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SOLVABILITY OF NONLOCAL PARABOLIC PROBLEM WITH NUMERICAL SOLUTION

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Abstract

This paper is devoted to the study of linear nonlocal problems Dirichlet condition and Neumann condition modeling integration condition a second class of a class of linear reaction-diffusion equations. We show the existence and uniqueness of weak solutions to problems Fadeo-Galarkin method developed to circumvent the resulting complexities due to the existence of integration conditions. We also seek numerical solutions using finite difference techniques.

1. Introduction

Nonlinear diffusion equations, an important class of parabolic equations, which have come from a variety of diffusion phenomena appearing widely in nature. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences [19].

Many natural phenomena can be modeled by partial differential equations with non-local conditions. However, many phenomena can better be

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described by integral conditions, which are of growing interest. A lot of modern physics and technology problems are stated using non-local and integral conditions for partial differential equations [2], [3], [5], [8], [16], [10], [11], [14] and [6]. The first type is given by

$$\int_{\Omega} k(x,t)u(x,t)dx = E(t),$$

where $t \in (0,T)$, $\Omega \subset \mathbb{R}^n$ and k is a given function, or the second type, where the Dirichlet or Neumann condition modelling by integral condition, for example

$$u(x,t)|_{\partial\Omega} = \int k(x,t)u(x,t)dx,$$

can be used when it is impossible to directly measure the sought quantity on the border, but its total value or its average is known.

The study of problems of evolution equations with different boundary conditions types (classical and non-classical condition) has been solved by many powerful and different methods in nonlinear analysis, i.e., fixed-point theorem, semi-group method, Galerkin [18] and energy inequality method [1] [9], [17], [13] and [15].

Motivated by this, we study parabolic equation with a classical Dirichlet condition and an integral condition of second type which is more general than any integral condition. We show the existence and uniqueness of the weak solution for the problem by the method of Faedo-Galerkin method.

2. Position of Problem

In the rectangular area $Q = \Omega \times (0,T)$, and $T < \infty$. Consider the following linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u + bu = f\,(x,t) & \forall\ (x,t) \in Q \\ u\,(x,0) = \varphi\,(x) & \forall\, x \in (0,1) \\ u(0,t) = 0 & \forall\, t \in (0,T) \\ u_x(1,t) = \displaystyle\int_0^1 k(x,t)u(x,t)dx & \forall\, t \in (0,T) \end{cases} \tag{P_1}$$

whose parabolic equation is given as follows

$$u = \frac{\partial u}{\partial t} - \Delta u + bu = f(x,t), \tag{1}$$

with the initial condition

$$\ell u = u(x,0) = \varphi\left(x\right), \qquad x \in \left(0,1\right),$$

the Dirichlet boundary condition

$$u(0,t) = 0, \qquad t \in \left(0,T\right),$$

and the integral condition of the second type

$$u_x(1,t) = \int_0^1 k(x,t)u\left(x,t\right)dx, \qquad t \in \left(0,T\right).$$

We define space V by :

$$V = \left\{u \in H^1\left(\Omega\right) \;\; \text{tq} : \, v(0) = 0\right\},$$

where the space V provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)}$ is a Hilbert space.

We are now able to formulate the problem (P_2) precisely to study it, we will need the following hypothesis :

$$(H) : \begin{cases} f \in L^2\left(0,T;\; L^2\left(\Omega\right)\right) & (H.1) \\ \varphi \in H^1\left(\Omega\right) & (H.2) \end{cases}.$$

Definition 1. The weak solution of the problem (P_1) is a function that checks :

(i) $u \in L^2\left(0,T;\; H^1\left(\Omega\right)\right) \cap L^\infty\left(0,T;\; H^1\left(\Omega\right)\right)$.

(ii) u admits a strong derivative $\frac{\partial u}{\partial t} \in L^2\left(0,T;\; L^2\left(\Omega\right)\right)$.

(iii) $u\left(0\right) = \varphi$.

(iv) Identity

$$\left(u_t,v\right) + a\left(u_x,v_x\right) + b\left(u,v\right) = \left(f,v\right) + u_x(1,t)v(1) \quad \forall\, v \in V,\ \forall\, t \in \left[0,T\right].$$

3. Variational Formulation

By multiplying the equation :

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} + bu = f\left(x,t\right), \tag{2}$$

By an element $v \in V$, by integrating it on Ω we obtain :

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v dx - a \int_\Omega \frac{\partial^2 u}{\partial x^2} \cdot v dx + b \int_\Omega u \cdot v dx = \int_\Omega f \cdot v dx. \tag{3}$$

By using the boundary conditions and using Green's formula, (3) becomes

$$\left(u_t, v\right) + a\left(u_x, v_x\right) + b\left(u, v\right) = \left(f, v\right) + u_x(1,t)v(1) \quad \forall\, v \in V, \tag{4}$$

Where $(\cdot,\cdot)$ denotes the scalar product $L^2\left(\Omega\right)$.

