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FRACTIONAL SOBOLEV SPACES AND BOUNDARY VALUE PROBLEMS VIA HADAMARD DERIVATIVE

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Abstract

This paper is devoted to the existence and uniqueness of solution to a class of Hadamard fractional differential equation under fractional Sobolev spaces. A novel form of fractional Sobolev space via Hadamard fractional operator is well proposed and related properties are also proved. Furthermore, a variational formulation of considered system is established and thereby the Lax-Milgram theorem is also employed to demonstrate the existence and uniqueness.

1. Introduction

Compared to the classical calculus reflecting the local changes in dynamic process, the fractional calculus can simulate the global variability well. Generally speaking, depicting the process involved with the historic dependence, long distance interaction in real phenomena, hereditary factor, and so on, one needs fractional calculus for the time being, and the obtained mathematical models are often characterized by fractional differential/integral equations. In literatures, there exist several versions of fractional integrals and fractional derivatives, such as, Riemann-Liouville, Caputo, Grunwald-Letnikov, Riesz, Erdlyi-Kober, Weyl and Hadamard, etc, in which they are

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not equivalent to each other with its own advantages and disadvantages used to describe certain specific phenomena [18, 26, 27, 10, 29, 8, 22, 23, 24].

The two-point problems associated with a second-order differential equation, which have been enlarged to equations of fractional order between 1 and 2, are among those that have been widely addressed in both ancient and modern mathematical research. The concept of the weak solution, for which several formulas were discovered in the case of fractional orders according to the equations that were studied, received a lot of attention from the researchers. As an illustration, we provide a few works that made use of the variational approach and critical point theory.

The presence of a weak solution for fractional Euler-Lagrange equations

$$\frac{\partial L}{\partial x}(u, D_{-}^{\alpha}u, t) + D_{+}^{\alpha}\left(\frac{\partial L}{\partial y}(u, D_{-}^{\alpha}u, t)\right) = 0,$$

was examined by Bourdin [3]. The authors of [14] explored the identical Euler-Lagrange problem using a generalized fractional derivative of the Caputo type.

Chen and Liu ensure the existence of three weak solutions for the p-Laplacian problem in their citation [7]. We discover additional works in [21] and [15] for systems, respectively.

In [19, 20], the authors studied a nonlinear boundary value problem of the type

$$\begin{cases} D_{-}^{\alpha}(D_{+}^{\alpha}u(t)) = \nabla F(t, u(t)), \text{ in } (0,T), \\ u(0) = u(T) = 0, \end{cases}$$

In a classic two-point problem with Dirichlet conditions

$$\begin{cases} -u''(x) + \lambda(x)u(x) = f(x), \text{ in } (a,b), \\ u(a) = u_a, \ u(b) = u_b, \end{cases} \tag{1}$$

if it exists a weak solution of problem (1), it belongs to a Sobolev space adapted to this type of problems, it is the space $H^1(a,b)$, which is a space among the Sobolev spaces $W^{1,p}(a,b)$ based on the Lebesgue spaces $L^p(a,b)$. It is obvious to try to find spaces adapted to similar boundary problems, but associated to fractional order equations

$$(P') \quad \left(D^{\alpha}_{b^-}\left(D^{\alpha}_{a^+}u\right)\right)(x) + \lambda(x)u\left(x\right) = f\left(x\right), \text{ in } (a,b).$$

A classical Sobolev space $W^{1,p}(a,b)$ where $(a,b) \subset \mathbb{R}$ is defined by

$$W^{1,p}(a,b) = \left\{u \in L^p(a,b), \exists g \in L^p(a,b); \int_a^b u\varphi' = -\int_a^b g\varphi, \ \forall \varphi \in C_c^{\infty}(a,b)\right\}, \tag{2}$$

where $C_c^{\infty}(a,b)$ is the set of infinitely differentiable functions on (a,b) with compact support in (a,b). (cf. [1, 6]).

We can define the Sobolev space $W^{1,p}(a,b)$ of order $n > 1$ by

$$W^{n,p}(a,b) = \{u \in W^{n-1,p}(a,b), \ u' \in W^{n-1,p}(a,b)\}. \tag{3}$$

However, a Sobolev space $W^{n,p}(a,b)$ can be described [13] as follows $W^{n,p}(a,b) = AC^{n,p}(a,b)$, where $AC^{n,p}(a,b)$ is space of functions f from $[a,b]$ to $\mathbb{R}$ such that there exist $c_0, c_1, \ldots, c_{n-1}$ and $\varphi \in L^p(a,b)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!}(x-a)^k + \int_a^x \varphi(t)dt, \ x \in [a,b] \ a.e. \tag{4}$$

This space has a generalization to the space $AC^{\alpha}(a,b); (n-1 < \alpha < n)$ of functions f from $[a,b]$ to $\mathbb{R}$ such that there exist $c_0, c_1, \ldots, c_{n-1}$ and $\varphi \in L^p(a,b)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{c_k}{\Gamma(\alpha-n+1+k)}(x-a)^{\alpha-n+k} + I^{\alpha}_{a^+}\varphi, \ x \in [a,b] \ a.e, \tag{5}$$

where $I^{\alpha}_{a^+}$ denote the left Riemann-Liouville integral. The work was initially focused on discovering containment relationships between spaces $AC^{\alpha}(a,b)$ (inclusion, embedding, integration by parts... c.f [11, 12]), and later a set of definitions of fractional Sobolev spaces was proposed (c.f [13]).

In [4], a definition of fractional Sobolev space based on $L^1(a,b)$ is given as follows

$$W^{\alpha,1}_{a^+}(a,b) := \{u \in L^1(a,b); I^{1-\alpha}_{a^+}u \in W^{1,1}(a,b)\},$$
$$W^{\alpha,1}_{b^-}(a,b) := \{u \in L^1(a,b); I^{1-\alpha}_{b^-}u \in W^{1,1}(a,b)\},$$

which is later extended to $W^{\alpha,p}$ (c.f [9]).

In [13], the authors establish a fractional Sobolev space via the Riemann-Liouville approach. They gave topological properties, inclusions and imbeddings, then applied these results to two-point problems related to these spaces. Other properties and results are treated in the articles [4], [5], [11], [12].

In our work, we will treat a two-point problem in a bounded interval via Hadamard approach with homogeneous boundary conditions

$$\begin{cases} \left({}^{H}D_{b^-}^{\alpha}\left({}^{H}D_{a^+}^{\alpha}u\right)\right)(x)+\lambda(x)u(x)=f(x), \text{ in } (a,b), \\ \left({}^{H}D_{a^+}^{\alpha}u\right)(a)=0,\ u(b)=0. \end{cases}$$

First, we will give a functional framework adapted to this type of problems, then we prove the existence and uniqueness under conditions on $a,\ b,\ \lambda,\ f$, according to the following methodology: In section two, we give some preliminaries according to some important space and the fractional calculus. In section three, we establish fractional Sobolev spaces ${}^{H}W_{a^+}^{\alpha,p}(a,b)$ in a bounded interval (a,b) for $0<\alpha<1$ and $1\leq p<+\infty$. Finally, we present two theorems of existence and uniqueness of the problem considered.

2. Preliminaries

Let $1\leq p<+\infty,\ 0<\alpha\leq 1,\ c\in\mathbb{R}$ and a,b two positive real numbers such that $a<b<+\infty$.

