Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **18** (2023), No. 1, pp. 85-103
DOI: 10.21915/BIMAS.2023105

STABILITY OF SOLUTION TO VISCOELASTIC WAVE EQUATION WITH FIRST ORDER PERTURBATION TERM

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Abstract

This work is concerned with a nonlinear viscoelastic wave system in a bounded domain. The system contains a first order perturbation term. Our main object is to prove the stability of the solution, under appropriate conditions on the parameters of this system.

1. Introduction

In this work, we study the following system

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + F(t,\nabla u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ u(\cdot,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot,0) = u_0(\cdot) \quad \text{and } u_t(\cdot,\ 0) = u_1(\cdot) \quad \text{in } \ \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, g is a positive function, $a : \Omega \to \mathbb{R}^+$ and $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$.

In the absence of the viscoelasticity $(g = 0)$, many results was established concerning global existence and nonexistence of wave equations of the form

$$u_{tt} - \Delta u + au_t \left|u_t\right|^m = b \left|u\right|^\gamma u \qquad \text{in } \Omega \times \mathbb{R}^+$$

Received December 21, 2022.

AMS Subject Classification: 35L05.

Key words and phrases: Wave equation, viscoelastic term, first order perturbation, exponential decay.

with m, $\gamma > 0$. When $a = 0$, the source term $b\,|u|^{\gamma}\,u$ causes finite time blow up of solutions with negative initial energy, we can refer to [2, 18]. For $b = 0$, the damping term $au_t\,|u_t|^{m}$ assures the global existence of the solution for arbitrary initial data see [16, 19]. The case $m = 0$, Levine [20, 21] proved that the solutions with negative initial energy blow up in finite time. The last Levine's result was extended by Georgiev and Todorova [13] to the nonlinear damping case $(m > 0)$, Messaoudi [22] extended the blow up result of [13] to solutions with negative initial energy only.

Dafermos [11, 12] studied one-dimensional viscoelastic problems. He proved an existence results, then he showed for smooth monotone decreasing relaxation function that the solutions go to zero as t goes to infinity.

The most results regarding existence, nonexistence and asymptotic behavior concerning the single following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + h(u_t) = f(u) \quad \text{in } \Omega \times \mathbb{R}^+$$

with initial and boundary conditions have been obtained. We refer the reader to [1, 4, 6, 17, 23, 24, 25, 26].

Cavalcanti et al. [6] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^{p-1}\,u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

where $a : \Omega \to \mathbb{R}^+$ is a function, which may vanish on a part of Ω. Assuming the condition that $a(x) \geq a_0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and for two positive constants ξ_1 and ξ_2 we have

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad \forall t \geq 0.$$

The authors proved an exponential decay result which extended the result of Zuazua [27]. In [4], Berrimi and Messaoudi established the result of [6], under weaker conditions on both a and g. Cavalcanti and Oquendo [7] considered the following equation

$$u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}\left[a(x)g(t-\tau)\Delta u(\tau)d\tau\right] + b(x)h(u_t) + f(u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

and established an exponential stability result under some conditions on the relaxation function g and for $a(x) + b(x) \geq \beta > 0$. Also, in [8], Cavalcanti et al. studied the equation

$$|u_t|^p \, u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = 0 \qquad \text{in } \Omega \times \mathbb{R}^+,$$

they showed that for $\gamma \geq 0$ a global existence result and for $\gamma > 0$ an exponential decay. Messaoudi and Tatar [25] extented the last result, where the source term is competing with the strong damping mechanism. In [23], Messaoudi showed that the solutions of the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a u_t \, |u_t|^m = b\,|u|^\gamma \, u \qquad \text{in } \Omega \times \mathbb{R}^+$$

with negative initial energy blow up in finite time if $\gamma > m$, and continue to exist if $m \geq \gamma$.

Cabanillas and Rivera [5] studied in a bounded domain, an anisotropic and inhomogeneous viscoelastic equation and showed that the sum of the first and the second energy decays polynomially when the relaxation function is of polynomial decay type. Baretto et al. [3] proved that the solution energy decays at the same decay rate of the relaxation function, which could be exponential or polynomial.

In the literature, the problem of existence and properties of solution of wave equation when $F \neq 0$ was considered, we can cite the work of [9, 10, 14, 15]. In all these works, the nonlinear term F, which depends on the gradient, represents a difficulty in the study since we do not have more information about the sign of the derivative of the energy.

The main goal of the present paper is to establish the exponential stability of (1). We give a positive answer about the influence of the perturbation on the energy. Precisely, we prove that if the dissipation terms dominate the term of pertubation, then the energy is decreasing in time and the solution is exponentially stable. The key of the proof is the good choice of the Liaponov functional wich makes the proving steps easier.

This paper is organised as follows: in Section 2, we present some assumptions and technical results needed for our work and we state the global existence theorem. Section 3 is devoted to the proof of the theorem of stability.

2. Assumptions and Technical Results

In this section, we introduce some assumptions and present some technical results used throughout this paper.

