}{a!b!}\, z^a\, \zeta^b, \qquad \zeta' = \zeta, \qquad w' = w,$$

gives us $u' = F'(z',\zeta',\overline{z'},\overline{\zeta'})$ with:

- $F'_{a,b,1,0} = 0$, $\forall a+b \geqslant 2$; $F'_{1,0,1,0} = 1$;
- $F'_{a,b,2,0} = 0$, $\forall 2 \leqslant a+b \leqslant 4$; $F'_{0,1,2,0} = 1$;
- $F'_{3,0,0,1} = 0$, $F'_{3,0,1,1} = F'_{1,1,3,0}$.

Then:

$$z'' = z', \qquad \zeta'' = \zeta' + \sum_{a+b\geqslant 5} \tfrac{F'_{a,b,2,0}}{a!b!}\, z^a\, \zeta^b, \qquad w' = w,$$

gives us $u'' = F''(z'',\zeta'',\overline{z''},\overline{\zeta''})$ with:

- $F''_{a,b,1,0} = 0$, $\forall a+b \geqslant 2$; $F''_{1,0,1,0} = 1$;
- $F''_{a,b,2,0} = 0$, $\forall a+b \geqslant 2$; $F''_{0,1,2,0} = 1$;
- $F''_{3,0,0,1} = 0$, $F''_{3,0,1,1} = F''_{1,1,3,0}$.

So $u'' = F''(z'',\zeta'',\overline{z''},\overline{\zeta''})$ is in $\mathscr{H}^{(7)}$. It is the form we want.

Now suppose that $(f,g,\rho) \in G$ sends one element $F^{(7)} \in \mathscr{H}^{(7)}$ to another $F'^{(7)} \in \mathscr{H}^{(7)}$. In the truncated setting, $\pi_4(f,g,\rho) \in G_4$ sends $\pi_5(F^{(7)}) \in \mathscr{H}_5^{(6)}$ to $\pi_5(F'^{(7)}) \in \mathscr{H}_5^{(6)}$. So the truncated action $\pi_4(f,g,\rho)$ should be in the stabilizer $G_4^{(6)}$. That is to say:

$$f(z,\zeta) = r\, e^{i\,\theta}\, z + \mathrm{O}(5), \qquad g(z,\zeta) = e^{2\,i\,\theta}\, \zeta + \mathrm{O}(4), \qquad \rho = r^2.$$

Recall the fundamental equation:

$$\rho\, F^{(7)}(z,\zeta,\overline{z},\overline{\zeta}) = F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big).$$

When we compare the coefficients of $z^j\,\zeta^{n-j}\,\overline{z}$ for any $n \geqslant 2$ and $0 \leqslant j \leqslant n$:

$$\begin{aligned}
0 &= \mathrm{Coef}_{z^j\,\zeta^{n-j}\,\overline{z}}\big\{F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big)\big\} \\
&= \mathrm{Coef}_{z^j\,\zeta^{n-j}\,\overline{z}}\big\{f(z,\zeta)\,\overline{f(z,\zeta)}\big\} + \mathrm{Coef}_{z^j\,\zeta^{n-j}\,\overline{z}}\big\{\sum_{c=0,d=1}(\cdots)\,\overline{g(z,\zeta)}^d\big\} \\
&\quad + \mathrm{Coef}_{z^j\,\zeta^{n-j}\,\overline{z}}\big\{\sum_{c+d\geqslant 2}(\cdots)\,\overline{f(z,\zeta)}^c\,\overline{g(z,\zeta)}^d\big\},
\end{aligned}$$

the last two terms are 0 because they only contain monomials with $\deg_{\overline{z}} = 0$ or $\deg_{\overline{z}} + \deg_{\overline{\zeta}} \geqslant 2$. The first term gives us $0 = r\,e^{-i\,\theta}\,\frac{f_{j,n-j}}{j!(n-j)!}$. Hence $f(z,\zeta) = r\,e^{i\,\theta}\,z$.