4. The Existence of Weak Solution of the Problem ($\mathbf{P}_2$)

The demonstration of the existence of the solution of the problem (P_2) is based on the Faedo-Galerkin method which consists of carrying out the following three steps :

4.1. Step 1: Construction of the approximate solutions

The space V is separable, then there exists a sequence $w_1, w_2, \ldots, w_m$, having the following properties :

$$\begin{cases} w_i \in V, & \forall\, i, \\ \forall\, m, w_1, w_2, \ldots, w_m & \text{are linearly independent,} \\ V_m = \langle\{w_1, w_2, \ldots, w_m\}\rangle & \text{is dense in } V. \end{cases} \tag{5}$$

In particular :

$$\forall\, \varphi \in V \implies \exists(\alpha_{km})_m \in IN^*,\ \varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{6}$$

Faedo-Galerkin's approximation consists in searching for any integer $m \geq 1$, a function

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\, w_i(x),$$

verifies

$$\begin{cases} u_m(t) \in V_m, & \forall\, t \in [0,T] \\ \Big(\big(u_m(t)\big)_t, w_k\Big) + A\big(u_m(t), w_k\big) + b\left(u_m(t), w_k\right) = \big(f(t), w_k\big) & \forall\, k = \overline{1,m} \end{cases}. \tag{P_2}$$

We have

$$\begin{aligned} \left((u_m(t))_t, w_k\right) &= \left(\left(\sum_{i=1}^{m} g_{im}(t)\, w_i\right)_t, w_k\right) \\ &= \left(\sum_{i=1}^{m} \frac{\partial g_{im}}{\partial t}(t)\, w_i(x), w_k\right) \\ &= \sum_{i=1}^{m} (w_i, w_k)\, \frac{\partial g_{im}}{\partial t}(t), \end{aligned} \tag{7}$$

and

$$\begin{aligned} A\left(u_m(t), w_k\right) &= A\left(\sum_{i=1}^{m} g_{im}(t)\, w_i, w_k\right) \\ &= a \sum_{i=1}^{m} g_{im}(t) \left[\int_\Omega \frac{\partial w_i}{\partial x} \frac{\partial w_k}{\partial x} dx - \frac{\partial w_i}{\partial x}(1)\, w_k(1)\right] \\ &= a \sum_{i=1}^{m} g_{im}(t) \int_\Omega \frac{\partial w_i(x)}{\partial x} \frac{\partial w_k(x)}{\partial x} dx - a \sum_{i=1}^{m} g_{im}(t)\, \frac{\partial w_i}{\partial x}(1)\, w_k(1) \\ &= \sum_{i=1}^{m} A(w_i, w_k)\, g_{im}(t). \end{aligned} \tag{8}$$

Also, we have

$$\begin{aligned} u_m(0) &= \sum_{i=1}^{m} g_{im}(0)\, w_i(x) \\ &= \varphi_m \\ &= \sum_{i=1}^{m} \alpha_{im} w_i(x). \end{aligned}$$

We obtain a system of first order nonlinear differential equations :

$$\begin{cases} \sum\limits_{i=1}^{m} (w_i, w_k) \frac{\partial g_{im}}{\partial t} (t) + a \sum\limits_{i=1}^{m} \left(\frac{\partial w_i}{\partial x}, \frac{\partial w_k}{\partial x} \right) g_{im} (t) + b \sum\limits_{i=1}^{m} g_{im} (t) (w_i, w_k) \\ \qquad = (f(t), w_k) + a \sum\limits_{i=1}^{m} g_{im} (t) \frac{\partial w_i}{\partial x} (l) \, w_k(l) \\ \qquad g_{im} (0) = \alpha_{im} \qquad \forall \, i = \overline{1, m}. \end{cases} \tag{P_3}$$

We consider the vector

$$g_m = (g_{1m}(t), \ldots, g_{mm}(t)) \, , f_m = ((f, w_1) \, , \ldots, (f, w_m))$$

and the matrix

$$B_m = ((w_i, w_j))_{\substack{1 \le i \le m \\ 1 \le j \le m}} \, , \qquad A_m = \left(\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_j}{\partial x} \right) \right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$$

and

$$C_m = \left(\frac{\partial w_i}{\partial x} (1) \cdot w_j(1) \right)_{\substack{1 \le i \le m \\ 1 \le j \le m}} .$$

We write the problem (P_3) in the matrix form, we obtain :

$$\begin{cases} B_m \frac{\partial g_m}{\partial t} (t) + a A_m g_m + b B_m g_m = f_m + a C_m g_m \\ \qquad g_m (0) = (\alpha_{im})_{1 \le i \le m} \end{cases}$$

As the matrix entries B_m are linearly independent (because it is a diagonal matrix) then $\det B_m \neq 0$, so it is invertible, then g_m is the solution of

$$\begin{cases} \frac{\partial g_m}{\partial t} (t) + \left(a B_m^{-1} A_m + b B_m^{-1} B_m - a B_m^{-1} C_m \right) g_m = B_m^{-1} f_m \\ \qquad g_m (0) = (\alpha_{im})_{1 \le i \le m} . \end{cases} \tag{P_4}$$