Definition 1. The space $L^p(a,b)$ is the set of measurable functions f on (a,b) such that $\|f\|_{L^p(a,b)}<0$, defined by

$$\|f\|_{L^p(a,b)}=\left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}},$$

for $1\leq p<+\infty$, and

$$\|f\|_{L^\infty(a,b)}=\sup_{x\in(a,b)} ess\,|f(x)|\,.$$

The space $L^p(a,b)$ is a Banach space. In addition, $L^2[a,b]$ is a Hilbert space with respect to the scalar product

$$< f, g >= \int_a^b f(x) . g(x) dx.$$

Definition 2 ([18])**.** The space $X_c^p(a,b)$ is the set of measurable functions f from (a,b) to $\mathbb{R}$ such that

$$\int_a^b |x^c f(x)|^p \frac{dx}{x} < +\infty.$$

It is a Banach space respect to the norm

$$\|f\|_{X_c^p(a,b)} = \left(\int_a^b |x^c f(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}}.$$

Proposition 1. *Let $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $f \in X_c^p(a,b)$ and $g \in X_c^q(a,b)$ we have a type of Hölder's inequality*

$$\|f.g\|_{X_c^1(a,b)} \leq \|f\|_{X_c^p(a,b)} . \|g\|_{X_c^q(a,b)} . \tag{6}$$

Proof. Let $f \in X_c^p(a,b)$, $g \in X_c^q(a,b)$. Then we have

$$\int_a^b |x^c f(x) g(x)| \frac{dx}{x} = \int_a^b \left| \frac{x^{\frac{c}{p}} f(x)}{x^{\frac{1}{p}}} \right| \left| \frac{x^{\frac{c}{q}} g(x)}{x^{\frac{1}{q}}} \right| dx.$$

Using the Hölder's inequality in $L^p(a,b)$, we obtain:

$$\int_a^b \left| \frac{x^{\frac{c}{p}} f(x)}{x^{\frac{1}{p}}} \right| \left| \frac{x^{\frac{c}{q}} g(x)}{x^{\frac{1}{q}}} \right| dx \leq \left(\int_a^b \left| \frac{x^{\frac{c}{p}} f(x)}{x^{\frac{1}{p}}} \right|^p dx \right)^{\frac{1}{p}} \left(\int_a^b \left| \frac{x^{\frac{c}{q}} g(x)}{x^{\frac{1}{q}}} \right|^q dx \right)^{\frac{1}{q}},$$

hence,

$$\int_a^b |x^c f(x) g(x)| \frac{dx}{x} \leq \left(\int_a^b |x^c f(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}} \left(\int_a^b |x^c g(x)|^q \frac{dx}{x} \right)^{\frac{1}{q}}. \qquad \square$$

Theorem 1. *Let $0 < a < b < +\infty$, $X_c^p(a,b) = L^p(a,b)$ and the norms $\|.\|_{L^p(a,b)}$, $\|.\|_{X_c^p(a,b)}$ are equivalent.*

Proof. Let f be a function from (a,b) into $\mathbb{R}$, then we have

$$\int_a^b |x^c f(x)|^p \frac{dx}{x} = \int_a^b x^{pc-1} |f(x)|^p \, dx.$$

If $pc-1 \geq 0$, then

$$a^{pc-1} \int_a^b |f(x)|^p \, dx \leq \int_a^b x^{pc-1} |f(x)|^p \, dx \leq b^{pc-1} \int_a^b |f(x)|^p \, dx,$$

hence $f \in X_c^p(a,b)$ if and only if $f \in L^p(a,b)$ and we have

$$a^{c-\frac{1}{p}} \|f\|_{L^p(a,b)} \leq \|f\|_{X_c^p[a,b]} \leq b^{c-\frac{1}{p}} \|f\|_{L^p(a,b)}.$$

If $pc-1 < 0$ then

$$b^{pc-1} \int_a^b |f(x)|^p \, dx \leq \int_a^b x^{pc-1} |f(x)|^p \, dx \leq a^{pc-1} \int_a^b |f(x)|^p \, dx,$$

hence $f \in X_c^p(a,b)$ if and only if $f \in L^p(a,b)$ and we have

$$b^{c-\frac{1}{p}} \|f\|_{L^p(a,b)} \leq \|f\|_{X_c^p(a,b)} \leq a^{c-\frac{1}{p}} \|f\|_{L^p(a,b)}. \qquad \square$$

Remark 1. If $a=0$ or $b=+\infty$ the above theorem is false.

Definition 3 ([18])**.** We define the following spaces

$$\begin{aligned} AC[a,b] = &\Big\{ f : [a,b] \to \mathbb{R},\ f(x) = c + \int_a^x \psi(x)dx,\ x \in [a,b],\ c \in \mathbb{R}, \\ &\quad \psi \in L^1[a,b] \Big\}, \\ AC_\delta[a,b] = &\{ f : [a,b] \to \mathbb{R} :\ \delta\psi \in AC[a,b] \}. \end{aligned}$$

where $\delta = x\frac{d}{dx}$.

Now, we give some definitions and properties of fractional calculus.

Definition 4 ([18])**.** Let $\alpha > 0$, the left and right Riemann-Liouville fractional integrals of order α of $f \in L^p(a,b)$ are respectively defined by

$$\left({}^{RL}I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{7}$$

$$\left({}^{RL}I_{b^-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt, \tag{8}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 5 ([16])**.** Let $\alpha>0$, the left and right Hadamard fractional integrals of order α of $f\in L^p(a,b)$ are respectively defined by

$$\left({}^{H}I_{a^+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}f(t)\frac{dt}{t}. \tag{9}$$

$$\left({}^{H}I_{b^-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}f(t)\frac{dt}{t}. \tag{10}$$

Proposition 2. *For all* $f\in L^p(a,b)$, *we have* ${}^{H}I_{a^+}^{\alpha}f,\ {}^{H}I_{b^-}^{\alpha}f\in L^p(a,b)$, *Moreover*

$$\left\|{}^{H}I_{a^+}^{\alpha}f\right\|_{L^p(a,b)}\leq\frac{(b-a)^{\alpha}}{a^pb^{(\alpha-1)p}\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}, \tag{11}$$

$$\left\|{}^{H}I_{b^-}^{\alpha}f\right\|_{L^p(a,b)}\leq\frac{(b-a)^{\alpha}}{a^pb^{(\alpha-1)p}\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}. \tag{12}$$

Proof. Let $f\in L^p(a,b)$, then for all $x\in(a,b)$ we have

$$\left|\left({}^{H}I_{a^+}^{\alpha}f\right)(t)\right|=\frac{1}{\Gamma(\alpha)}\left|\int_a^x(\ln x-\ln t)^{\alpha-1}f(t)\frac{dt}{t}\right|.$$