We assume that

$(H1)$ The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1(\mathbb{R}^+)$ positive decreasing function such that

$$g(0) > 0 \text{ and } 1 - \int_0^\infty g(s)ds = l > 0.$$

$(H2)$ There exist two positive constants ξ_1 and ξ_2 such that

$$-\xi_1 g(t) \leq g^{'}(t) \leq -\xi_2 g(t) \text{ for all } t \geq 0.$$

$(H3)$ The function $a : \Omega \to \mathbb{R}_+$, satisfying for all $x \in \Omega$

$$0 < a_1 < a(x) < a_2 < \infty \quad \text{for some } a_1 > 0 \text{ and } a_2 > 0.$$

$(H4)$ The function $F \in C^1\left(\mathbb{R}^{n+1},\ \mathbb{R}\right)$ and

$$|F(t,\ U)|^2 \leq \frac{a_1}{2} g(t) |U|^2,\ \forall t \geq 0,\ \forall U \in \mathbb{R}^n.$$

In the next theorem, we state a result on existence and uniqueness of weak and strong global solutions of system (1).

Theorem 2.1.

1. *For any initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *there exists a unique weak solution of* (1) *in the class*

$$u \in C\left([0,\ \infty);\ H_0^1(\Omega)\right);\ u^{'} \in C\left([0,\ \infty);\ L^2(\Omega)\right).$$

2. *For any initial data* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$ *there exists a unique global strong solution of* (1) *in the class*

$$u \in L^\infty([0,\ \infty);\ H_0^1(\Omega) \cap H^2(\Omega));\ u^{'} \in L^\infty\left([0,\ \infty);\ H_0^1(\Omega)\right);$$

$$u^{''} \in L^{\infty}\left([0,\ \infty);\ L^{2}(\Omega)\right).$$

The energy functional E associated to the system (1) is defined for all $t \geq 0$ by

$$E(t) = \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\left\|\nabla u(t)\right\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)$$

where

$$(g \circ u)(t) = \int_0^t g(t-\tau)\left\|u(\tau) - u(t)\right\|_2^2 d\tau.$$

In the following Lemma, we show that the energy functional E is a decreasing function.

Lemma 2.1. *Suppose that* (H_1)*,* (H_2)*,* (H_3) *and* (H_4) *hold. Let* u *be the solution of the system* (1)*, then the energy functional* E *is decreasing in* t*.*

Proof. By multiplying the first equation in (1) by u_t and integrating it over Ω, we get

$$\begin{aligned}&\int_\Omega a(x)\left|u_t(t)\right|^2 dx + \int_\Omega F(t,\nabla u)u_t(t)dx\\ &\quad = -\int_\Omega u_{tt}u_t dx + \int_\Omega \Delta u(t)u_t(t)dx - \int_\Omega \Big(\int_0^t g(t-\tau)\Delta u(\tau)u_t(t)d\tau\Big)dx.\end{aligned}$$

The first term.

$$2\int_\Omega u_{tt}u_t dx = \frac{d}{dt}\int_\Omega u_t^2 dx = \frac{d}{dt}\left\|u_t\right\|_2^2$$

we find

$$-\int_\Omega u_{tt}u_t dx = -\frac{1}{2}\frac{d}{dt}\left\|u_t\right\|_2^2.$$

The second term.

The Green formula and the assumption on the boundary yield

$$\int_\Omega \Delta u(t)u_t(t)dx = -\int_\Omega \nabla u(t)\nabla u_t(t)dx = -\frac{1}{2}\frac{d}{dt}\left\|\nabla u(t)\right\|_2^2.$$

The third term.

By the Green formula, we obtain

$$
\begin{aligned}
&-\int_\Omega \Big(\int_0^t g(t-\tau)\Delta u(\tau) u_t(t) d\tau \Big) dx \\
&= -\int_0^t g(t-\tau)(\int_\Omega (-\nabla u(\tau) + \nabla u(t)) \nabla u_t(t) dx)\, d\tau \\
&\quad + \int_0^t g(t-\tau) \left(\int_\Omega \nabla u(t) \nabla u_t(t) dx \right).
\end{aligned}
$$

We have

$$
\frac{d}{dt} \left| \nabla u(t) - \nabla u(\tau) \right|^2 = 2 \left(\nabla u(t)) - \nabla u(\tau) \right) \nabla u_t(t)
$$

then

$$
(\nabla u(t)) - \nabla u(\tau)) \nabla u_t(t) = \frac{1}{2} \frac{d}{dt} \left| \nabla u(t) - \nabla u(\tau) \right|^2
$$

so

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-\tau) \Delta u(\tau) u_t(t) d\tau \Big) dx \\
&= \frac{1}{2} \int_0^t g(t-\tau) \left(\int_\Omega \frac{d}{dt} \left| (\nabla u(t)) - \nabla u(\tau) \right|^2 dx \right) d\tau \\
&\quad - \frac{1}{2} \int_0^t g(t-\tau) \left(\int_\Omega \frac{d}{dt} \left| \nabla u_t(t) \right|^2 dx \right) d\tau \\
&= \frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt} \left\| (\nabla u(t)) - \nabla u(\tau) \right\|_2^2 d\tau - \frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt} \left\| (\nabla u(t)) \right\|_2^2 d\tau.
\end{aligned}
$$