Thanks to the usual existence and uniqueness theorems used for ordinary differential systems [4, Theorem 3.1, Chapter 3], we have the matrix $(a B_m^{-1} A_m + b B_m^{-1} B_m - a B_m^{-1} C_m)$ with constant coefficients and the vector $B_m^{-1} f_m$ with continuous functions and majorized by integrable fuctions on $(0, T)$. Then we can conclude that there exists a t_m depends only on $|\alpha_{im}|$ such that in the interval $[0, t_m]$, the nonhomogeneous problem (P_4) admits a unique local solution $g_m (t) \in C [0, t_m]$ and $g_m^{'} (t) \in L^2 [0, T]$. But as the elements of the vector $B_m^{-1} f_m$ are majorized by integrable functions on $(0, T)$, the solution can be extended to $[0,T]$.

4.2. Step 2: A priori estimate

Lemma 1. *For all $m \in \mathbb{N}^*$, if*

$$b-\frac{\varepsilon}{2}-\frac{a\varepsilon}{l}-\frac{aK}{2\varepsilon}>0,\ \frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}>0,\ \frac{b}{2}-\|k\|_{\infty}^{2}>0 \text{ and } 1-\frac{aK}{2\varepsilon}>0,$$

the solution $u_m \in L^2(0,T\ ;V_m)$ of the problem (P_1) checks

$$\begin{aligned}\|u_m\|_{L^2(0,T;H^1(\Omega))} &\leq c_1\\ \left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(0,T;\ L^2(\Omega))} &\leq c_2\end{aligned}$$

where $c_{1,}c_2$ are two positive constants independent of m.

Proof. Multiply the equation of (P_2) by $g_{km}(t)$ and we sum over k, we find

$$\begin{aligned}&\sum_{k=1}^{m}\left((u_m(t))_t\,,w_k\right)\cdot g_{km}(t)+a\sum_{k=1}^{m}\left(\frac{\partial u_m}{\partial x}(t),\frac{\partial w_k}{\partial x}\right)\cdot g_{km}(t)\\ &\quad+b\sum_{k=1}^{m}\left(u_m(t),w_k\right)\cdot g_{km}(t)\\ &=\sum_{k=1}^{m}\left(f(t),w_k\right)\cdot g_{km}(t)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1).\end{aligned}$$

So, we obtain

$$\begin{aligned}&\Big((u_m(t))_t,u_m(t)\Big)+a\left(\frac{\partial u_m}{\partial x}(t),\frac{\partial u_m}{\partial x}(t)\right)+b\left(u_m(t),u_m(t)\right)\\ &=\left(f(t),u_m(t)\right)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1).\end{aligned}$$

Thus, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2(\Omega)}^{2}+a\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^{2}+b\|u_m\|_{L^2(\Omega)}^{2}\\ &=\left(f(t),u_m(t)\right)+a\sum_{i=1}^{m}g_{im}\left(t\right)\frac{\partial w_i}{\partial x}\left(1\right)\sum_{k=1}^{m}g_{km}(t)w_k(1).\end{aligned}$$

By using the Cauchy inequality with ε, ($|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$), we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|^2_{L^2(\Omega)} + a\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} + b\left\|u_m\right\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\left\|u_m\right\|^2_{L^2(\Omega)} + a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(1)\sum_{k=1}^{m} g_{km}(t)w_k(1), \quad (9)$$

integrating from 0 to t, we get

$$\frac{1}{2}\left\|u_m\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|u_m(0)\right\|^2_{L^2(\Omega)} + a\int_0^t \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} d\tau + b\int_0^t \left\|u_m\right\|^2_{L^2(\Omega)} d\tau$$
$$\leq \frac{1}{2\varepsilon}\int_0^t \left\|f\right\|^2_{L^2(\Omega)} d\tau + \frac{\varepsilon}{2}\int_0^t \left\|u_m\right\|^2_{L^2(\Omega)} d\tau$$
$$+a\int_0^t \left(\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(1)\sum_{k=1}^{m} g_{km}(t)w_k(1)\right) d\tau.$$

Now, we must give an estimate of the third part of right hand side of previous inequality. So, we have

$$\int_0^t \left(\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(1)\sum_{k=1}^{m} g_{km}(t)w_k(1)\right) d\tau = \int_0^t \left(\frac{\partial u_m}{\partial x}(1,t)\cdot u_m(1,t)\right) d\tau;$$

By using the Cauchy inequality with ε; we obtain

$$\int_0^t \left(\frac{\partial u_m}{\partial x}(1,\tau)\cdot u_m(1,\tau)\right) d\tau < \frac{\varepsilon}{2}\int_0^t u^2(1,\tau)\, d\tau + \frac{1}{2\varepsilon}\int_0^t u_x^2(1,\tau)\, d\tau;$$