Using the finite increase theorem for the function ln in the interval $[t,x]$, and taking into account that for all $\tau\in[t,x]\subset[a,b]$ we have $\frac{1}{a^{\alpha-1}}\leq\frac{1}{\tau^{\alpha-1}}\leq\frac{1}{b^{\alpha-1}}$. Then

$$\begin{aligned}\left|\left({}^{H}I_{a^+}^{\alpha}f\right)(t)\right| &\leq \frac{1}{b^{\alpha-1}\Gamma(\alpha)}\left|\int_a^x(x-t)^{\alpha-1}f(t)\frac{dt}{t}\right|\\ &\leq \frac{1}{ab^{\alpha-1}\Gamma(\alpha)}\left|\int_a^x(x-t)^{\alpha-1}f(t)\,dt\right|\\ &= \frac{1}{ab^{\alpha-1}}\left|\left({}^{RL}I_{a^+}^{\alpha}f\right)(t)\right|,\end{aligned}$$

hence

$$\left|\left({}^{H}I_{a^+}^{\alpha}f\right)(t)\right|^p\leq\frac{1}{a^pb^{(\alpha-1)p}}\left|\left({}^{RL}I_{a^+}^{\alpha}f\right)(t)\right|^p,$$

which deduce that

$$\left\|{}^{H}I_{a+}^{\alpha}f\right\|_{L^{p}(a,b)} \leq \frac{1}{a^{p}b^{(\alpha-1)p}}\left\|{}^{RL}I_{a+}^{\alpha}f\right\|_{L^{p}(a,b)}.$$

Noting that (from theorem 2.6 in [17]) we have

$$\left\|{}^{RL}I_{a+}^{\alpha}f\right\|_{L^{p}(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left\|f\right\|_{L^{p}(a,b)},$$

then,

$$\left\|{}^{H}I_{a+}^{\alpha}f\right\|_{L^{p}(a,b)} \leq \frac{(b-a)^{\alpha}}{a^{p}b^{(\alpha-1)p}\Gamma(\alpha+1)}\left\|f\right\|_{L^{p}(a,b)}.$$

The second inequality is made in the same way. □

The following proposition gives a similar formulation of integration by parts as to the relation (2.20) in [17]

Proposition 3. *Let $1 \leq p,q \leq +\infty$ be such that $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$. Then, for all $f \in L^{p}(a,b)$, $g \in L^{p}(a,b)$ we have*

$$\int_{b}^{a} f(x)\left({}^{H}I_{b-}^{\alpha}g\right)(x)\frac{dx}{x} = \int_{a}^{b} g(x)\,{}^{H}I_{a+}^{1-\alpha}f(x)\frac{dx}{x}. \tag{13}$$

Definition 6 ([16])**.** Let $0<\alpha<1$, the left and right Hadamard fractional derivatives of order α of $f \in L^{p}(a,b)$ are respectively defined by

$$\begin{aligned}\left({}^{H}D_{a+}^{\alpha}f\right)(x) &= \delta\left({}^{H}I_{a^{+}}^{1-\alpha}f\right)(x)\\ &= \frac{x}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\left(\ln\frac{x}{t}\right)^{-\alpha}f(t)\frac{dt}{t},\ a<x<b. \quad (14)\\ \left({}^{H}D_{b-}^{\alpha}f\right)(x) &= (-\delta)\left({}^{H}I_{b^{-}}^{n-\alpha}f\right)(x)\\ &= -\frac{x}{\Gamma(n-\alpha)}\frac{d}{dx}\int_{x}^{b}\left(\ln\frac{t}{x}\right)^{-\alpha}f(t)\frac{dt}{t},\ a<x<b. \quad (15)\end{aligned}$$

Theorem 2 ([16])**.** *Let $0<\alpha<1$, $f \in AC_{\delta}[a,b]$, we have*

$$\left({}^{H}I_{a^{+}}^{\alpha}\,{}^{H}D_{a+}^{\alpha}f\right)(x) = f(x) - \frac{\left({}^{H}I_{a^{+}}^{1-\alpha}f\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1}. \tag{16}$$

We define the following sets

Definition 7.

$$
\begin{aligned}
{}^{H}AC_{a^{+}}^{\alpha,p}\left(a,b\right) = \Big\{ u \;/\; u\left(x\right) = \frac{A}{\Gamma\left(\alpha\right)}\left(\ln\frac{x}{a}\right)^{\alpha-1} + \left({}^{H}I_{a^{+}}^{\alpha}\xi\right)\left(x\right), \\
A\in\mathbb{R},\ \xi\in L^{p}\left(a,b\right)\Big\}, \quad (17)
\end{aligned}
$$

$$
\begin{aligned}
{}^{H}AC_{b^{-}}^{\alpha,p}\left(a,b\right) = \Big\{ u \;/\; u\left(x\right) = \frac{B}{\Gamma\left(\alpha\right)}\left(\ln\frac{b}{x}\right)^{\alpha-1} + \left({}^{H}I_{b^{-}}^{\alpha}\zeta\right)\left(x\right), \\
B\in\mathbb{R},\ \zeta\in L^{p}\left(a,b\right)\Big\}. \quad (18)
\end{aligned}
$$

Theorem 3 (Integration by parts)**.** *Let $1 \le p,q < +\infty$ be such that $\frac{1}{p} < \alpha,\ \frac{1}{q} < \alpha$. Then for all $f \in {}^{H}AC_{a^{+}}^{\alpha,p}\left[a,b\right]$ and $g \in {}^{H}AC_{b^{-}}^{\alpha,q}\left[a,b\right]$, we have*

$$
\begin{aligned}
\int_{a}^{b} f\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}g\right)\left(x\right)\frac{dx}{x} &= \left({}^{H}I_{a^{+}}^{1-\alpha}f\right)\left(a\right)g\left(a\right) - \left({}^{H}I_{b^{-}}^{1-\alpha}g\right)\left(b\right)f\left(b\right) \\
&\quad + \int_{a}^{b}\left({}^{H}D_{a^{+}}^{\alpha}f\right)\left(x\right)g\left(x\right)\frac{dx}{x}. \qquad (19)
\end{aligned}
$$

Proof. From the definition 7, we have

$$
f\left(x\right) = \frac{A}{\Gamma\left(\alpha\right)}\left(\ln\frac{x}{a}\right)^{\alpha-1} + \left({}^{H}I_{a^{+}}^{\alpha}\xi\right)\left(x\right)
$$

and

$$
g\left(x\right) = \frac{B}{\Gamma\left(\alpha\right)}\left(\ln\frac{b}{x}\right)^{\alpha-1} + \left({}^{H}I_{b^{-}}^{\alpha}\zeta\right)\left(x\right),
$$