When we compare the coefficients of $z^j\,\zeta^{n-j}\,\overline{z}^2$ for any $n \geqslant 2$ and $0 \leqslant j \leqslant n$:

$$\begin{aligned}
0 &= \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{F'^{(7)}\big(f(z,\zeta), g(z,\zeta), \overline{f(z,\zeta)}, \overline{g(z,\zeta)}\big)\big\} \\
&= \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{f(z,\zeta)\overline{f(z,\zeta)}\big\} + \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{\sum_{c=0,d=1}(\cdots)\overline{g(z,\zeta)}\big\} \\
&\quad + \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{\tfrac{1}{2}g(z,\zeta)\overline{f(z,\zeta)}^2\big\} + \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{\sum_{c=1,d=1}(\cdots)\overline{f(z,\zeta)g(z,\zeta)}\big\} \\
&\quad + \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}^2}\big\{\sum_{c=0,d=2}(\cdots)\overline{g(z,\zeta)}^2\big\} + \mathrm{Coef}_{z^j\zeta^{n-j}\overline{z}}\big\{\sum_{c+d\geqslant 3}(\cdots)\overline{f(z,\zeta)}^c\,\overline{g(z,\zeta)}^d\big\},.
\end{aligned}$$

each term, except the third, is 0. The third term gives us $0 = \frac{1}{2}r^2\,\frac{g_{j,n-j}}{j!(n-j)!}$. Hence $g(z,\zeta) = e^{2\,i\,\theta}\,\zeta$. □

5. Branches $I_0 \neq 0$, $V_0 \neq 0$ and $I_0 \equiv 0 \equiv V_0$

To get a normal form under the full rigid transformation group, including rotations and dilations:

$$z' = r\,e^{i\theta}\,z, \qquad \zeta' = e^{2\,i\,\theta}\,\zeta, \qquad \rho = r^2,$$

we should normalize I_0 or V_0. Such a rotation and a dilation would send

$(\boldsymbol{I}_0, \boldsymbol{V}_0, \boldsymbol{Q}_0)$ to $(\boldsymbol{I}_0', \boldsymbol{V}_0', \boldsymbol{Q}_0')$ with:

$$\boldsymbol{I}_0' = r^{-1}\, e^{-i\,\theta}\, \boldsymbol{I}_0, \qquad \boldsymbol{V}_0' = r^{-2}\, e^{2\,i\,\theta}\, \boldsymbol{V}_0, \qquad \boldsymbol{Q}_0' = r^{-2}\, \boldsymbol{Q}_0.$$

We avoid the mixed type and focus on the 3 possible branches:

- $\boldsymbol{I}_0 \neq 0$;
- $\boldsymbol{I}_0 \equiv 0$ but $\boldsymbol{V}_0 \neq 0$;
- $\boldsymbol{I}_0 \equiv 0 \equiv \boldsymbol{V}_0$.

5.1. Branch $\boldsymbol{I}_0 \neq 0$

In this branch we can normalize $\boldsymbol{I}_0$ to 1 by choosing $r\, e^{i\,\theta} = \boldsymbol{I}_0$. More precisely, for any surface in $\mathscr{H}^{(7)}$ graphed by:

$$\begin{aligned} F^{(7)}(z,\zeta,\overline{z},\overline{\zeta}) = z\,\overline{z} &+ \tfrac{1}{2}\, z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\boldsymbol{Q}_0\, z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\boldsymbol{Q}_0\, z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\,\boldsymbol{V}_0\,\zeta\,\overline{z}^4 \\ &+ \tfrac{1}{24}\,\overline{\boldsymbol{V}_0}\, z^4\,\overline{\zeta} + \tfrac{1}{12}\,\boldsymbol{I}_0\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,\overline{\boldsymbol{I}_0}\, z^3\,\overline{\zeta}^2 + \zeta\,\overline{\zeta}\,(\cdots) + \mathrm{O}(6), \end{aligned}$$

where $\boldsymbol{I}_0 \neq 0$, after the transformation

$$z' = \boldsymbol{I}_0\, z, \qquad \zeta' = \frac{\boldsymbol{I}_0^2}{|\boldsymbol{I}_0|^2}\,\zeta, \qquad \rho = |\boldsymbol{I}_0|^2,$$

the polynomial $F^{(7)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes:

$$\begin{aligned} F^{(8,1)}(z',\zeta',\overline{z'},\overline{\zeta'}) = z'\,\overline{z'} &+ \tfrac{1}{2}\, z'^2\,\overline{\zeta'} + \tfrac{1}{2}\,\zeta'\,\overline{z'}^2 + \tfrac{1}{6}\,\tilde{\boldsymbol{Q}}_0\, z'\,\zeta'\,\overline{z'}^3 + \tfrac{1}{6}\,\tilde{\boldsymbol{Q}}_0\, z'^3\,\overline{z'}\,\overline{\zeta'} \\ &+ \tfrac{1}{24}\,\tilde{\boldsymbol{V}}_0\,\zeta'\,\overline{z'}^4 + \tfrac{1}{24}\,\overline{\tilde{\boldsymbol{V}}_0}\, z'^4\,\overline{\zeta'} + \tfrac{1}{12}\,\zeta'^2\,\overline{z'}^3 + \tfrac{1}{12}\, z'^3\,\overline{\zeta'}^2 \\ &+ \zeta'\,\overline{\zeta'}\,(\cdots) + \mathrm{O}(6), \end{aligned}$$

where

$$\tilde{\boldsymbol{V}}_0 = \frac{\boldsymbol{V}_0}{\overline{\boldsymbol{I}_0}^2}, \qquad \tilde{\boldsymbol{Q}}_0 = \frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2},$$