We apply the Leibniz integral rule to obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} (g \circ \nabla u)(t) &= \int_0^t \frac{d}{dt} g(t-\tau) \left\| (\nabla u(t)) - \nabla u(\tau) \right\|_2^2 d\tau \\
&= \frac{1}{2} \left(g' \circ \nabla u \right)(t) + \frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt} \left\| (\nabla u(t)) - \nabla u(\tau) \right\|_2^2 d\tau.
\end{aligned}
$$

This yields

$$
-\frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt} \left\| (\nabla u(t)) - \nabla u(\tau) \right\|_2^2 d\tau = -\frac{1}{2} \frac{d}{dt} \left(g \circ \nabla u \right)(t) + \frac{1}{2} \left(g' \circ \nabla u \right)(t).
$$

For the term $\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|(\nabla u(t))\|_2^2\,d\tau$, we apply the Leibniz integral,

again, to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^t g(t-\tau)\|(\nabla u(t))\|_2^2\,d\tau =& \frac{1}{2}\frac{d}{dt}\int_0^t g(s)\|\nabla u(t)\|_2^2\,ds\\ =& \frac{1}{2}\int_0^t g(\tau)\frac{d}{dt}\|\nabla u(t)\|_2^2\,d\tau+\frac{1}{2}g(t)\|\nabla u(t)\|_2^2.\end{aligned}$$

This yields

$$\begin{aligned}&\int_\Omega\Big(\int_0^t g(t-\tau)\Delta u(\tau)u_t(t)d\tau\Big)dx\\ =&\ -\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t)+\frac{1}{2}\left(g^{'}\circ\nabla u\right)(t)+\frac{1}{2}\frac{d}{dt}\int_0^t g(t-\tau)\|(\nabla u(t))\|_2^2\,d\tau\\ &-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2.\end{aligned}$$

Using the definition of E, we obtain

$$\begin{aligned}\frac{d}{dt}E(t)\leq& \frac{1}{2}\left(g^{'}\circ\nabla u\right)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2-a_1\|u_t\|_2^2\\ &-\int_\Omega F(t,\nabla u(t))u_t(t)dx.\end{aligned}$$

By the Young inequality, we find, for all $\epsilon>0$

$$\begin{aligned}\frac{d}{dt}E(t)\leq&-\frac{\xi_2}{2}(g\circ\nabla u)(t)+\frac{\epsilon}{2}\int_\Omega|F(t,\nabla u|^2\,dx+\frac{1}{2\epsilon}\int_\Omega|u_t(t)|^2\,dx\\ &-\frac{1}{2}\int_\Omega g(t)|\nabla u(t)|^2\,dx-a_1\|u_t\|_2^2\\ \leq&-\frac{\xi_2}{2}(g\circ\nabla u)(t)+\left(\frac{1}{2\epsilon}-a_1\right)\|u_t\|_2^2+\frac{1}{2}\left(\epsilon\frac{a_1}{2}-1\right)g(t)\|\nabla u(t)\|_2^2,\end{aligned}$$

we take $\epsilon=\frac{1}{a_1}$ to get

$$\frac{d}{dt}E(t)\leq-\frac{\xi_2}{2}(g\circ\nabla u)(t)+\left(\frac{1}{2\epsilon}-a_1\right)\|u_t\|_2^2+\frac{1}{2}\left(-\frac{1}{2}\right)g(t)\|\nabla u(t)\|_2^2,$$

this yields

$$\frac{d}{dt}E(t) \leq -\frac{\xi_2}{2}(g \circ \nabla u)(t) - \frac{a_1}{2}\|u_t\|_2^2 - \frac{1}{4}g(t)\|\nabla u(t)\|_2^2 . \tag{2}$$

3. Theorem of the Stability

In this section, we prove our main stability result. For this end, we introduce the Lyapunov functional which is the basis of the proof.

$$\mathfrak{F}(t) := E(t) + \epsilon_1 \psi(t) + \epsilon_2 \chi(t) \qquad \text{for all } t \geq 0,$$

where ϵ_1 and ϵ_2 are positive constants to be chosen later.

$$\psi(t) := \int_\Omega u u_t dx \qquad \text{for all } t \geq 0$$

and

$$\chi(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \qquad \text{for all } t \geq 0.$$

We need the following lemmas.

Lemma 3.1. *The functional $\mathfrak{F}$ is equivalent to the energy functional E, which means that there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 \mathfrak{F}(t) \leq E(t) \leq \alpha_2 \mathfrak{F}(t) \qquad \textit{for all } t \geq 0$$