where $\xi,\ \zeta \in L^{p}\left(a,b\right)$. Then

$$
\begin{aligned}
&\int_{a}^{b} f\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}g\right)\left(x\right)\frac{dx}{x} \\
&= \int_{a}^{b}\left[\frac{A}{\Gamma\left(\alpha\right)}\left(\ln\frac{x}{a}\right)^{\alpha-1} + \left({}^{H}I_{a^{+}}^{\alpha}\xi\right)\left(x\right)\right]\zeta\left(x\right)\frac{dx}{x} \\
&= \frac{A}{\Gamma\left(\alpha\right)}\int_{a}^{b}\left(\ln\frac{x}{a}\right)^{\alpha-1}\zeta\left(x\right)\frac{dx}{x} + \int_{a}^{b}\left({}^{H}I_{a^{+}}^{\alpha}\xi\right)\left(x\right)\zeta\left(x\right)\frac{dx}{x} \\
&= A\left({}^{H}I_{b^{-}}^{\alpha}\zeta\right)\left(a\right) + \int_{a}^{b}\left({}^{H}I_{a^{+}}^{\alpha}\xi\right)\left(x\right)\zeta\left(x\right)\frac{dx}{x}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_a^b \left({}^H D_{a+}^{\alpha} f\right)(x)\, g(x)\, \frac{dx}{x} \\
&= \int_a^b \xi(x) \left[\frac{B}{\Gamma(\alpha)} \left(\ln \frac{b}{x} \right)^{\alpha-1} + \left({}^H I_{b-}^{\alpha} \zeta\right)(x) \right] \frac{dx}{x} \\
&= \frac{B}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{x} \right)^{\alpha-1} \xi(x) \frac{dx}{x} + \int_a^b \xi(x) \left({}^H I_{b-}^{\alpha} \zeta\right)(x) \frac{dx}{x} \\
&= B \left({}^H I_{a+}^{\alpha} \xi\right)(b) + \int_a^b \left({}^H I_{a+}^{\alpha} \xi\right)(x)\, \zeta(x)\, \frac{dx}{x}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_a^b f(x) \left({}^H D_{b-}^{\alpha} g\right)(x) \frac{dx}{x} - \int_a^b \left({}^H D_{a+}^{\alpha} f\right)(x)\, g(x) \frac{dx}{x} \\
&\quad = A \left({}^H I_{b-}^{\alpha} \zeta\right)(a) - B \left({}^H I_{a+}^{\alpha} \xi\right)(b).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
A \left({}^H I_{b-}^{\alpha} \zeta\right)(a) &= A g(a) - \frac{AB}{\Gamma(\alpha)} \left(\ln \frac{b}{a} \right)^{\alpha-1}, \\
B \left({}^H I_{a+}^{\alpha} \xi\right)(b) &= B f(b) - \frac{AB}{\Gamma(\alpha)} \left(\ln \frac{b}{a} \right)^{\alpha-1},
\end{aligned}
$$

then

$$
\begin{aligned}
A \left({}^H I_{b-}^{\alpha} \zeta\right)(a) - B \left({}^H I_{a+}^{\alpha} \xi\right)(b) &= A g(a) - B f(b) \\
&= \left({}^H I_{a^+}^{1-\alpha} f\right)(a)\, g(a) - \left({}^H I_{b^-}^{1-\alpha} g\right)(b)\, f(b).
\end{aligned}
$$

Finally, we get (19). □

3. Fractional Sobolev Spaces via Hadamard Operator

Let $0 < \alpha < 1$, $1 \leq p < +\infty$ and $0 < a < b < +\infty$. As in the classical Sobolev spaces, following the definitions in [13], and taking into account theorem 1 we establish the following space

Definition 8. A fractional Sobolev space via Hadamard operator is given by

$$ {}^{H}W_{a^{+}}^{\alpha,p}(a,b)=\left\{ \begin{array}{l} u\in L^{p}\left(a,b\right)\ \ /\ \exists g\in L^{p}\left(a,b\right); \\ \displaystyle\int_{a}^{b}u\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\frac{dx}{x}=\int_{a}^{b}g\left(x\right)\varphi\left(x\right)\frac{dx}{x},\ \forall\varphi\in C_{c}^{\infty}(a,b) \end{array} \right\}. \tag{20} $$

Proposition 4. *The function g coincides with ${}^{H}D_{a+}^{\alpha}u$ in $[a,b]$.*

Proof. Let $u\in {}^{H}W_{a^{+}}^{\alpha,p}\left(a,b\right)$, and $\varphi\in C_{c}^{\infty}(a,b)$. It is easy to show that $\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\in L^{q}\left(a,b\right)$, where $\frac{1}{p}+\frac{1}{q}=1$.
Since $\varphi\left(a\right)=\varphi\left(b\right)=0$, we get

$$ \left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)=-\left({}^{H}I_{b^{-}}^{1-\alpha}\delta\varphi\left(t\right)\right)\left(x\right)=-\left({}^{H}I_{b^{-}}^{1-\alpha}t\varphi'\left(t\right)\right)\left(x\right), $$

and

$$ \int_{a}^{b}u\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\frac{dx}{x}=-\int_{a}^{b}u\left(x\right)\,{}^{H}I_{b^{-}}^{1-\alpha}\left(x\varphi'\left(x\right)\right)\frac{dx}{x}. $$

Applying 13, we obtain

$$ \begin{aligned} \int_{a}^{b}u\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\frac{dx}{x} &= -\int_{a}^{b}\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)\left(x\right)\left(x\varphi'\left(x\right)\right)\frac{dx}{x} \\ &= -\int_{a}^{b}\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)\left(x\right)\varphi'\left(x\right)dx. \end{aligned} $$

Using the classical integration by parts, we get

$$ \begin{aligned} &\int_{a}^{b}u\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\frac{dx}{x} \\ &= \left[-\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)\left(x\right)\varphi\left(x\right)\right]_{a}^{b}+\int_{a}^{b}\left[\frac{d}{dx}\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)\right]\left(x\right)\varphi\left(x\right)dx \\ &= \int_{a}^{b}\left(\delta^{H}I_{a^{+}}^{1-\alpha}u\right)\left(x\right)\varphi\left(x\right)\frac{dx}{x}=\int_{a}^{b}\left({}^{H}D_{a+}^{\alpha}u\right)\left(x\right)\varphi\left(x\right)\frac{dx}{x}. \end{aligned} $$

Since

$$ \int_{a}^{b}u\left(x\right)\left({}^{H}D_{b^{-}}^{\alpha}\varphi\right)\left(x\right)\frac{dx}{x}=\int_{a}^{b}g\left(x\right).\varphi\left(x\right)\frac{dx}{x}, $$

then

$$ g\left(x\right)=\left({}^{H}D_{a+}^{\alpha}u\right)\left(x\right)\ \text{ a e in }\ [a,b]. $$

□

Theorem 4. ${}^{H}W_{a^{+}}^{\alpha,p}(a,b) = {}^{H}AC_{a^{+}}^{\alpha,p}(a,b) \cap L^{p}(a,b)$.

Proof. Let $u \in {}^{H}W_{a^{+}}^{\alpha,p}(a,b)$, then $u \in L^{p}(a,b)$, ${}^{H}D_{a^{+}}^{\alpha}u \in L^{p}(a,b)$ and

$$\left({}^{H}I_{a^{+}}^{\alpha}\ {}^{H}D_{a^{+}}^{\alpha}u\right)(x) = u(x) - \frac{\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1},$$

hence

$$u(x) = \frac{\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1} + \left({}^{H}I_{a^{+}}^{\alpha}\ {}^{H}D_{a^{+}}^{\alpha}u\right)(x). \tag{21}$$

Setting $A = \left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a)$ and $\xi = {}^{H}D_{a^{+}}^{\alpha}u$, we get the direct inclusion.

Reciprocally, let $u \in {}^{H}AC_{a^{+}}^{\alpha,p}(a,b) \cap L^{p}(a,b)$, then $u \in L^{p}(a,b)$ and there exists $A \in \mathbb{R}$ and $\xi \in L^{p}(a,b)$ such that

$$u(x) = \frac{A}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1} + \left({}^{H}I_{a^{+}}^{\alpha}\xi\right)(x),$$

so ${}^{H}D_{a^{+}}^{\alpha}u = \xi \in L^{p}(a,b)$, which gives the reciprocal inclusion. □

Corollary 1. *From* (11) *and the above result, we obtain*

$$\left\| u - \frac{\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1} \right\|_{L^{p}(a,b)} \leq \frac{(b-a)^{\alpha}}{a^{p}b^{(\alpha-1)p}\Gamma(\alpha+1)}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{p}(a,b)}, \tag{22}$$

$\forall u \in {}^{H}W_{a^{+}}^{\alpha,p}(a,b)$.