We define $\mathscr{H}^{(8,1)}$ a codimension 2 submanifold of $\mathscr{H}^{(7)}$ by requiring $\boldsymbol{I}_0 = 1$.

For any fixed element $F^{(8,1)} \in \mathscr{H}^{(8,1)}$, the stabilizer $G^{(8,1)}$ is the identity.

5.2. Branch $I_0 \equiv 0$ but $V_0 \neq 0$

In this branch we can normalize V_0 to 1 by choosing $r^2\, e^{-2\,i\,\theta} = V_0$. This equation has two solutions: $r\, e^{i\,\theta} = \pm x$, where $x^2 = \overline{V_0}$ and $\arg(x) \in [0, \pi)$. More precisely, for any surface in $\mathscr{H}^{(7)}$ graphed by:

$$\begin{aligned} F^{(7)}(z,\zeta,\overline{z},\overline{\zeta}) = {} & z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\boldsymbol{Q}_0\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\boldsymbol{Q}_0\,z^3\,\overline{z}\,\overline{\zeta} \\ & + \tfrac{1}{24}\,\boldsymbol{V}_0\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,\overline{\boldsymbol{V}_0}\,z^4\,\overline{\zeta} + \underbrace{\tfrac{1}{12}\,\boldsymbol{I}_0\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,\overline{\boldsymbol{I}_0}\,z^3\,\overline{\zeta}^2}_{=0,\ \text{when } \boldsymbol{I}_0 \equiv 0} \\ & + \zeta\,\overline{\zeta}\,(\cdots) + \mathrm{O}(6), \end{aligned}$$

where $\boldsymbol{V}_0 \neq 0$, after the transformation

$$z' = x\,z, \qquad \zeta' = \frac{\overline{\boldsymbol{V}_0}}{|\boldsymbol{V}_0|}\,\zeta, \qquad \rho = |\boldsymbol{V}_0|,$$

the polynomial $F^{(7)}(z,\zeta,\overline{z},\overline{\zeta})$ becomes

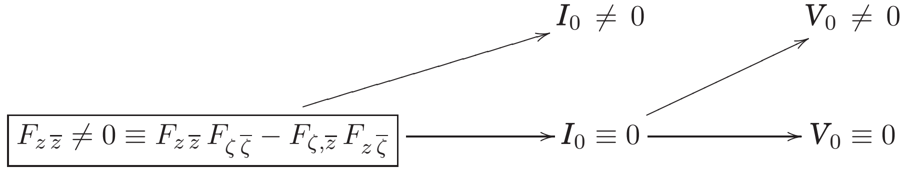
$$\begin{aligned} F^{(8,2)}(z',\zeta',\overline{z'},\overline{\zeta'}) = {} & z'\,\overline{z'} + \tfrac{1}{2}\,z'^2\,\overline{\zeta'} + \tfrac{1}{2}\,\zeta'\,\overline{z'}^2 + \tfrac{1}{6}\,\tilde{\boldsymbol{Q}}_0\,z'\,\zeta'\,\overline{z'}^3 + \tfrac{1}{6}\,\tilde{\boldsymbol{Q}}_0\,z'^3\,\overline{z'}\,\overline{\zeta'} \\ & + \tfrac{1}{24}\,\zeta'\,\overline{z'}^4 + \tfrac{1}{24}\,z'^4\,\overline{\zeta'} + \zeta'\,\overline{\zeta'}\,(\cdots) + \mathrm{O}(6), \end{aligned}$$

where $\tilde{\boldsymbol{Q}}_0 = \frac{\boldsymbol{Q}_0}{|\boldsymbol{V}_0|}$. We define $\mathscr{H}^{(8,2)}$ a codimension 2 submanifold of $\mathscr{H}^{(7)}$ by requiring $\boldsymbol{V}_0 = 1$. For any fixed element $F^{(8,2)} \in \mathscr{H}^{(8,2)}$, the stabilizer $G^{(8,2)}$ is a group of two elements: the identity and $(-z, \zeta, 1)$.