Proof. We have

$$\begin{aligned}
\mathfrak{F}(t) &= E(t) + \epsilon_1 \int_\Omega u.u_t dx - \epsilon_2 \int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\
&\leq E(t) + \frac{\epsilon_1}{2}\int_\Omega |u|^2 dx + \frac{\epsilon_1}{2}\int_\Omega |u_t|^2 dx + \frac{\epsilon_2}{2}\int_\Omega |u_t|^2 dx \\
&\quad + \frac{\epsilon_2}{2}\int_\Omega \Big(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\Big)^2 dx \\
&\leq E(t) + \frac{\epsilon_1}{2}c_*^2 \|\nabla u(t)\|_2^2 + (\epsilon_1+\epsilon_2)E(t) \\
&\quad + \frac{\epsilon_2}{2}(1-l)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_2^2 d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le E(t) + \frac{\epsilon_1 c_*^2}{l} E(t) + (\epsilon_1 + \epsilon_2) E(t) \\
&\quad + \epsilon_2 (1-l) c_* \frac{1}{2} \int_0^t g(t-\tau) \|\nabla u(t) - \nabla u(\tau)\|_2^2 \, d\tau \\
&\le (1 + \frac{\epsilon_1 c_*^2}{l} E(t) + (\epsilon_1 + \epsilon_2)) E(t) + \epsilon_2 (1-l) c_* \frac{1}{2} (g \circ \nabla u)(t) \\
&\le \left(1 + \frac{\epsilon_1 c_*^2}{l} + \epsilon_1 + \epsilon_2 + \epsilon_2 (1-l) c_*\right) E(t).
\end{aligned}
$$

Then

$$\alpha_1 \mathfrak{F}(t) \le E(t).$$

Where: $\alpha_1 = \left(1 + \frac{\epsilon_1 c_*^2}{l} + \epsilon_1 + \epsilon_2 + \epsilon_2(1-l)c_*\right)^{-1}$ and c_* is the Poincaré constant.

We have

$$
\begin{aligned}
\mathfrak{F}(t) &\ge E(t) - \frac{\epsilon_1}{2} \int_\Omega |u|^2 \, dx - \frac{\epsilon_1}{2} \int_\Omega |u_t|^2 \, dx - \frac{\epsilon_2}{2} \int_\Omega |u_t|^2 \, dx \\
&\quad - \frac{\epsilon_2}{2} c_* (1-l) (g \circ \nabla u)(t) \\
&\ge E(t) - \frac{\epsilon_1}{2} c_*^2 \|\nabla u(t)\|_2^2 - \frac{\epsilon_1 + \epsilon_2}{2} \|u_t(t)\|_2^2 - \frac{\epsilon_2}{2} c_* (1-l) (g \circ \nabla u)(t) \\
&\ge E(t) - \frac{\epsilon_1}{l} c_*^2 E(t) - \frac{\epsilon_1 + \epsilon_2}{2} \|u_t\|_2^2 - \frac{\epsilon_2}{2} c_* (1-l) (g \circ \nabla u)(t) \\
&\ge \left(1 - \left(\frac{1}{l} c_*^2 + 1\right) \epsilon_1 - \epsilon_2 (1 + c_* (1-l))\right) E(t).
\end{aligned}
$$

Then

$$E(t) \le \alpha_2 \mathfrak{F}(\mathfrak{t})$$

where $\alpha_2 = \left(1 - (\frac{1}{l} c_*^2 + 1)\epsilon_1 - \epsilon_2(1 + c_*(1-l))\right)^{-1}$

By the last two inequalities, we get

$$\alpha_1 \mathfrak{F}(\mathfrak{t}) \le E(t) \le \alpha_2 \mathfrak{F}(\mathfrak{t}) \quad \text{for all } t \ge 0. \tag{3}$$

Lemma 3.2. *We have for all* $t \ge 0$

$$\psi'(t) \le -\frac{l}{4} \|\nabla u(t)\|_2^2 + \frac{1-l}{2l} (g \circ \nabla u)(t) + \left(1 + \frac{1 + a_2}{l} a_2 c_*\right) \|u_t(t)\|_2^2$$

$$+\frac{1+a_2}{2l}a_1c_*g(t)\left\|\nabla u(t)\right\|_2^2$$

Proof. We have

$$\psi^{'}(t)=\int_\Omega u_t^2dx+\int_\Omega u_{tt}udx.$$

For the term $\int_\Omega u_{tt}udx$, multiplying the first equation of (1) by u and using the Green formula to find

$$\begin{aligned}\int_\Omega u_{tt}u(t)dx &= \int_\Omega \Delta u(t)u(t)dx-\int_\Omega\left(\int_0^t g(t-\tau)\Delta u(\tau)d\tau\right)u(t)dx\\ &\quad -\int_\Omega a(x)u_tu(t)dx-\int_\Omega F(t,\nabla u)u(t)dx\\ &= -\int_\Omega\left|\nabla u(t)\right|^2dx+\int_\Omega\nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)d\tau dx\\ &\quad -\int_\Omega F(t,\nabla u)u(t)dx-\int_\Omega a(x)u_tu(t)dx\end{aligned}$$

Estimate of the term $\displaystyle\int_\Omega \nabla u(t)\int_0^t g(t-\tau)\nabla u(t)d\tau dx.$