Remark 2. Note that if u take the form (21), then $u \in L^{p}(a,b)$ if and only if

$$\frac{\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1} \in L^{p}(a,b).$$

Hence,

If $(1-\alpha)p \geq 1$, then $u \in L^{p}(a,b)$ if and only if $\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a) = 0$.

If $\left({}^{H}I_{a^{+}}^{1-\alpha}u\right)(a) \neq 0$, then $u \in L^{p}(a,b)$ if and only if $(1-\alpha)p < 1$.

Definition 9. We define in ${}^{H}W_{a^{+}}^{\alpha,p}(a,b)$, two norms

$${}^{1}\|u\|_{{}^{H}W_{a^{+}}^{\alpha,p}(a,b)}^{p} = \|u\|_{L^{p}(a,b)}^{p} + \left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{p}(a,b)}^{p}. \tag{23}$$

$$^{2}\left\|u\right\|^{p}_{^{H}W^{\alpha,p}_{a^{+}}(a,b)}=\left|^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)\right|^{p}+\left\|^{H}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}. \tag{24}$$

Theorem 5. *The norm* $^{1}\left\|.\right\|_{^{H}W^{\alpha,p}_{a^{+}}(a,b)}$ *is equivalent to the norm* $^{2}\left\|.\right\|_{^{H}W^{\alpha,p}_{a^{+}}(a,b)}$.

Proof. Let $u\in {}^{H}W^{\alpha,p}_{a^{+}}\left(a,b\right)$. We distinguish two cases:
Case 1: $(1-\alpha p)<1$, u can be expressed as

$$u=\frac{^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)}{\Gamma(\alpha)}(\ln\frac{x}{a})^{\alpha-1}+{}^{H}I^{\alpha H}_{a^{+}}D^{\alpha}_{a^{+}}u(x).$$

Using the same arguments of theorem 24 in [13], we obtain

$$\begin{aligned}\left\|u\right\|^{p}_{L^{p}(a,b)} &= \int_{a}^{b}\left|\frac{^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1}+{}^{H}I^{\alpha H}_{a^{+}}D^{\alpha}_{a^{+}}u(x)\right|^{p}dx\\ &\leq 2^{p-1}\left(\left|\frac{^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)}{\Gamma(\alpha)}\right|^{p}\int_{a}^{b}\left(\ln\frac{x}{a}\right)^{(\alpha-1)p}dx+\left\|^{H}I^{\alpha H}_{a^{+}}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}\right)\\ &\leq \frac{2^{p-1}}{((\alpha-1)p+1)\Gamma^{p}(\alpha)}\left(\ln\frac{b}{a}\right)^{(\alpha-1)p+1}\left|^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)\right|^{p}\\ &\quad+\frac{2^{p-1}(b-a)^{\alpha}}{a^{p}b^{(\alpha-1)p}\Gamma(\alpha+1)}\left\|^{H}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}.\end{aligned}$$

Hence

$$\begin{aligned}^{1}\left\|u\right\|^{p}_{^{H}W^{\alpha,p}_{a^{+}}(a,b)} &= \left\|u\right\|^{p}_{L^{p}(a,b)}+\left\|^{H}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}\\ &\leq \frac{2^{p-1}}{((\alpha-1)p+1)\Gamma^{p}(\alpha)}\left(\ln\frac{b}{a}\right)^{(\alpha-1)p+1}\left|^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)\right|^{p}\\ &\quad+\left(1+\frac{2^{p-1}(b-a)^{\alpha}}{a^{p}b^{(\alpha-1)p}\Gamma(\alpha+1)}\right)\left\|^{H}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}\\ &\leq M_{1}\left(\left|^{H}I^{1-\alpha}_{a^{+}}u\left(a\right)\right|^{p}+\left\|^{H}D^{\alpha}_{a^{+}}u\right\|^{p}_{L^{p}(a,b)}\right)\\ &= M_{1}\,^{2}\left\|u\right\|^{p}_{^{H}W^{\alpha,p}_{a^{+}}(a,b)},\end{aligned}$$

where

$$M_{1}=\max\left\{\frac{2^{p-1}}{((\alpha-1)p+1)\Gamma^{p}(\alpha)}\left(\ln\frac{b}{a}\right)^{(\alpha-1)p+1},1+\frac{2^{p-1}(b-a)^{\alpha}}{a^{p}b^{(\alpha-1)p}\Gamma(\alpha+1)}\right\}.$$

Reciprocally, from the integral mean theorem there exists $x_0\in[a,b]$ such that

$$ {}^{H}I_{a^{+}}^{1-\alpha}u\left(x_{0}\right)=\frac{1}{b-a}\int_{a}^{b}{}^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)dx. $$

Noting that we can write

$$ \begin{aligned} {}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right) &= {}^{H}I_{a^{+}}^{1-\alpha}u\left(x_{0}\right)-\int_{a}^{x_{0}}\frac{d}{dx}\,{}^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)dx \\ &= \frac{1}{b-a}\int_{a}^{b}{}^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)dx-\int_{a}^{x_{0}}\delta^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)\frac{dx}{x} \\ &= \frac{1}{b-a}\int_{a}^{b}{}^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)dx-\int_{a}^{x_{0}}{}^{H}D_{a^{+}}^{\alpha}u(x)\,\frac{dx}{x}. \end{aligned} $$

Then,

$$ \begin{aligned} \left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right)\right| &\leq \frac{1}{b-a}\int_{a}^{b}\left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(x\right)\right|dx+\int_{a}^{x_{0}}\left|{}^{H}D_{a^{+}}^{\alpha}u(x)\right|\frac{dx}{x} \\ &\leq \frac{1}{b-a}\left\|{}^{H}I_{a^{+}}^{1-\alpha}u\right\|_{L^{1}(a,b)}+\frac{1}{a}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{1}(a,b)}. \end{aligned} $$

Using the Hölder's inequality, we obtain

$$ \left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right)\right|\leq\frac{1}{(b-a)^{\frac{1}{p}}}\left\|{}^{H}I_{a^{+}}^{1-\alpha}u\right\|_{L^{p}(a,b)}+\frac{1}{a(b-a)^{\frac{1}{p}-1}}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{1}(a,b)}. $$

Let's apply (11) to the first term of the right-hand side above, we find

$$ \left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right)\right|\leq\frac{(b-a)^{1-\alpha-\frac{1}{p}}}{a^{p}b^{-\alpha p}\Gamma(2-\alpha)}\left\|u\right\|_{L^{p}(a,b)}+\frac{1}{a(b-a)^{\frac{1}{p}-1}}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{1}(a,b)}, $$

hence,

$$ \begin{aligned} \left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right)\right|^{p} &\leq \left(\frac{(b-a)^{1-\alpha-\frac{1}{p}}}{a^{p}b^{-\alpha p}\Gamma(2-\alpha)}\left\|u\right\|_{L^{p}(a,b)}+\frac{(b-a)^{1-\frac{1}{p}}}{a}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{1}(a,b)}\right)^{p} \\ &\leq 2^{p-1}\left(\frac{(b-a)^{(1-\alpha)p-1}}{a^{2p}b^{-2\alpha p}(\Gamma(2-\alpha))^{p}}\left\|u\right\|_{L^{p}(a,b)}^{p}\right. \\ &\qquad\qquad \left.+\frac{(b-a)^{p-1}}{a^{p}}\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{1}(a,b)}^{p}\right). \end{aligned} $$