5.3. Branch $I_0 \equiv 0 \equiv V_0$

Since $\boldsymbol{Q}_0$ can be generated by $\boldsymbol{I}_0$, $\boldsymbol{V}_0$ and their differentials, we have $\boldsymbol{Q}_0 \equiv 0$. The structure equations degenerate to the model case. The surface is equivalent as the Gaussier-Merker model $u = \frac{z\,\overline{z} + \frac{1}{2}\zeta^2\,\overline{z} + \frac{1}{2}z^2\,\overline{\zeta}}{1 - \zeta\,\overline{\zeta}}$.

To conclude, we draw the branches from our root assumption.



where $\boldsymbol{I}_0$ and $\boldsymbol{V}_0$ are relative invariants of order 5.

Theorem 5.1. *The following three statements hold true.*

(1) *Within the branch $\boldsymbol{I}_0 \neq 0$, the surface is, in a unique way, equivalent to:*

$$\begin{aligned} u = z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}\,z^3\,\overline{z}\,\overline{\zeta} \\ + \tfrac{1}{24}\,\frac{\boldsymbol{V}_0}{\overline{\boldsymbol{I}_0}^2}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,\frac{\overline{\boldsymbol{V}_0}}{\boldsymbol{I}_0^2}\,z^4\,\overline{\zeta} + \tfrac{1}{12}\,\zeta^2\,\overline{z}^3 + \tfrac{1}{12}\,z^3\,\overline{\zeta}^2 \\ + \zeta\,\overline{\zeta}\,(\cdots) + \sum_{a+b+c+d\geqslant 6,\, b\,d=0} \frac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d, \end{aligned}$$

without any harmonic monomial $z^j\,\zeta^{n-j}$, $\forall\,n \geqslant 0$, $0 \leqslant j \leqslant n$ and any monomial $z^a\,\zeta^b\,\overline{z}^c$, $\forall a+b \geqslant 2$, $c \in \{1,2\}$. Collections of coefficients: $\frac{\boldsymbol{V}_0}{\overline{\boldsymbol{I}_0}^2}$, $\frac{\boldsymbol{Q}_0}{|\boldsymbol{I}_0|^2}$ and $\big\{F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\, b\,d=0}$, are in one-to-one correspondence with biholomorphic equivalent classes.

(2) *When $\boldsymbol{I}_0 \equiv 0 \neq \boldsymbol{V}_0$, the surface is, up to $z \mapsto -z$, equivalent to:*

$$\begin{aligned} u = z\,\overline{z} + \tfrac{1}{2}\,z^2\,\overline{\zeta} + \tfrac{1}{2}\,\zeta\,\overline{z}^2 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{V}_0|}\,z\,\zeta\,\overline{z}^3 + \tfrac{1}{6}\,\frac{\boldsymbol{Q}_0}{|\boldsymbol{V}_0|}\,z^3\,\overline{z}\,\overline{\zeta} + \tfrac{1}{24}\,\zeta\,\overline{z}^4 + \tfrac{1}{24}\,z^4\,\overline{\zeta} \\ + \zeta\,\overline{\zeta}\,(\cdots) + \sum_{a+b+c+d\geqslant 6,\, b\,d=0} \frac{F_{a,b,c,d}}{a!b!c!d!}\,z^a\,\zeta^b\,\overline{z}^c\,\overline{\zeta}^d, \end{aligned}$$

without any harmonic monomial $z^j\,\zeta^{n-j}$, $\forall\,n \geqslant 0$, $0 \leqslant j \leqslant n$ and any monomial $z^a\,\zeta^b\,\overline{z}^c$, $\forall a+b \geqslant 2$, $c \in \{1,2\}$. Pairs of collection of coefficients:

$$\frac{\boldsymbol{Q}_0}{|\boldsymbol{V}_0|},\ \big\{F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\, b\,d=0}, \qquad \frac{\boldsymbol{Q}_0}{|\boldsymbol{V}_0|},\ \big\{(-1)^{a+c}\,F_{a,b,c,d}\big\}_{a+b+c+d\geqslant 6,\, b\,d=0}$$

are in one-to-one correspondence with biholomorphic equivalent classes.