We have

$$\begin{aligned}&\int_\Omega\nabla u(t)\int_0^t g(t-\tau)\nabla u(\tau)d\tau dx\\ &\le\frac12\int_\Omega\left|\nabla u(t)\right|^2dx+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)\right|d\tau\Big)^2dx\\ &\le\frac12\int_\Omega\left|\nabla u(t)\right|^2dx+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)+\nabla u(t)-\nabla u(t)\right|d\tau\Big)^2dx\\ &\le\frac12\int_\Omega\left|\nabla u(t)\right|^2dx+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)(\left|\nabla u(\tau)-\nabla u(t)\right|+\left|\nabla u(t)\right|)d\tau\Big)^2dx\\ &\le\frac12\int_\Omega\left|\nabla u(t)\right|^2dx+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\\ &\quad+\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)^2dx\\ &\le\frac12\int_\Omega\left|\nabla u(t)\right|^2dx+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\Big)^2dx\\ &\quad+\frac12\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)^2dx\end{aligned}$$

$$+\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\Big)\Big(\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)dx$$

Applying the Young inequality to get for all $\delta>0$

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-\tau)\nabla u d\tau dx\\
&\quad\le \frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2dx+\frac{1}{2}\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\Big)^2dx\\
&\qquad+\frac{1}{2}\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)^2dx\\
&\qquad+\frac{\delta}{2}\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\Big)^2dx\\
&\qquad+\frac{1}{2\delta}\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)^2dx\\
&\quad\le \frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2dx+\frac{1}{2}\Big(1+\frac{1}{\delta}\Big)\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\Big)^2dx\\
&\qquad+\frac{1}{2}(1+\delta)\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(t)\right|d\tau\Big)^2dx.
\end{aligned}$$

But

$$\begin{aligned}
\int_0^t g(t-\tau)d\tau \le \int_0^\infty g(t-\tau)d\tau = 1-l \text{ and } (g\circ\nabla u)(t)\\
=\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|^2d\tau.
\end{aligned}$$

This yields

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-\tau)\nabla u d\tau dx\\
&\quad\le \frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2dx+\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t)\\
&\qquad+\frac{1}{2}(1+\delta)\int_\Omega\left|\nabla u(t)\right|^2\Big(\int_0^t g(t-\tau)d\tau\Big)^2dx\\
&\quad\le \frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2dx+\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t)\\
&\qquad+\frac{1}{2}(1+\delta)(1-l)^2\int_\Omega\left|\nabla u(t)\right|^2dx.
\end{aligned}$$

So, for all $\eta > 0$ and $\delta > 0$, we have

$$\begin{aligned}
\psi'(t) &\leq -\frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{2}(1+\delta)(1-l)^2\int_\Omega |\nabla u(t)|^2\,dx \\
&\quad +\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t) - \int_\Omega a(x)u_t(t)u(t)dx \\
&\quad -\int_\Omega F(t,\nabla u)u(t)dx + \int_\Omega |u_t|^2\,dx \\
&\leq -\frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{2}(1+\delta)(1-l)^2\int_\Omega |\nabla u(t)|^2\,dx \\
&\quad +\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t) - \int_\Omega a(x)u_t(t)u(t)dx \\
&\quad +\frac{1}{2\eta}\int_\Omega |F(t,\nabla u)|^2\,dx + \frac{\eta}{2}\int_\Omega |u(t)|^2\,dx + \int_\Omega |u_t|^2\,dx \\
&\leq -\frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{2}((1+\delta)(1-l)^2)\int_\Omega |\nabla u(t)|^2\,dx \\
&\quad +\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t) + \frac{a_2}{2\eta}\int_\Omega |u_t|^2\,dx + a_2\frac{\eta}{2}\int_\Omega |u(t)|^2\,dx \\
&\quad +\frac{\eta}{2}\int_\Omega |u(t)|^2\,dx + \frac{a_1}{4\eta}g(t)\int_\Omega |\nabla u(t)|^2\,dx + \int_\Omega |u_t|^2\,dx.
\end{aligned}$$

This yields

$$\begin{aligned}
\psi'(t) \;&\leq\; -\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{2}(1+(1+\delta)(1-l)^2)\int_\Omega |\nabla u(t)|^2\,dx \\
&\quad +\frac{1}{2}(1+\frac{1}{\delta})(1-l)(g\circ\nabla u)(t) + \frac{\eta}{2}(a_2+1)\|u(t)\|_2^2 \\
&\quad +\frac{a_1}{4\eta}g(t)\int_\Omega |\nabla u(t)|^2\,dx + \left(1+\frac{a_2}{2\eta}\right)\int_\Omega |u_t|^2\,dx.
\end{aligned}$$

By choosing $\delta = \frac{l}{1-l}$, the last inequality becomes

$$\begin{aligned}
\psi'(t) \;&\leq\; \frac{1}{2}\left(\eta\left(1+a_2\right)c_* - l\right)\|\nabla u(t)\|_2^2 + \frac{1-l}{2l}\left(g\circ\nabla u\right)(t) \\
&\quad +(1+\frac{a_2}{2\eta})\|u_t(t)\|_2^2 + \frac{a_1}{4\eta}g(t)\|\nabla u(t)\|_2^2.
\end{aligned}$$

By choosing $\eta = \frac{l}{2(1+a_2)c_*}$, we obtain

$$\psi'(t) \;\leq\; -\frac{l}{4}\|\nabla u(t)\|_2^2 + \frac{1-l}{2l}\left(g\circ\nabla u\right)(t)$$

$$+(1+\frac{1+a_2}{l}a_2c_*)\left\|u_t(t)\right\|_2^2+\frac{1+a_2}{2l}a_1c_*g(t)\left\|\nabla u(t)\right\|_2^2. \quad (4)$$

Lemma 3.3. *There exists t_0 such that the derivative of the functional χ satisfies for all $t \geq t_0$*

$$\chi^{'}(t) \leq \frac{g_0}{1+a_2}(1+2(1-l)^2)\left\|\nabla u(t)\right\|_2^2 + a_1\frac{g_0}{4(1+a_2)}g(t)\left\|\nabla u\right\|_2^2+c(g\circ\nabla u)(t).$$

for some $c > 0$.

