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LOCAL LINEAR ESTIMATION FOR ERGODIC DATA UNDER RANDOM CENSORSHIP MODEL IN HIGH DIMENSIONAL STATISTICS

RACHIDA ROUANE[1,a], SOMIA AYAD[1,b],
ALI LAKSACI[2,c] AND SAÂDIA RAHMANI[1,d]

[1]Laboratory of Stochastic Models, Statistic and Applications, Dr. Taher Moulay University of Saida, 20000 Saida, Algeria.

[a]E-mail: rachdia.rouane@univ-saida.dz

[b]E-mail: somia.ayad@univ-saida.dz

[d]E-mail: saadia.rahmani@gmail.com and saadia.rahmani@univ-saida.dz

[2]Department of Mathematics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia.

[c]E-mail: alilak@yahoo.fr

Abstract

This paper addresses the problem of estimating the conditional density function of a randomly censored scalar response variable given a functional random variable. Furthermore, we suppose that the data are sampled from stationary ergodic process. We introduce a local linear type estimator of the conditional density function. Under less restrictive assumptions closely related to the concentration of the probability of small balls of the underlying covariate we state the almost complete convergence with explicit rates as well as the asymptotic normality of the constructed estimator. As a direct application, the same properties are established for the conditional mode function.

1. Introduction and Motivations

Let $(X_i, Y_i)_{i=1,\dots,n}$ be a sequence of stationary and ergodic functional random samples identically distribution as the process (X, Y), where the covariate X takes its values in a semi-metric space $\mathcal{F}$ equipped with a semi-metric d and the response Y takes its values in $\mathbb{R}$. For a fixed $x \in \mathcal{F}$ and if the probability distribution of Y given X is absolutely continuous with

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respect to the Lebesgue measure, under a differentiability assumption on the conditional cumulative distribution function $F_Y^X(x,.)$, we denote $f_Y^X(x,y)$ the value of the corresponding conditional density function at (x,y) which can be written as:

$$\text{for all } y\in\mathbb{R},\quad f_Y^X(x,y)=\frac{\partial}{\partial y}F_Y^X(x,y).$$

In this work, we study the problem of the local linear estimation of the conditional density of Y given $X=x$ when the response variable $(Y_i)_{1\le i\le n}$ is a sequence representing the survival time which has a common unknown continuous distribution function L, $(\forall\, t\in\mathbb{R},\ L(t)=\mathbb{P}(Y\le t))$.
In the censoring case, instead of observing the lifetimes Y, we observe the censored lifetimes of items under study, denoted by C. More precisely, let $(C_i)_{1\le i\le n}$ be a sequence of i.i.d. censoring random variables with a common unknown continuous distribution function G, $(\forall\, t\in\mathbb{R},\ G(t)=\mathbb{P}(C\le t))$.
Thus, in contrast to the complete data, censored model involves the triplets $(X_i,T_i,\delta_i)_{1\le i\le n}$ of i.i.d. observations, with $T_i=min(Y_i,C_i)$ and $\delta_i=1\!\!1_{\{Y_i\le C_i\}}$, where

$$\delta_i=\begin{cases}1, & \text{if } Y_i \text{ is observed } \ (T_i=Y_i),\\ 0, & \text{if } Y_i \text{ is censored } \ (T_i=C_i), \text{ we observe incomplete data.}\end{cases}$$

The aim of this paper is to add some new results to the local linear estimation of the conditional density function for right-censored and stationary ergodic data. As far as we know, this problem has not been studied in the literature before and the results obtained here are believed to be novel. Moreover, our motivation for studying this type of estimator is due to its easy implementation compared to other local linear estimators introduced in the literature as well as its interest in some practical applications.

Recall that the framework of the local polynomial fitting was introduced for the first time by Stone [35] and developed and illustrated substantially via a variety of examples in Cleveland and Devlin [11]. However, vast efforts have been devoted to this method in the monograph of Fan and Gijbels [18]. Their interest was on methodologies rather than on theory, with a particular focus on applications of nonparametric techniques to various statistical problems.

In the last decade, many statistical researchers preferred to use the kernel method for studying nonparametric functional data due to its easiness of implementation. We can refer to the pioneer work of Ferraty and Vieu [20], in which they asked their famous question "How can the local polynomial ideas be adapted to infinite dimensional settings?" The problem has received special attention in the literature by several scenarios. Firstly and exactly in 2009, Baillo and Grané [3] constructed a local linear estimator for the regression operator in the case where the covariate takes its values in a Hilbert space. One year later, that question has been responded by the same authors who asked it in collaboration with Barrientos-Marin [4]. They have introduced an estimator of the regression function considering the case of a polynomial of order one called local linear approach which is flexible and more general than the kernel method. They studied the almost complete convergence with a rate of the proposed estimator. The case of the conditional distribution function was addressed by Demongeot et al. [13]. In this work, the authors studied the almost complete convergence as well as the mean square error with rates of the constructed estimator in the i.i.d case. Recently Bouanani et al. [6] and [7] established the asymptotic normality of several conditional models in the both cases: $\alpha-$mixing and independent. Other local linear estimation procedures have also been developed for functional data. For instance, Berlinet et al. [5] introduced another local linear estimator, but their method is restricted because they considered that the explanatory variable belongs in a Hilbert space. In the case when the data are observed as ergodic functional times series, Ayad et al. established the pointwise almost complete consistency of local linear estimators of the conditional density function the conditional mode.