(3) *When $\boldsymbol{I}_0 \equiv 0 \equiv \boldsymbol{V}_0$, the surface is equivalent to the Gaussier-Merker model $u = \frac{z\,\overline{z} + \frac{1}{2}\zeta^2\,\overline{z} + \frac{1}{2}\,z^2\,\overline{\zeta}}{1-\zeta\,\overline{\zeta}}$, and conversely.*

6. Finalized Expression of $\boldsymbol{Q}_0$

In this section, we briefly revisit the secondary invariant $\boldsymbol{Q}_0$. Our goal is to transform $\boldsymbol{Q}_0$ into a new expression which makes transparent two interesting features of $\boldsymbol{Q}_0$: that it is real-valued and that it is of order 5 (not 6 as it was first obtained by Cartan's method in [16]).

Proposition 6.1. *The secondary invariant* $\boldsymbol{Q}_0$ *can be brought into the following form*

$$\boldsymbol{Q}_0 = \boldsymbol{B}\boldsymbol{I}_0 + \overline{\boldsymbol{B}\boldsymbol{I}_0} - \boldsymbol{B}\overline{\boldsymbol{B}} + \frac{2}{3}\,\mathrm{Re}\left\{\mathscr{L}_1\Big[\frac{\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\Big]\right\} + \frac{1}{3}\,\mathrm{Re}\Big(\overline{\mathscr{L}}_1(\boldsymbol{P})\Big). \tag{6.2}$$

Let us first recall the formulas of $\boldsymbol{I}_0, \boldsymbol{V}_0, \boldsymbol{Q}_0$ from [16].

$$\boldsymbol{I}_0 = -\frac{1}{3}\frac{\mathscr{K}\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^2} + \frac{1}{3}\frac{\mathscr{K}\overline{\mathscr{L}}_1(\boldsymbol{k})\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^3} + \frac{2}{3}\frac{\mathscr{L}_1\mathscr{L}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})} + \frac{2}{3}\frac{\mathscr{L}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}, \tag{6.3}$$

$$\boldsymbol{V}_0 = -\frac{1}{3}\frac{\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{5}{9}\frac{(\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k}))^2}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^2} - \frac{1}{9}\frac{\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})\overline{\boldsymbol{P}}}{\overline{\mathscr{L}}_1(\boldsymbol{k})} + \frac{1}{3}\overline{\mathscr{L}}_1(\overline{\boldsymbol{P}}) - \frac{1}{9}\overline{\boldsymbol{P}\boldsymbol{P}}, \tag{6.4}$$

and

$$\boldsymbol{Q}_0 = \frac{1}{2}\left\{\boldsymbol{B}\,\boldsymbol{I}_0 + \overline{\mathscr{L}}_1(\boldsymbol{I}_0) - \frac{\overline{\boldsymbol{B}}\,\overline{\mathscr{K}}(\boldsymbol{I}_0)}{\mathscr{L}_1(\overline{\boldsymbol{k}})} - \frac{\mathscr{K}(\boldsymbol{V}_0)}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right\}, \tag{6.5}$$

where

$$\boldsymbol{B} = \frac{1}{3}\Big(\frac{\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})} - \overline{\boldsymbol{P}}\Big) \quad \text{and} \quad \overline{\boldsymbol{B}} = \frac{1}{3}\Big(\frac{\mathscr{L}_1\mathscr{L}_1(\overline{\boldsymbol{k}})}{\mathscr{L}_1(\overline{\boldsymbol{k}})} - \boldsymbol{P}\Big).$$

In order to transform the expression of $18\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2\boldsymbol{Q}_0$, one makes use of the following identities.

Lemma 6.6. *We have the following identities:*

(1) $\mathscr{K}(\overline{\boldsymbol{P}}) = -\boldsymbol{P}\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k})$,

(2) $\mathscr{K}\overline{\mathscr{L}}_1(\overline{\boldsymbol{P}}) = -\overline{\mathscr{L}}_1(\boldsymbol{k})\cdot 2\,\mathrm{Re}\,\big(\overline{\mathscr{L}}_1(\boldsymbol{P})\big) - \boldsymbol{P}\,\overline{\mathscr{L}}_1\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k})$,

(3) $\overline{\mathscr{K}}(\boldsymbol{I}_0) = (-2)\,\overline{\boldsymbol{I}_0}\cdot\mathscr{L}_1(\overline{\boldsymbol{k}})$.

Proof. The identities (1) and (3) are obtained in Lemma 2.7 and Lemma 10.6 of [16], respectively.