Proof. We have

$$\begin{aligned}\chi^{'}(t) =& -\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\ &-\int_\Omega u_t\Big(\int_0^t g(t-\tau)(u(t)-u(\tau))^{'}d\tau\Big)dx\\ =& -\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx - \int_\Omega u_t\int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau dx\\ &-\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2dx\end{aligned}$$

For the term $-\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$, we multiply the first equation of (1) by $\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau$ to obtain

$$\begin{aligned}&-\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\ &= -\int_\Omega \Delta u(t)\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\ &\quad+\int_\Omega\Big(\int_0^t g(t-\tau)\Delta u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\Big)dx\\ &\quad+\int_\Omega a(x)u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\ &\quad+\int_\Omega F(t,\nabla u)\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\end{aligned}$$

This implies that

$$\chi^{'}(t) \leq \int_\Omega \nabla u(t)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx$$

$$
\begin{aligned}
&-\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx\\
&+\int_\Omega a(x)u_t(t)\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\
&+\int_\Omega F(t,\nabla u)\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\\
&-\int_\Omega u_t\int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau dx-\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2dx.
\end{aligned}
$$

For the first term, Young inequality yields

$$
\int_\Omega\nabla u(t)\bigg(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\bigg)\,dx\le\delta\int_\Omega|\nabla u|^2\,dx+\frac{1-l}{4\delta}(g\circ\nabla u)(t).
$$

For the term

$$
\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx,
$$

we apply again, the Young inequality to find

$$
\begin{aligned}
&\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx\\
&\le\delta\int_\Omega\left|\int_0^t g(t-\tau)\nabla u(\tau)d\tau\right|^2dx+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau)d\tau)\right|^2dx\\
&\le\delta\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)\right|d\tau\Big)^2dx\\
&\quad+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-\tau)\left|(\nabla u(t)-\nabla u(\tau)\right|d\tau)\Big)^2dx\\
&\le\delta\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)+\nabla u(t)-\nabla u(t)\right|d\tau\Big)^2dx\\
&\quad+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-\tau)\left|(\nabla u(t)-\nabla u(\tau)\right|d\tau\Big)\Big(\int_0^t g(t-\tau)\left|(\nabla u(t)-\nabla u(\tau)\right|d\tau\Big)dx.
\end{aligned}
$$

So

$$
\begin{aligned}
&\int_\Omega\Big(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\Big)dx\\
&\quad\le\delta\int_\Omega\Big(\int_0^t g(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|+\left|\nabla u(t)\right|d\tau\Big)^2dx
\end{aligned}
$$

$$+\frac{1}{4\delta}(1-l)\int_0^t g(t-\tau)\int_\Omega |\nabla u(t)-\nabla u(\tau)|^2\,dxd\tau$$
$$\leq 2\delta(1-l)^2\int_\Omega |\nabla u|^2\,dx+(2\delta+\frac{1}{4\delta})(1-l)(g\circ\nabla u)(t).$$

For the term $\int_\Omega a(x)u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$, using the assumption $(H3)$, we get

$$\int_\Omega a(x)u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$\leq a_2\delta\int_\Omega |u_t|^2\,dx+\frac{a_2}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)(u(t)-u(\tau)d\tau\right)^2 dx$$
$$\leq a_2\delta\int_\Omega |u_t|^2\,dx+\frac{a_2c*}{4\delta}(1-l)(g\circ\nabla u)(t).$$

We have

$$\int_\Omega F(t,\nabla u)\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$\leq\frac{\delta}{2}\int_\Omega |F(t,\nabla u)|^2\,dx+\frac{(1-l)c_*}{2\delta}(g\circ\nabla u)(t)$$
$$\leq\delta\frac{a_1}{4}g(t)\left\|\nabla u\right\|_2^2++\frac{(1-l)c_*}{2\delta}(g\circ\nabla u)(t)$$

and

$$-\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx$$
$$\leq\frac{\delta}{2}\int_\Omega |u_t|^2\,dx+\frac{1}{2\delta}\int_\Omega\Big(\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\Big)^2 dx$$
$$\leq\frac{\delta}{2}\int_\Omega |u_t|^2\,dx-\frac{c_*g(0)}{2\delta}(g'\circ\nabla u)(t).$$