In all the papers which we mentioned, the results have been established in the case when the data are functional with complete observed response. In the local constant approach when the data are functional but incomplete observed such as missing at random or censored, Khardani et al. [28] studied the almost sure and asymptotic normality of the kernel estimate of the conditional mode when the data are censored and independent. However, in the dependent case, Horrigue and Ould Saïd [26] established the uniform strong consistency with rates of a kernel conditional quantile estimator. The asymptotic normality of the previous estimator was studied by the same authors in [26]. Ling et al. [33] constructed a kernel estimator of the regression operator for functional stationary ergodic data with the fact that the responses are missing at random and they established the convergence in probability with rate and the asymptotic normality of the constructed estimator. The case when the response variable is a subject of left-truncation by another random variable was studied by Derrar et al. [15]. Recently, Fetitah et al. [23] established asymptotic properties of a kernel estimator of the relative error regression for randomly censored data. Concerning the local linear approach for incomplete data, there are few results such as Chahad et al. [9]. We can refer, also, to the recent paper of Rahmani et al. [34]. It should be noted that, to the best of our knowledge, no asymptotic results have been available in the literature for local linear conditional density fitting neither in the special framework considered in this paper nor for the general case of polynomial fitting for functional covariate when the data are randomly censored and assumed to be sampled from a stationary and ergodic process. Moreover, as will become clear from the following sections, establishing the almost complete convergence for the functional local linear fitting estimators is technically more involved than for the classical kernel method.

This paper is aimed at constructing a new local linear estimator of the conditional density function and establishing its asymptotic properties. The latter extend previous results established by Ayad et al. [1] to the censored case.

This paper is organized as follows. Section 2 introduces the construction of our local linear estimator. The precise goals will be stated in Section 3 after we introduce some notations and hypotheses which are needed to obtain our main results. An application of the conditional mode function is stated in the same section. Finally, the detailed proofs of our theoretical results and all technical lemmas needed are gathered in the Appendix.

2. Local Linear Estimator Construction

Our approach is based on the functional local-linear smoother; see, for instance, Barrientos-Marin [4]. Let $\rho(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ be known bi-functional operators defined from $\mathcal{F}^2$ into $\mathbb{R}$, where $\delta(\cdot,\cdot)$ is lay with the topological structure of the functional space $\mathcal{F}$, that means $|\delta(x,z)| = d(x,z)$; and ρ controls the local shape of our model. K and J are kernels, where the first one is a density function and the second one is a distribution function. $h_K = h_{K,n}$ (resp. $h_J = h_{J,n}$) is a sequence of positive real numbers called the

smoothing parameter. In addition, we denote $\bar{G}(\cdot)$ (resp. $\bar{L}(\cdot)$) the survival function of the censoring variable C (resp. the real random variable Y) defined by: $\bar{G}(t) = 1 - G(t)$ (resp. $\bar{L}(t) = 1 - L(t)$), $\forall\, t \in \mathbb{R}$.
A local-linear estimator of the conditional density function is given by $\widehat{f^x}(y) = \widehat{\alpha}$, where

$$(\widehat{\alpha}, \widehat{\beta}) = \arg\min_{(\alpha,\beta)\in\mathbb{R}^2} \sum_{i=1}^{n} \left(\frac{\delta_i}{\overline{G}(T_i)} J(h_J^{-1}(y - T_i)) - \alpha - \beta\rho(X_i, x) \right)^2 K(h_K^{-1}\delta(x, X_i)),$$

Furthermore, let $\tau_L = \sup\{y, L(y) < 1\}$ (resp.$\tau_G = \sup\{y, G(y) < 1\}$) is the upper endpoint of $\bar{L}$ (resp. $\bar{G}$).
We assume that $(C_i)_{1\leq i\leq n}$ and $(X_i, Y_i)_{1\leq i\leq n}$ are independent. The "pseudo" estimator of $f^x(y)$ is defined by:

$$\widetilde{f^x}(y) = \frac{\displaystyle\sum_{j=1}^{n} \delta_j \bar{G}^{-1}(T_j)\Gamma_j K_j J_j}{\displaystyle h_J \sum_{j=1}^{n} \Gamma_j K_j} := \frac{\widehat{f_1^x}(y)}{\widehat{f_0^x}(y)}, \tag{1}$$

where

$$\widehat{f_l^x}(y) = \frac{1}{n h_J^l \mathbb{E}(\Gamma_1 K_1)} \sum_{j=1}^{n} \delta_j^l \bar{G}^{-l}(T_j)\Gamma_j K_j J_j^l \quad \text{for} \quad l = 0, 1,$$

with

$$\Gamma_j = \sum_{i=1}^{n} \rho_i^2 K_i - \left(\sum_{i=1}^{n} \rho_i K_i \right) \rho_j,$$

where

$$\rho_i = \rho(X_i, x), K_i = K\left(\frac{\delta(x, X_i)}{h_K}\right), \; J_j = J\left(\frac{y - T_j}{h_J}\right)$$

with the convention $0/0 := 0$.