For the identity (2), we use the relation $[\mathscr{K}, \overline{\mathscr{L}}_1] = \mathscr{K}\,\overline{\mathscr{L}}_1 - \overline{\mathscr{L}}_1\,\mathscr{K} = -\overline{\mathscr{L}}_1(\boldsymbol{k})\,\mathscr{L}_1$ from (2.9) of [16] to deduce that

$$\begin{aligned}\mathscr{K}\overline{\mathscr{L}}_1(\overline{\boldsymbol{P}}) &= \overline{\mathscr{L}}_1\,\mathscr{K}(\overline{\boldsymbol{P}}) - \overline{\mathscr{L}}_1(\boldsymbol{k})\,\mathscr{L}_1(\overline{\boldsymbol{P}}) \\ &= \overline{\mathscr{L}}_1\Big[-\boldsymbol{P}\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k})\Big] - \overline{\mathscr{L}}_1(\boldsymbol{k})\,\mathscr{L}_1(\overline{\boldsymbol{P}}) \qquad \text{(using (1))}\end{aligned}$$

$$= -\overline{\mathscr{L}}_1(\boldsymbol{P})\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \boldsymbol{P}\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1(\boldsymbol{k})\,\mathscr{L}(\overline{\boldsymbol{P}})$$
$$= -\overline{\mathscr{L}}_1(\boldsymbol{k})\Big[\overline{\mathscr{L}}_1(\boldsymbol{P}) + \mathscr{L}_1(\overline{\boldsymbol{P}})\Big] - \boldsymbol{P}\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k})$$
$$= -\overline{\mathscr{L}}_1(\boldsymbol{k})\cdot 2\,\mathrm{Re}\,\big(\overline{\mathscr{L}}_1(\boldsymbol{P})\big) - \boldsymbol{P}\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k}) - \overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k}). \qquad \square$$

***Proof of Proposition* 6.1.** Intermediate computations appear at the end of [4], until we reach

$$\begin{aligned} 18\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2\boldsymbol{Q}_0 = 2\,\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2\,\Big(3B\;3I_0 + 3\overline{B}\;3\overline{I_0} - 3\overline{B}\;3B\Big)& \\ + 12\,\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2\;\mathrm{Re}\left\{\mathscr{L}_1\Big[\frac{\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\Big]\right\}& \\ + 6\,\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2\;\mathrm{Re}\Big(\overline{\mathscr{L}}_1(\boldsymbol{P})\Big).& \end{aligned} \tag{6.7}$$

Finally, simplifying the factor $18\big|\overline{\mathscr{L}}_1(\boldsymbol{k})\big|^2$ on both sides of (6.7) gives us the desired expression (6.2) of $\boldsymbol{Q}_0$. $\square$

When we fully expand $\boldsymbol{Q}_0$ from the expression (6.2) using the formulas of $\boldsymbol{I}_0$ and $\boldsymbol{B}$, we arrive at the following long expression of $\boldsymbol{Q}_0$, which only involves in the fundamental functions $\mathbf{k}$ and $\boldsymbol{P}$, and their derivatives:

$$\begin{aligned} \boldsymbol{Q}_0 = \;& \frac{2}{9}\,\mathrm{Re}\left\{\frac{\mathscr{K}\,\overline{\mathscr{L}}_1(\boldsymbol{k})\;(\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k}))^2}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^4}\right\} \\ & - \frac{2}{9}\,\mathrm{Re}\left\{\frac{\mathscr{K}\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k}) + \mathscr{K}\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\boldsymbol{P}}}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^3}\right\} \\ & + \frac{2}{9}\,\mathrm{Re}\left\{\frac{2\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\mathscr{L}}_1\,\mathscr{L}_1(\boldsymbol{k}) + \mathscr{K}\,\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\boldsymbol{P}}}{(\overline{\mathscr{L}}_1(\boldsymbol{k}))^2}\right\} \\ & - \frac{2}{9}\,\mathrm{Re}\left\{\frac{2\,\mathscr{L}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\overline{\boldsymbol{P}} + \overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})\,\boldsymbol{P}}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right\} \\ & - \frac{1}{9}\,|\boldsymbol{P}|^2 + \frac{1}{3}\left|\frac{\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\right|^2 \\ & + \frac{2}{3}\,\mathrm{Re}\left\{\mathscr{L}_1\Big[\frac{\overline{\mathscr{L}}_1\,\overline{\mathscr{L}}_1(\boldsymbol{k})}{\overline{\mathscr{L}}_1(\boldsymbol{k})}\Big]\right\} + \frac{1}{3}\,\mathrm{Re}\Big(\overline{\mathscr{L}}_1(\boldsymbol{P})\Big). \end{aligned} \tag{6.8}$$

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