Since g is a positive function then there exists $t_0>0$ such that

$$\int_0^t g(s)ds\geq\int_0^{t_0} g(s)ds=g_0>0 \text{ for all } t\geq t_0.$$

Then, replacing in χ', we arrive at

$$\chi'(t)\leq\delta(1+2(1-l)^2)\left\|\nabla u(t)\right\|_2^2+a_1\frac{\delta}{4}g(t)\left\|\nabla u(t)\right\|_2^2$$

$$+(\delta(a_2+1)-g_0)\int_\Omega |u_t|^2\,dx+(1-l)\left(2\delta+\frac{1}{2\delta}(1+c_*(\frac{a_2}{2}+1)\right)(g\circ\nabla u)(t)$$
$$-\frac{c_*g(0)}{2\delta}(g^{'}\circ\nabla u)(t).$$

This gives

$$\begin{aligned}\chi^{'}(t) \;\leq\; & \delta(1+2(1-l)^2)\left\|\nabla u(t)\right\|_2^2+a_1\frac{\delta}{4}g(t)\left\|\nabla u(t)\right\|_2^2\\ & +(\delta(a_2+1)-g_0)\int_\Omega |u_t|^2\,dx+c_1(\delta)(g\circ\nabla u)(t),\end{aligned}$$

where

$$c_1(\delta)=(1-l)\left(2\delta+\frac{1}{2\delta}(1+c_*)+\frac{c_*g(0)}{2\delta}\xi_1\right)>0.$$

We choose $\delta=\frac{g_0}{1+a_2}$ to find

$$\chi^{'}(t)\leq\frac{g_0}{1+a_2}(1+2(1-l)^2)\left\|\nabla u(t)\right\|_2^2+a_1\frac{g_0}{4(1+a_2)}g(t)\left\|\nabla u\right\|_2^2+c(g\circ\nabla u)(t), \tag{5}$$

where

$$c=c_1\left(\frac{g_0}{1+a_2}\right)=(1-l)\left(2\frac{g_0}{1+a_2}+\frac{1+a_2}{2g_0}(1+c_*)+\frac{(1+a_2)c_*g(0)}{2g_0}\xi_1\right)>0.$$

We then state our main exponential decay result.

Theorem 3.1. *There exist two positive constants c and w such that*

$$E(t)\leq ce^{-wt}\quad\textit{for all}\quad t\geq t_0.$$

Proof. By using the assumption $(H2)$, exploiting (4) and (5), we easily get

$$\begin{aligned}\mathfrak{F}^{'}(t) \;\leq\; & E^{'}(t)+\epsilon_1\psi^{'}(t)+\epsilon_2\chi^{'}(t)\\ \leq\; & -\frac{\xi_2}{2}(g\circ\nabla u)(t)-\frac{1}{2}\left\|u_t\right\|_2^2-\frac{1}{4}g(t)\left\|\nabla u(t)\right\|_2^2\\ & -\epsilon_1\frac{l}{4}\left\|\nabla u(t)\right\|_2^2+\epsilon_1\frac{1-l}{2l}\left(g\circ\nabla u\right)(t)+\epsilon_1(1+\frac{1+a_2}{l}a_2c_*)\left\|u_t(t)\right\|_2^2\\ & +\epsilon_1\frac{(1+a_2)}{4l}a_1c_*g(t)\left\|\nabla u(t)\right\|_2^2+\epsilon_2\frac{g_0}{1+a_2}(1+2(1-l)^2)\left\|\nabla u(t)\right\|_2^2\end{aligned}$$

$$+\epsilon_2 a_1 \frac{g_0}{4(1+a_2)} g(t) \|\nabla u\|_2^2 + \epsilon_2 c (g \circ \nabla u)(t)).$$

Then

$$\begin{aligned}
\mathfrak{F}'(t) \;\le\; & \left(-\frac{\xi_2}{2} + \epsilon_1 \frac{1-l}{2l} + \epsilon_2 c\right)(g \circ \nabla u)(t) \\
& + \left(-\frac{l\epsilon_1}{4} + \epsilon_2 \frac{g_0}{1+a_2}(1+2(1-l)^2)) \|\nabla u(t)\|_2^2\right) \\
& + \left(-\frac{1}{2} + \epsilon_1 \Big(1 + \frac{1+a_2}{l} a_2 c_*\Big)\right) \int_\Omega |u_t|^2 \, dx \\
& + \left(\frac{-1}{4} + \epsilon_1 \frac{(1+a_2)}{4l} a_1 c_* + \epsilon_2 a_1 \frac{g_0}{4(1+a_2)}\right) g(t) \|\nabla u(t)\|_2^2 .
\end{aligned}$$

We choose ϵ_1 and ϵ_2 such that

$$\begin{aligned}
\rho_1 &= \Big(\frac{\xi_2}{2} - \epsilon_1 \frac{1-l}{2l} - \epsilon_2 c\Big) > 0, \\
\rho_2 &= \Big(\frac{\epsilon_1}{4} - \epsilon_2 \frac{g_0}{1+a_2}\Big(1+2(1-l)^2\Big)\Big) > 0, \\
\rho_3 &= \Big(\frac{1}{2} - \epsilon_1 \Big(1 + \frac{1+a_2}{l} a_2 c_*\Big)\Big) > 0
\end{aligned}$$

and

$$\rho_4 = \Big(\frac{1}{4} - \epsilon_1 \frac{(1+a_2)}{4l} a_1 c_* - \epsilon_2 a_1 \frac{g_0}{4(1+a_2)}\Big) > 0.$$