To obtain the estimate, we need the inequalities

$$u^2(1,t) \leqslant 2\int_x^1 u_x^2\, dx + 2u^2,$$

which easily followed from the equalities

$$u(1,t) = \int_x^1 u_x(x,t)\, dx + u(x,t).$$

Also by second integral condition kind, we have

$$\int_0^t u_x(1,\tau)\, u(1,\tau)\, d\tau$$

$$\leqslant \frac{\varepsilon}{2}\int_0^t u^2(1,\tau)\,d\tau + \frac{1}{2\varepsilon}\int_0^t u_x^2(1,\tau)\,d\tau$$
$$\leqslant \frac{\varepsilon}{2}\int_0^t \left[2\int_x^1 u_x^2 dx + 2u^2\right] d\tau + \frac{1}{2\varepsilon}\int_0^t \left[\int_0^1 k(x,t)\,u(x,t)\,dx\right]^2 d\tau.$$

So, by using Hölder's inequality, we have

$$\int_0^t u_x(1,t)\,u(1,t)\,dt \;\leqslant\; \varepsilon\int_Q u_x^2 dxdt + \varepsilon\int_0^T u^2 dt + \frac{K}{2\varepsilon}\int_Q u^2 dxdt;$$

where the constant $K = \max\int_Q k^2(x,t)\ dxdt$. Then, we obtain

$$\frac{1}{2}\|u_m\|^2_{L^\infty(0,T;\ L^2(\Omega))} + a\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(0,T;\ L^2(\Omega))} + b\|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$\leq \frac{1}{2\varepsilon}\|f\|^2_{L^2(0,T;\ L^2(\Omega))} + \frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$+a\varepsilon\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(0,T;\ L^2(\Omega))} + a\varepsilon\|u_m\|^2_{L^2(0,T)} + \frac{aK}{2\varepsilon}\|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}.$$

integrating over Ω, we get

$$\frac{1}{2}\|u_m\|^2_{L^\infty(0,T;\ L^2(\Omega))} + (a-a\varepsilon)\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$+\left(b-\frac{\varepsilon}{2}-a\varepsilon-\frac{aK}{2\varepsilon}\right)\|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$\leq \frac{1}{2\varepsilon}\|f\|^2_{L^2(0,T;\ L^2(\Omega))} + \frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}$$

$$\|u_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(0,T;L^2(\Omega))} + \|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$\leq \frac{1}{2\varepsilon\min\left\{\frac{1}{2}, a(1-\varepsilon), \left(b-\frac{\varepsilon}{2}-a\varepsilon-\frac{aK}{2\varepsilon}\right)\right\}}\left(\|f\|^2_{L^2(0,T;\ L^2(\Omega))} + \|\varphi_m\|^2_{L^2(\Omega)}\right).$$

So, we get

$$\|u_m\|^2_{L^\infty(0,T;\ L^2(\Omega))} + \left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(0,T;\ L^2(\Omega))} + \|u_m\|^2_{L^2(0,T;\ L^2(\Omega))}$$
$$\leq C_1\left(\|f\|^2_{L^2(0,T;\ L^2(\Omega))} + \|\varphi_m\|^2_{L^2(\Omega)}\right) = c_1, \tag{10}$$

where

$$C_1 = \frac{1}{2\varepsilon \min\left\{\frac{1}{2}, a\left(1-\varepsilon\right), \left(b - \frac{\varepsilon}{2} - a\varepsilon - \frac{aK}{2\varepsilon}\right)\right\}}.$$

Now, using the same formulation variational (4) and by multiplying the new equation by $g_{km}^{'}(t)$ and we sum over k, we find

$$\begin{aligned}
&\int_Q \left(\frac{\partial u_m}{\partial t}\right)^2 dxdt + a\int_Q \frac{\partial u_m}{\partial x}\cdot\frac{\partial\left(u_m\right)_t}{\partial x}dx - a\int_0^\tau \frac{\partial u_m}{\partial x}\cdot\frac{\partial u_m}{\partial t}\bigg|_{x=0}^{x=1} dt \\
&\quad + b\int_Q u_m\cdot\frac{\partial u_m}{\partial t}dx \\
&= \int_Q f\cdot\frac{\partial u_m}{\partial t}dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|\frac{\partial u_m}{\partial x}\left(\tau\right)\right\|^2_{L^2(\Omega)} + \frac{b}{2}\left\|u_m\right\|^2_{L^2(\Omega)} \\
&= \int_0^t \left(f\left(t\right), \frac{\partial u_m}{\partial t}\right) + a\int_0^\tau \frac{\partial u_m}{\partial x}\left(l,t\right)\cdot\frac{\partial u_m}{\partial t}\left(l,t\right)dt \\
&\quad + \frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}. \qquad (11)
\end{aligned}$$