We deduce that

$$ {}^{2}\left\|u\right\|_{{}^{H}W_{a^{+}}^{\alpha,p}(a,b)}^{p}=\left|{}^{H}I_{a^{+}}^{1-\alpha}u\left(a\right)\right|^{p}+\left\|{}^{H}D_{a^{+}}^{\alpha}u\right\|_{L^{p}(a,b)}^{p} $$

$$
\begin{aligned}
&\leq \frac{2^{p-1}(b-a)^{(1-\alpha)p-1}}{a^{2p}b^{-2\alpha p}(\Gamma(2-\alpha))^p} \|u\|_{L^p(a,b)}^p \\
&\quad + \left(1+\frac{2^{p-1}(b-a)^{p-1}}{a^p}\right) \left\| {}^H D_{a^+}^{\alpha} u \right\|_{L^1(a,b)}^p \\
&\leq M_2 \left(\|u\|_{L^p(a,b)}^p + \left\| {}^H D_{a^+}^{\alpha} u \right\|_{L^1(a,b)}^p \right) \\
&= M_2 {}^1 \|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^p ,
\end{aligned}
$$

where

$$
M_2 = \max \left\{ \frac{2^{p-1}(b-a)^{(1-\alpha)p-1}}{a^{2p}b^{-2\alpha p}(\Gamma(2-\alpha))^p}, 1+\frac{2^{p-1}(b-a)^{p-1}}{a^p} \right\}
$$

Case 2: $(1-\alpha p) \geq 1$. From the Remark 2 we have ${}^H I_{a^+}^{1-\alpha} u(a) = 0$. Then,

$$
{}^2 \|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^p = \left\| {}^H D_{a^+}^{\alpha} u \right\|_{L^p(a,b)}^p \leq {}^1 \|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^p ,
$$

and

$$
\begin{aligned}
{}^1 \|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^p &= \|u\|_{L^p(a,b)}^p + \left\| {}^H D_{a^+}^{\alpha} u \right\|_{L^p(a,b)}^p , \\
&\leq \left(1+\frac{2^{p-1}(b-a)^{\alpha}}{a^p b^{(\alpha-1)p}\Gamma(\alpha+1)}\right) \left\| {}^H D_{a^+}^{\alpha} u \right\|_{L^p(a,b)}^p , \\
&= \left(1+\frac{2^{p-1}(b-a)^{\alpha}}{a^p b^{(\alpha-1)p}\Gamma(\alpha+1)}\right) {}^2 \|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^p . \qquad \square
\end{aligned}
$$

Theorem 6. *The space ${}^H W_{a^+}^{\alpha,p}(a,b)$ is complete respect to each norm ${}^1 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$ or ${}^2 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$.*

Proof. Using the same arguments from the proof of Theorem 25 in [13], it follows that is complete respect to each norm ${}^2 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$. From the equivalence of norms ${}^1 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$ or ${}^2 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$, we deduce that ${}^H W_{a^+}^{\alpha,p}(a,b)$ is complete respect to each norm ${}^1 \|.\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}$. □

The following two theorems are established in the same way as the theorem 3.5 in [1] (see also Theorem 8.1 in [6], theorem 26 and theorem 27 in [13]).

Theorem 7. *The space ${}^H W_{a^+}^{\alpha,p}(a,b)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The space ${}^H W_{a^+}^{\alpha,2}(a,b)$ is a separable Hilbert space.*

4. A Boundary Value Problem

In this section we assume that $\frac{1}{2} < \alpha < 1$, $0 < a < b < +\infty$. Considering the following two points problem with homogenous mixed boundary conditions

$$(P) \quad \begin{cases} \left({}^{H}D_{b-}^{\alpha}\left({}^{H}D_{a+}^{\alpha}u\right)\right)(x) + \lambda(x)u(x) = f(x), \text{ in } (a,b), \\ \left({}^{H}D_{a+}^{\alpha}u\right)(a) = 0,\ u(b) = 0. \end{cases}$$

wher $\lambda \in L^{\infty}(a,b)$ and $f \in L^{2}(a,b)$.

Let K be the closed subspace defined by

$$K = \left\{ v \in {}^{H}W_{a^{+}}^{\alpha,p}(a,b),\ \left({}^{H}D_{a+}^{\alpha}v\right)(a) = 0,\ v(b) = 0 \right\}.$$

To establish a variational formulation of this problem, we will multiply the two terms of the first equation by a sufficiently regular function v, so we get

$$\left({}^{H}D_{b-}^{\alpha}\left({}^{H}D_{a+}^{\alpha}u\right)\right)(x)\, v(x) + \lambda(x)u(x)\, v(x) = f(x)\, v(x).$$

By integrating over the interval (a,b)

$$\int_a^b \left({}^{H}D_{b-}^{\alpha}\left({}^{H}D_{a+}^{\alpha}u\right)\right)(x)\, v(x)\frac{dx}{x} + \int_a^b \lambda(x)u(x)\, v(x)\frac{dx}{x} = \int_a^b f(x)\, v(x)\frac{dx}{x}.$$

Using the formula (13) of integration by parts in the first term, we obtain

$$\begin{aligned} &\int_a^b \left({}^{H}D_{b-}^{\alpha}\left({}^{H}D_{a+}^{\alpha}u\right)\right)(x)\, v(x)\frac{dx}{x} \\ &\quad= \left({}^{H}I_{a^{+}}^{1-\alpha}v\right)(a)\left({}^{H}D_{a+}^{\alpha}u\right)(a) - {}^{H}I_{b^{-}}^{1-\alpha}\left({}^{H}D_{a+}^{\alpha}u\right)(b)\, v(b) \\ &\qquad+ \int_a^b \left({}^{H}D_{a+}^{\alpha}u\right)(x)\left({}^{H}D_{a+}^{\alpha}v\right)(x)\frac{dx}{x} \\ &\quad= \int_a^b \left({}^{H}D_{a+}^{\alpha}u\right)(x)\left({}^{H}D_{a+}^{\alpha}v\right)(x)\frac{dx}{x}. \end{aligned}$$

At this step, we can assume that $v \in K$. Taking into account the above assumption and the boundary conditions, we arrive to the variational problem

$$(PV) \int_a^b \left({}^{H}D_{a+}^{\alpha}u\right)(x)\left({}^{H}D_{a+}^{\alpha}v\right)(x)\frac{dx}{x} + \int_a^b \lambda(x)u(x)v(x)\frac{dx}{x} = \int_a^b f(x)v(x)\frac{dx}{x}.$$

Reciprocally, let $u \in K$ such that

$$\int_a^b \left({}^H D_{a+}^{\alpha} u\right)(x)\left({}^H D_{a+}^{\alpha} \varphi\right)(x)\frac{dx}{x} + \int_a^b \lambda(x) u\,(x)\,\varphi\,(x)\,\frac{dx}{x} = \int_a^b f\,(x)\,\varphi\,(x)\frac{dx}{x},$$

$\forall \varphi \in C_c^{\infty}(a,b)$, then

$$\int_a^b \left({}^H D_{b-}^{\alpha H} D_{a+}^{\alpha} u\right)(x)\,\varphi\,(x)\,\frac{dx}{x} + \int_a^b \lambda(x) u\,(x)\,\varphi\,(x)\frac{dx}{x} = \int_a^b f\,(x)\,\varphi\,(x)\frac{dx}{x},$$

hence

$$\int_a^b \left[\frac{\left({}^H D_{b-}^{\alpha H} D_{a+}^{\alpha} u\right)(x) + \lambda(x) u\,(x) - f\,(x)}{x}\right] \varphi\,(x)\,dx = 0,\ \ \forall \varphi \in C_c^{\infty}(a,b), \tag{25}$$

which deduce that

$$\left({}^H D_{b-}^{\alpha H} D_{a+}^{\alpha} u\right)(x) + \lambda(x).u\,(x) = f\,(x) \ \text{ a e in } \ (a,b).$$

The following theorems ensure the existence and uniqueness of the solution of problem (PV).