Putting

$$\rho = \inf(\rho_1,\ \rho_2,\ \rho_3,\ \rho_4)$$

This yields

$$\begin{aligned}
\mathfrak{F}'(t) \le & -2\rho E(t) - \rho\Big(1 - \int_0^t g(\tau) d\tau\Big) \|\nabla u(t)\|_2^2 - \rho \|\nabla u(t)\|_2^2 - \rho g(t) \|\nabla u(t)\|_2^2 \\
\le & -2\rho E(t) \le -2\rho\alpha_1 \mathfrak{F}(\mathfrak{t}).
\end{aligned}$$

By a simple integration, we obtain

$$\mathfrak{F}(\mathfrak{t}) \le \mathfrak{F}(t_0) e^{2\alpha_1 \rho t_0} e^{-2\alpha_1 \rho t}.$$

By exploiting (3), we arrive at the desired result

$$E(t) \leq \lambda e^{-\mu t}, \ \forall t \geq t_0,$$

where

$$\lambda = \alpha_2 \mathfrak{F}(\mathfrak{t}_0) e^{2\alpha_1 \rho t_0} \text{ and } \mu = 2\alpha_1 \rho.$$

References

1. F. Alabau-Boussouira, P. Cannarsa and D. Sforza, Decay estimates for second order evolution equations with memory, *J. Funct. Anal.*, **254** (2008), No.5, 1342-1372.
2. J. Ball, Remarks on blow up and nonexistence theorems for nonlinear evolutions equations, *Quart. J. Math. Oxford.*, **28** (1977), 473-486.
3. R. Barreto, E. Lapa and J. Muňoz Rivera, Decay rates for viscoelastic plates with memory, *J. Elasticity*, **44** (1996), No.1, 61-87.
4. S. Berrimi and S. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, *Nonl. Anal.*, **64** (2006), 2314-2331.
5. E. Cabanillas and J. Muňoz Rivera, Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels, *Commun. Math. Phys.*, **177** (1996), 583-602.
6. M. Cavalcanti, V. Domingos Cavalcanti and J. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *E. J. Differ. Equ.*, **44**(2002), 1-14.
7. M. Cavalcanti and H. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.*, **42** (2003), No.4, 1310-1324.
8. M. Cavalcanti, V. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, *Math. Methods Appl. Sci.*, **24** (2001), 1043-1053.
9. M. Cavalcanti and J. Soriano. On solvability and stability of the wave equation with lower order terms and boundary feedback, *Rev, Mat. Apl.*, **18** (1997), 61-78.
10. M. Cavalcanti and A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, *Differential and Integral Equations*, **18** (2005), No.5, 583-600.
11. C. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, **37** (1970), 297-308.
12. C. Dafermos, On abstract volterra equations with applications to linear viscoelasticity, *J. Differ. Equ.*, **7** (1970), 554-569.
13. V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term, *J. Differ. Equ.*, **109** (1994), 295-308.

14. A. Guesmia, Nouvelles inégalités integrales et applications à la stabilisation des systèmes distribués non dissipatifs, *C. R. Acad. Sci. Sér I. Math, Paris.*, **336** (2003), 801-804.

15. A. Guesmia, A new approach of stabilization of nondissipative distributed systems, *SIAM J. Control Optim.*, **42** (2003), 24-52.

16. A. Haraux and E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Rational Mech. Anal.*, **150** (1988), 191-206.

17. M. Kafini and S. Messaoudi, A blow up result for a viscoelastic system in $\mathbb{R}^N$, *E. J. Differ. Equ.*, **113** (2006), 1-7.

18. V. Kalantarov and O. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, *J. Soviet Math.*, **10** (1978), 53-70.

19. M. Kopackova, Remarks on bounded solutions of a semilinear dissipative hyperbolic equation, *Comment. Math. Univ. Carolin.*, **30**(1989), 713-719.

20. H. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$, *Trans. Amer. Math. Soc.*, **192** (1974), 1-21.

21. H. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equation, *SIAM J. Math. Anal.*, **5** (1974), 138-146.

22. S. Messaoudi, Blow up in a nonlinearly damped wave equation, *Mathematische Nachrichten*, **231** (2001), 1-7.

23. S. Messaoudi, Blow up and global existence in nonlinear viscoelastic wave equations, *Math. Nachrich* **260** (2003), 58-66.

24. S. Messaoudi, Blow up of solutions with positive initial energy in a nonlinear viscoelastic wave equations, *J. Math. Anal. Appl.*, **320** (2006), 902-915.

25. S. Messaoudi and N. Tatar, Global existence asymptotic behavior for a Nonlinear Viscoelastic Problem, *Math. Meth. Sci. Research J.*, **7** (2003), No.4, 136-149.

26. S. Messaoudi and N. Tatar, Global existence and uniform stability of solutions for a quasilinear viscoelastic problem, *Math. Methods Appl. Sci.*, **30** (2007), 665-680.

27. E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Comm. PDE.*, **15**(1990), 205-235.