With (11), using the Cauchy inequality with ε, we get

$$\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|\frac{\partial u_m}{\partial x}\left(\tau\right)\right\|^2_{L^2(\Omega)} + \frac{b}{2}\left\|u_m\right\|^2_{L^2(\Omega)} \\
&= \int_0^t \left(f\left(t\right), \frac{\partial u_m}{\partial t}\right) + a\int_0^\tau \left(\int_0^1 k(x,t)u_m(x,t)dx\right)\cdot\frac{\partial u_m}{\partial t}\left(1,t\right)dt \\
&\quad + \frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\frac{\partial \varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}
\end{aligned}$$

Then we get

$$\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|\frac{\partial u_m}{\partial x}\left(\tau\right)\right\|^2_{L^2(\Omega)} + \frac{b}{2}\left\|u_m\right\|^2_{L^2(\Omega)} \\
&\le \int_0^t \left(f\left(t\right), \frac{\partial u_m}{\partial t}\right) + a\left\|k\right\|_\infty \int_0^\tau \left(\int_0^1 u_m(x,t)dx\right)\cdot\frac{\partial u_m}{\partial t}\left(1,t\right)dt
\end{aligned}$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t\left(f(t),\frac{\partial u_m}{\partial t}\right)+a\left\|k\right\|_\infty\left\|u_m\right\|_{L^\infty(0,T;\,L^2(\Omega))}\int_0^\tau\frac{\partial u_m}{\partial t}(1,t)\,dt$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t\left(f(t),\frac{\partial u_m}{\partial t}\right)$$

$$+a\left\|k\right\|_\infty\left\|u_m\right\|_{L^\infty(0,T;\,L^2(\Omega))}\int_0^\tau\left[\int_0^1\frac{\partial^2 u_m}{\partial x\partial t}dx+\frac{\partial u_m}{\partial t}(x,t)\right]dt$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t\left(f(t),\frac{\partial u_m}{\partial t}\right)$$

$$+a\left\|k\right\|_\infty\left\|u_m\right\|_{L^\infty(0,T;\,L^2(\Omega))}\left[\int_0^1\int_0^\tau\frac{\partial^2 u_m}{\partial x\partial t}dx+\int_0^\tau\frac{\partial u_m}{\partial t}(x,t)\right]dt$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t\left(f(t),\frac{\partial u_m}{\partial t}\right)$$

$$+a\left\|k\right\|_\infty\left\|u_m\right\|_{L^\infty(0,T;\,L^2(\Omega))}\left[\int_0^1\frac{\partial u_m}{\partial x}dx-\int_0^l\frac{\partial\varphi_m}{\partial x}dx+\int_0^\tau\frac{\partial u_m}{\partial t}(x,t)\right]$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \int_0^t\left(f(t),\frac{\partial u_m}{\partial t}\right)$$

$$+a\left[\frac{\varepsilon}{2}\left\|k\right\|^2_\infty\left\|u_m\right\|^2_{L^\infty(0,T;\,L^2(\Omega))}+\frac{1}{\varepsilon}\left[\left(\int_0^1\frac{\partial u_m}{\partial x}dx+\int_0^\tau\frac{\partial u_m}{\partial t}(x,t)\right)^2\right]\right]$$

$$+\frac{a}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)}+\left(\frac{a}{2}+\frac{a}{\varepsilon}\right)\left\|\frac{\partial\varphi_m}{\partial x}\right\|^2_{L^2(\Omega)}$$

$$\leq \frac{1}{2\delta}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{\delta}{2}\left\|f\right\|^2_{L^2(Q)}$$

$$+a\left(\left\|k\right\|^2_\infty\left\|u_m\right\|^2_{L^\infty(0,T;\,L^2(\Omega))}+\left[\frac{2}{\varepsilon}\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)}+\frac{2T}{\varepsilon}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}\right]\right)$$

$$+\frac{a}{2}\left\|\varphi_m\right\|_{L^2(\Omega)}^2+\left(\frac{a}{2}+\frac{a}{\varepsilon}\right)\left\|\frac{\partial\varphi_m}{\partial x}\right\|_{L^2(\Omega)}^2.$$

Therefore, we get

$$\left(\frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}\right)\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(Q)}^2+\left(\frac{a}{2}-\frac{2a}{\varepsilon}\right)\left\|\frac{\partial u_m}{\partial x}(\tau)\right\|_{L^\infty(0,T;\ L^2(\Omega))}^2$$
$$+\left(\frac{b}{2}-\|k\|_\infty^2\right)\|u_m\|_{L^\infty(0,T;\ L^2(\Omega))}^2$$
$$\leq\ \frac{a}{2}\left\|\varphi_m\right\|_{L^2(\Omega)}^2+\left(\frac{a}{2}+\frac{a}{\varepsilon}\right)\left\|(\varphi_m)_x\right\|_{L^2(\Omega)}^2+\frac{\delta}{2}\|f\|_{L^2(Q)}^2.$$

So, we finally get

$$C_2=\frac{\max\left\{\frac{a}{2},\left(\frac{a}{2}+\frac{a}{\varepsilon}\right),\frac{\delta}{2}\right\}}{\min\left\{\left(\frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}\right),\left(\frac{a}{2}-\frac{2a}{\varepsilon}\right),\left(\frac{b}{2}-\|k\|_\infty^2\right)\right\}}$$

and

$$c_2=C_2\left(\|f\|_{L^2(Q)}^2+\left\|(\varphi_m)_x\right\|_{L^2(\Omega)}^2+\left\|\varphi_m\right\|_{L^2(\Omega)}^2\right).$$