Theorem 8. *Assume that there exists $\lambda_0 > 0$ such that*

$$\lambda(x) \geq \lambda_0 \ \ a.e \ in \ (a,b). \tag{26}$$

Then, the problem (PV) admits a unique solution $u \in K$.

Proof. Putting

$$\begin{aligned} A(u,v) &= \int_a^b \left({}^H D_{a+}^{\alpha} u\right)(x)\left({}^H D_{a+}^{\alpha} v\right)(x)\,\frac{dx}{x} \\ &\quad + \int_a^b \lambda(x) u\,(x)\, v\,(x)\,\frac{dx}{x},\ \ (u,v) \in K\,, \end{aligned} \tag{27}$$

$$L(v) = \int_a^b f\,(x)\, v\,(x)\,\frac{dx}{x},\ \ v \in K. \tag{28}$$

It is easy to see that A is a bilinear operator and L is a linear operator.

Moreover, we have

$$\begin{aligned}|A(u,v)| &\leq \int_a^b \left|\left({}^H D_{a^+}^{\alpha} u\right)(x)\left({}^H D_{a^+}^{\alpha} v\right)(x)\right| \frac{dx}{x} + \int_a^b |\lambda(x) u(x) v(x)| \frac{dx}{x} \\ &\leq \frac{1}{a}\left\|{}^H D_{a^+}^{\alpha} u\right\|_{L^2(a,b)} \left\|{}^H D_{a^+}^{\alpha} v\right\|_{L^2(a,b)} + \frac{\|\lambda\|_{L^\infty(a,b)}}{a}\|u\|_{L^2(a,b)}\|v\|_{L^2(a,b)} \\ &\leq \frac{\max\left\{1, \|\lambda\|_{L^\infty(a,b)}\right\}}{a}\Big(\|u\|_{L^2(a,b)}\|v\|_{L^2(a,b)} \\ &\qquad\qquad + \left\|{}^H D_{a^+}^{\alpha} u\right\|_{L^2(a,b)}\left\|{}^H D_{a^+}^{\alpha} v\right\|_{L^2(a,b)}\Big) \\ &\leq \frac{2\max\left\{1, \|\lambda\|_{L^\infty(a,b)}\right\}}{a}\|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}\|v\|_{{}^H W_{a^+}^{\alpha,p}(a,b)},\end{aligned}$$

and

$$|L(v)| \leq \frac{\|f\|_{L^2(a,b)}}{a}\|v\|_{L^2(a,b)} \leq \frac{\|f\|_{L^2(a,b)}}{a}\|v\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}.$$

Then, A and L are continuous.

Now, we prove that A is coercive. We have

$$\begin{aligned}A(u,u) &= \int_a^b \left({}^H D_{a^+}^{\alpha} u\right)^2(x) . \left({}^H D_{a^+}^{\alpha} v\right) \frac{dx}{x} + \int_a^b \lambda(x) u^2(x) \frac{dx}{x} \\ &\geq \frac{1}{b}\left\|{}^H D_{a^+}^{\alpha} u\right\|_{L^2(a,b)}^2 + \frac{\lambda_0}{b}\|u\|_{L^2(a,b)}^2 \\ &\geq \frac{\min\{1, \lambda_0\}}{b}\|u\|_{{}^H W_{a^+}^{\alpha,p}(a,b)}^2.\end{aligned}$$

Hence, A is coercive. From the Lax-Milgram theorem, the problem (PV) has a unique solution $u \in K$. □

Theorem 9. *Assume that*

$$\frac{b}{a} < e, \tag{29}$$

and

$$\frac{(b-a)^{\alpha}\|\lambda\|_{L^\infty(a,b)}}{a^5 b^{4(\alpha-1)}\Gamma^2(\alpha+1)\left[1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}\right]} < \frac{1}{b}. \tag{30}$$

Then, the problem (PV) admits a unique solution $u \in K$.

Proof. We have proved the continuity of A and L, it remains to prove the coercivity of A. For this we have

$$A(u,u) = \frac{1}{b}\left\|{}^H D^\alpha_{a^+} u\right\|^2_{L^2(a,b)} - \frac{\|\lambda\|_{L^\infty(a,b)}}{a}\|u\|^2_{L^2(a,b)}.$$

From (22), we have

$$\left\|u - \frac{\left({}^H I^{1-\alpha}_{a^+} u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1}\right\|^2_{L^2(a,b)} \le \frac{(b-a)^{2\alpha}}{a^4 b^{4(\alpha-1)}\Gamma^2(\alpha+1)}\left\|{}^H D^\alpha_{a^+} u\right\|^2_{L^2(a,b)}.$$

Note that

$$\begin{aligned}
&\left\|u - \frac{\left({}^H I^{1-\alpha}_{a^+} u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1}\right\|^2_{L^2(a,b)} \\
&\quad = \|u\|^2_{L^2(a,b)} + \left\|\frac{\left({}^H I^{1-\alpha}_{a^+} u\right)(a)}{\Gamma(\alpha)}\left(\ln\frac{x}{a}\right)^{\alpha-1}\right\|^2_{L^2(a,b)} \\
&\qquad - \frac{2\left({}^H I^{1-\alpha}_{a^+} u\right)(a)}{\Gamma(\alpha)}\int_a^b u(x)\left(\ln\frac{x}{a}\right)^{\alpha-1}.
\end{aligned}$$

From the Hölder's inequality

$$\int_a^b u(x)\left(\ln\frac{x}{a}\right)^{\alpha-1} \le \sqrt{\frac{b}{2\alpha-1}}\left(\ln\frac{b}{a}\right)^{\alpha-\frac{1}{2}}\|u\|_{L^2(a,b)},$$

then

$$\begin{aligned}
&-\frac{2\left({}^H I^{1-\alpha}_{a^+} u\right)(a)}{\Gamma(\alpha)}\int_a^b u(x)\left(\ln\frac{x}{a}\right)^{\alpha-1} \\
&\quad \ge -\frac{2\sqrt{b}}{\sqrt{2\alpha-1}\Gamma(\alpha)}\left(\ln\frac{b}{a}\right)^{\alpha-\frac{1}{2}}\left|\left({}^H I^{1-\alpha}_{a^+} u\right)(a)\right|\|u\|_{L^2(a,b)} \\
&\quad = -2\left[\frac{\sqrt{b}}{\sqrt{2\alpha-1}\Gamma(\alpha)}\left|\left({}^H I^{1-\alpha}_{a^+} u\right)(a)\right|\right]\left[\left(\ln\frac{b}{a}\right)^{\alpha-\frac{1}{2}}\|u\|_{L^2(a,b)}\right] \\
&\quad \ge -\frac{b}{(2\alpha-1)\Gamma^2(\alpha)}\left|\left({}^H I^{1-\alpha}_{a^+} u\right)(a)\right|^2 - \left(\ln\frac{b}{a}\right)^{2\alpha-1}\|u\|^2_{L^2(a,b)},
\end{aligned}$$