Hence, we have

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(0,T;\ L^2(\Omega))}\leq c_2. \tag{12}$$

It follows from (10) that the solution to the initial value problem for the system of ODE (P_3) can be extended to $[0,T]$. This confirms what we have demonstrated in the first step. When $m\to+\infty$ in (12), we obtained

$$\left\{\begin{array}{ll} u_m & \text{uniformly bounded in } L^\infty(0,T;\ L^2(\Omega)) \\ u_m & \text{uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t & \text{uniformly bounded in } L^2\left(0,T;\ L^2(\Omega)\right) \end{array}\right. . \tag{13}$$

4.3. Step 3: Convergence and result of existence

Theorem 1. *There is a function* $u \in L^2(0,T;\ H^1(\Omega)) \cap L^\infty(0,T;\ L^2(\Omega))$ *with* $\frac{\partial u}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right)$ *and a subsequence denoted by* $(u_{m_k})_k \subseteq (u_m)_m$, *such that*

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0,T\ ;\ H^1(\Omega)) \\ \dfrac{\partial u_{m_k}}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{in } L^2\left(0,T\ ;\ L^2(\Omega)\right) \end{cases},$$

when $m \longrightarrow +\infty$.

Proof. We deduce from lemma (1.2) there are subsequences denoted by (u_{m_k}), $\left(\frac{\partial u_{m_k}}{\partial t}\right)$ of (u_m) and $(u_m)_t$ respectively, such that

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T;\ H^1(\Omega)). \tag{14}$$

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \qquad \text{in } L^2\left(0,T;\ L^2(\Omega)\right). \tag{15}$$

We know that according to Relikh-Kondrachoff's theorem that the injection of $H^1(Q)$ into $L^2(Q)$ is compact, and like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. So

$$u_{m_k} \longrightarrow u \qquad \text{in } L^2(Q). \tag{16}$$

On the other hand, from lemma (1.3) there is a subsequence of $(u_{m_k})_k$ is still denoted by u_{m_k} converges almost everywhere to u, such that

$$u_{m_k} \longrightarrow u \qquad \text{almost everywhere } Q. \tag{17}$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$, for that it suffices to prove :

$$u(t) = \varphi + \int_0^t w(\tau)d\tau. \tag{18}$$

As

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0,T;L^2(\Omega)),$$

then, the proof of (18), is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega)),$$

which means

$$\lim\left(u_{m_k}-\varphi-\chi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}=0,\ \ \forall\, v\in L^2(0,T\ ;\ L^2\left(\Omega\right)),$$

as

$$\chi\left(t\right)=\int_0^t w(\tau)d\tau.$$

Using equality

$$u_{m_k}-\varphi_{m_k}=\int_0^t\frac{\partial u_{m_k}}{\partial\tau}d\tau,\ \text{ for all } t\in\left[0,T\right],$$

which results from $u_{m_k}\in L^2\left(0,T;V_{m_k}\right)$ and $\left(u_{m_k}\right)_t\in L^2\left(0,T;V_{m_k}\right)$ that

$$\begin{aligned}
&\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}\\
&=\left(u_{m_k}-\varphi_{m_k}-\int_0^t w(\tau)d\tau,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}+\left(\varphi_{m_k}-\varphi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}\\
&=\left(\int_0^t\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau)\right)d\tau,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}+\left(\varphi_{m_k}-\varphi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))},\\
&\hspace{20em}\text{for all } t\in\left[0,T\right],
\end{aligned}$$

by virtue of **(ii)** of Lemma (1.6), it comes

$$\begin{aligned}
&\left(u_{m_k}-\varphi-\int_0^t w(\tau)d\tau,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}\\
&=\int_0^t\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau),v\right)_{L^2(0,T\ ;\ L^2(\Omega))}d\tau+\left(\varphi_{m_k}-\varphi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))},\\
&\hspace{20em}\text{for all } t\in\left[0,T\right].
\end{aligned}$$

On the one hand, we have

$$\lim_{k\longrightarrow\infty}\int_0^t\left(\frac{\partial u_{m_k}}{\partial\tau}-w(\tau),v\right)_{L^2(0,T\ ;\ L^2(\Omega))}d\tau=0,\text{for } t\in\left[0,T\right]. \tag{19}$$

Also, we have

$$\lim_{k\longrightarrow\infty}\left(\varphi_m-\varphi,v\right)_{L^2(0,T\ ;\ L^2(\Omega))}=0. \tag{20}$$

So we get

$$\lim_{k\longrightarrow\infty}(u_{m_k}-\varphi-\chi,v)_{L^2(0,T\,;\,L^2(\Omega))}=0,\quad \forall\, v\in L^2\left(0,T;L^2\left(\Omega\right)\right).$$

Theorem 2. *The function u of the theorem (1) is the weak solution to the problem (P_2) in the sense of the definition (1).*

Proof. From the theorem(1), we have shown that the limit function u satisfies the first two conditions of the definition (1).

Now we will demonstrate (**iii**). According to the theorem (1), we have

$$u_{m_k}(0)\rightharpoonup u(0)\qquad \text{in } L^2(\Omega).$$

On the other hand, we have

$$u_{m_k}(0)\longrightarrow\quad\varphi\quad\text{in } L^2(\Omega),$$

so

$$u_{m_k}(0)\ \rightharpoonup\quad\varphi\quad\text{in } L^2(\Omega).$$

From the uniqueness of the limit, we get

$$u(0)=\varphi.$$

Still to demonstrate (**iv**) :

$$(u_t,v)+a(u,v)+b(u,v)=(f,v)\qquad\forall\, v\in V,\ \text{and}\ \forall\, t\in[0,T]\,.$$

Integrating (P_2) on $(0,T)$, we find $\forall\, k=\overline{1,m}$, and $\forall\, t\in[0,T]$

$$\int_0^t\left((u_m(t))_t\,,w_k\right)d\tau+\int_0^t a\left(u_m(t),w_k\right)d\tau+b\int_0^t\left(u_m(t),w_k\right)d\tau =\int_0^t\left(f(t),w_k\right)d\tau \tag{21}$$

using (12) and that V_m dense in V and passing to the limit in (21), we find

$$\int_0^T(u_t,w_k)\,d\tau+\int_0^T a\left(u,w_k\right)d\tau+b\int_0^T(u,w_k)\,d\tau=\int_0^T(f,w_k)\,d\tau,\ \forall\, t\in[0,T]\,,$$

so (**iv**) is verified.

Corollary 1. *The uniqueness of the solution of problem* (P_1) *comes straight through the estimate* (10) .

5. Numerical Experiments

For the numerical solution of the considered main problem we apply the finite difference technique. First, we simplify the presentation of the interval $[0,1]$ in m by taking $\Delta x = \frac{1}{m}$ and the interval $[0,T]$ and by taking $\Delta t = \frac{T}{m}$. By u_i^k we denote the approximation to u at the i^{th} grid-point and k^{th} time step, the grid point (x_i, t_n) are given by : $x_i = i\Delta x$, $i = 0, 1, \ldots, m$. $t_k = k\Delta t, k = 0, \ldots, m$. $u_i^k = u(i\Delta x, k\Delta t)$. The notations u_i^k are used for approximations respectively. By using the finite difference scheme, we obtain:

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\Delta t} - a\frac{u_{i-1}^k - 2u_i^k + u_{i-1}^k}{\Delta x^2} + bu_i^k &= f_i^k \quad \forall\, i, k = 0, \ldots, m \\ u_i^{k+1} - u_i^k - \frac{a\Delta t}{\Delta x^2}\left(u_{i-1}^k - 2u_i^k + u_{i+1}^k\right) + b\Delta t u_i^k &= \Delta t f_i^k \quad \forall\, i, k = 0, \ldots, m \end{aligned}$$

by putting

$$r = \frac{a\Delta t}{\Delta x^2}$$

we get

$$u_i^{k+1} = (1 - b\Delta t - 2r)\, u_i^k + ru_{i-1}^k + ru_{i+1}^k + \Delta t f_i^k \quad \forall\, i, k = 0, \ldots, m$$

for the initial condition, we obtain

$$u_i^0 = \varphi_i \quad \forall\, i = 0, \ldots, m$$

and the boundary conditions, we find

$$\begin{aligned} u_0^k &= 0 \quad \forall\, k = 0, \ldots, m, \\ u_x(0, t_k) &= \int_0^1 k(x_i, t_k) u(x_i, t_k) dx \\ &= \frac{\Delta x}{2}\left[k(0, t_k)u_0^k + k(1, t_k)u_m^k + 2\sum_{i=1}^{m-1} k(x_i, t_k)u_i^k\right] + o(\Delta x)^2, \quad (22) \end{aligned}$$

$$\begin{aligned}
\frac{u_{m+1}^k - u_m^k}{\Delta x} &= \frac{\Delta x}{2}\Big[k_0^k u_0^k + k_1^k u_m^k + 2\sum_{i=1}^{m-1} k_i^k u_i^k\Big] \\
u_{m+1}^k &= u_m^k + \frac{(\Delta x)^2}{2}\Big[k_1^k u_m^k + 2\sum_{i=1}^{m-1} k_i^k u_i^k\Big] \\
&= \Big(1 + \frac{(\Delta x)^2}{2}k_1^k\Big)u_m^k + (\Delta x)^2 \sum_{i=1}^{m-1} k_i^k u_i^k.
\end{aligned}$$