and since

$$\left\| \frac{\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)} \left(\ln \frac{x}{a}\right)^{\alpha-1} \right\|_{L^2(a,b)}^2 \geq \frac{a}{(2\alpha-1)\Gamma^2(\alpha)} \left(\ln \frac{b}{a}\right)^{2\alpha-1} \left|\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)\right|^2,$$

we get

$$\begin{aligned} &\frac{a\left(\ln \frac{b}{a}\right)^{2\alpha-1} - b}{(2\alpha-1)\Gamma^2(\alpha)} \left|\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)\right|^2 + \left[1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}\right] \|u\|_{L^2(a,b)}^2 \\ &\quad \leq \frac{(b-a)^{2\alpha}}{a^4 b^{4(\alpha-1)}\Gamma^2(\alpha+1)} \left\|{}^{H}D_{a^+}^{\alpha}u\right\|_{L^2(a,b)}^2 . \end{aligned}$$

Then,

$$\begin{aligned} \left[1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}\right] \|u\|_{L^2(a,b)}^2 \leq\ & \frac{b - a\left(\ln \frac{b}{a}\right)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)} \left|\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)\right|^2 \\ & + \frac{(b-a)^{2\alpha}}{a^4 b^{4(\alpha-1)}\Gamma^2(\alpha+1)} \left\|{}^{H}D_{a^+}^{\alpha}u\right\|_{L^2(a,b)}, \end{aligned}$$

From (29) we deduce that $\left(\ln \frac{b}{a}\right)^{2\alpha-1} < 1$, then

$$\begin{aligned} \|u\|_{L^2(a,b)}^2 \leq\ & \frac{b - a\left(\ln \frac{b}{a}\right)^{2\alpha-1}}{\left[1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}\right](2\alpha-1)\Gamma^2(\alpha)} \left|\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)\right|^2 \\ & + \frac{(b-a)^{2\alpha}}{a^4 b^{4(\alpha-1)}\Gamma^2(\alpha+1)\left[1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}\right]} \left\|{}^{H}D_{a^+}^{\alpha}u\right\|_{L^2(a,b)}, \end{aligned}$$

hence

$$\begin{aligned} & -\frac{\|\lambda\|_{L^\infty(a,b)}}{a} \|u\|_{L^2(a,b)}^2 \\ & \quad \geq \left[\frac{b - a\left(\ln \frac{b}{a}\right)^{2\alpha-1}}{1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}}\right] \frac{\|\lambda\|_{L^\infty(a,b)}}{a(2\alpha-1)\Gamma^2(\alpha)} \left|\left({}^{H}I_{a^+}^{1-\alpha}u\right)(a)\right|^2 \\ & \qquad - \frac{(b-a)^{2\alpha}\|\lambda\|_{L^\infty(a,b)}}{a^5 b^{4(\alpha-1)}\Gamma^2(\alpha+1)\left[1 - \left(\ln \frac{b}{a}\right)^{2\alpha-1}\right]} \left\|{}^{H}D_{a^+}^{\alpha}u\right\|_{L^2(a,b)}^2, \end{aligned}$$

then

$$\begin{aligned} A(u,u) &\geq \left[\frac{b-a\left(\ln\frac{b}{a}\right)^{2\alpha-1}}{1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}}\right]\frac{\|\lambda\|_{L^\infty(a,b)}}{a(2\alpha-1)\Gamma^2(\alpha)}\left|\left({}^H I_{a^+}^{1-\alpha}u\right)(a)\right|^2 \\ &\quad+\left[\frac{1}{b}-\frac{(b-a)^\alpha\|\lambda\|_{L^\infty(a,b)}}{a^5 b^{4(\alpha-1)}\Gamma^2(\alpha+1)\left[1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}\right]}\right]\left\|{}^H D_{a^+}^\alpha u\right\|_{L^2(a,b)}^2 \\ &\geq M\|u\|^2_{{}^H W_{a^+}^{\alpha,p}(a,b)}, \end{aligned}$$

Since $\left(\ln\dfrac{b}{a}\right)^{2\alpha-1}<\dfrac{b}{a}$ we get $b-a\left(\ln\dfrac{b}{a}\right)^{2\alpha-1}>0$, $\dfrac{b-a\left(\ln\frac{b}{a}\right)^{2\alpha-1}}{1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}}>0.$
From (30) we deduce that $1-\dfrac{(b-a)^\alpha\|\lambda\|_{L^\infty(a,b)}}{a^5 b^{4\alpha-3}\Gamma^2(\alpha+1)\left[1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}\right]}>0$, then

$$A(u,u)\geq M\|u\|^2_{{}^H W_{a^+}^{\alpha,p}(a,b)},$$

where

$$\begin{aligned} M=\min\Bigg\{&\left(\frac{b-a\left(\ln\frac{b}{a}\right)^{2\alpha-1}}{1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}}\right)\frac{\|\lambda\|_{L^\infty(a,b)}}{a(2\alpha-1)\Gamma^2(\alpha)}, \\ &\frac{1}{b}-\frac{(b-a)^\alpha\|\lambda\|_{L^\infty(a,b)}}{a^5 b^{4(\alpha-1)}\Gamma^2(\alpha+1)\left[1-\left(\ln\frac{b}{a}\right)^{2\alpha-1}\right]}\Bigg\}. \end{aligned}$$

Hence, A is coercive. So, one can apply the Lax-Milgram theorem, it follows that the problem (PV) has a unique solution $u\in K$. □

Remark 3. In the same conditions to f and λ we can prove the existence and uniqueness from the following problem

$$(P')\quad\begin{cases}\left({}^H D_{b^-}^\alpha\left({}^H D_{a^+}^\alpha u\right)\right)(x)+\lambda(x)u(x)=f(x),\ \text{in }(a,b),\\ \left({}^H I_{a^+}^{1-\alpha}u\right)(a)=0,\ u(b)=0.\end{cases}$$

The unique solution belongs to

$$K'=\left\{v\in{}^H W_{a^+}^{\alpha,p}(a,b),\ \left({}^H I_{a^+}^{1-\alpha}v\right)(a)=0,\ v(b)=0\right\}.$$

Remark 4. Since A is symmetric, we deduce from the Lax-Milgram theorem [6] that the unique solution of (P) is also the solution of minimization problem

$$\min_{v\in K}\left\{\frac{1}{2}\int_a^b\left(\left({}^H D_{a^+}^{\alpha}v\right)^2(x)+\lambda(x)v^2(x)\right)dx-\int_a^b f(x)v(x)dx\right\}.$$

The unique solution of (P') is also the solution of minimization problem

$$\min_{v\in K'}\left\{\frac{1}{2}\int_a^b\left(\left({}^H D_{a^+}^{\alpha}v\right)^2(x)+\lambda(x)v^2(x)\right)dx-\int_a^b f(x)v(x)dx\right\}.$$

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