So, we obtain

for $i = 0$:

$$u_0^{k+1} = r u_1^k + \Delta t f_1^k \quad \forall\, k = 0, \ldots, m$$

for $i \in \{0, \ldots, m-1\}$:

$$u_i^{k+1} = (1 - b\Delta t - 2r)\, u_i^k + r u_{i-1}^k + r u_{i+1}^k + \Delta t f_i^k \quad \forall\, k = 0, \ldots, m, \\ \forall\, i = 1, \ldots, m-1$$

for $i = m$:

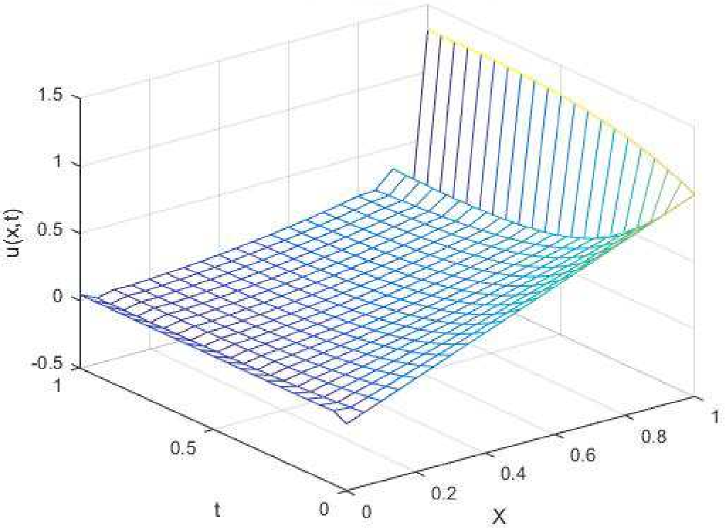
$$\begin{aligned}
u_m^{k+1} &= (1 - b\Delta t - 2r)\, u_m^k + r u_{m-1}^k + r u_{m+1}^k + \Delta t f_m^k \\
&= r\left(1 + \frac{(\Delta x)^2}{2}k_m^k\right) u_m^k + r\,(\Delta x)^2 \sum_{i=1}^{m-1} k_i^k u_i^k + (1 - b\Delta t - 2r)\, u_m^k \\
&\quad + r u_{m-1}^k + \Delta t f_m^k \\
&= \left(r\Big(1+\frac{(\Delta x)^2}{2}k_m^k\Big) + (1-b\Delta t-2r)\right) u_m^k + \Big(r\,(\Delta x)^2\, k_{m-1}^k + r\Big)\, u_{m-1}^k \\
&\quad + r\,(\Delta x)^2 \sum_{i=1}^{m-2} k_i^k u_i^k + \Delta t f_m^k
\end{aligned}$$

6. Examples

To test the above algorithm described in Section 3.3 , we use two examples with known analytical solutions as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - 0.002\Delta u + 0.09u = x^2 & \forall\ (x,t) \in Q \\ u\,(x,0) = x & \forall\, x \in (0,1) \\ u(0,t) = 0 & \forall\, t \in (0,T) \\ u_x(1,t) = \displaystyle\int_0^1 2u(x,t)dx & \forall\, t \in (0,T) \end{cases} \qquad (P_*)$$

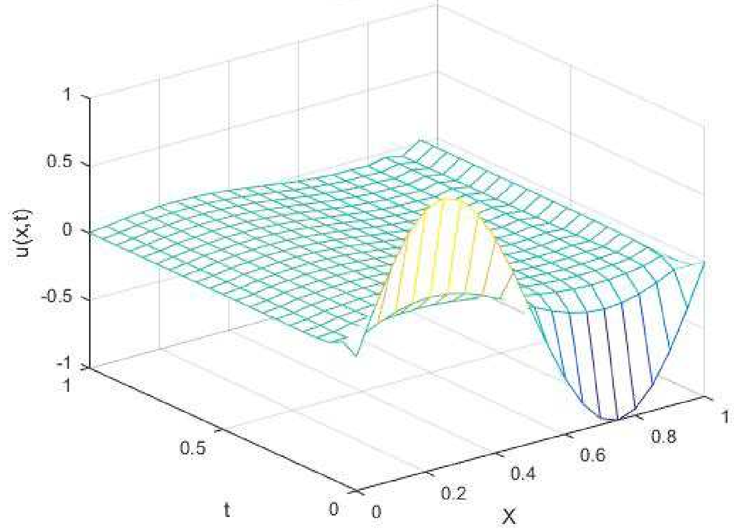
for a=0.002, b=0.09 and



The second test example to be solved is

$$\begin{cases} \dfrac{\partial u}{\partial t} - 0.001\Delta u + u = \sin(2\pi x) & \forall\ (x,t) \in Q \\ u\,(x,0) = \sin(2\pi x) & \forall\, x \in (0,1) \\ u(0,t) = 0 & \forall\, t \in (0,T) \\ u_x(1,t) = \displaystyle\int_0^1 \cos((t+1)\pi)xu(x,t)dx & \forall\, t \in (0,T). \end{cases} \qquad (P_{**})$$

for $a = 0.001$, $b = 1$



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