Since G is unknown in practice, it is not possible to use the estimator (1). Thus, in order to obtain the following explicit formula of our local linear estimator, we use the *Kaplan Meier* [27] estimator of G given by:

$$\bar{G}_n(y) = \begin{cases} \displaystyle\prod_{j}^{n} \left(1 - \frac{1 - \delta_{(j)}}{n - j + 1}\right)^{\mathbb{1}_{\{T_{(j)} \leq y\}}} & \text{if} \quad y < T_{(n)} \\ 0 & \text{otherwise} \end{cases},$$

where $T_{(1)} < T_{(2)} < \cdots < T_{(n)}$ are order statistics of T_j and $\delta_{(j)}$ is concomitant with $T_{(j)}$.

Thus a feasible estimator of $f^x(y)$ is given by

$$\widehat{f}^x(y) = \frac{\sum_{j=1}^{n} \delta_j \bar{G}_n^{-1}(T_j)\Gamma_j K_j J_j}{h_J \sum_{j=1}^{n} \Gamma_j K_j} := \frac{\widehat{f}_{1,n}^x(y)}{\widehat{f}_0^x(y)}, \tag{2}$$

where

$$\widehat{f}_{1,n}^x(y) = \frac{1}{nh_J \mathbb{E}(\Gamma_1 K_1)} \sum_{j=1}^{n} \delta_j \bar{G}_n^{-1}(T_j)\Gamma_j K_j J_j.$$

Moreover, we assume that there exists a certain compact set $\mathcal{C}_{\mathbb{R}} \subset \mathbb{R}$, such that $f^x(y)$ is unimodal and its conditional unique mode is denoted by $\Theta(x)$ on $\mathcal{C}_{\mathbb{R}}$. A natural and usual estimator of $\Theta(x)$ is defined by:

$$\widehat{\Theta}(x) = \arg\sup_{y \in \mathcal{C}_{\mathbb{R}}} \widehat{f}^x(y).$$

3. Asymptotic Theory

3.1. Pointwise almost complete convergence

The main aim of this section is to establish the pointwise almost complete convergence of the estimator $\widehat{f}^x(y)$ under some mild regularity conditions. Note that we only give the main results; detailed proofs can be found in the appendix.

Throughout this paper, when no confusion is possible, we will denote by C and C' some strictly positive generic constants and we fix a point x in $\mathcal{F}$, (respectively, a compact $\mathcal{C}_{\mathbb{R}} \subset \mathbb{R}$) and $\mathcal{N}_x$ denotes a fixed neighborhood of this point. Moreover, for $i = 1, \ldots, n$, let $\mathfrak{F}_i$ be the σ-field generated by $((X_1, Y_1), \ldots, (X_i, Y_i))$, and $\mathcal{G}_i$ the one generated by $((X_1, Y_1), \ldots, (X_i, Y_i), X_{i+1})$.

We assume that $\Theta(x) \in \mathcal{C}_{\mathbb{R}} \subset (-\infty, \tau]$ where $\tau < \tau_G \wedge \tau_L$ and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$ is the small ball probability function.

To state our results we need the following hypotheses:

(H.1) We suppose that the strictly stationary ergodic process $(X_i, Y_i)_{i\in\mathbb{N}^*}$ satisfies:

(i) For all $r > 0$, $\phi_x(r) = \mathbb{P}(X \in B(x,r)) > 0$, where $B(x,r) = \{x' \in \mathcal{F}/|\delta(x',x)| < r\}$.

(ii) For all $i = 1,\ldots,n$ there exists a deterministic function $\phi_{i,x}(\cdot)$ such that
$0 < \mathbb{P}(X_i \in B(x,r)|\mathfrak{F}_{i-1}) \leq \phi_{i,x}(r)$, $\forall\, r > 0$,

(iii) For any $r > 0$, $\frac{1}{n\phi_x(r)}\sum_{i=1}^{n}\phi_{i,x}(r) \xrightarrow{P} 1$ and $n\phi_x(r) \to \infty$ as $r \to 0$.

(iv) C and (X,Y) are independent and the first derivative of the survival distribution function G is bounded.

(H.2) (i) $\exists\, b_1 > 0$ and $b_2 > 0$, $\forall\, x' \in \mathcal{N}_x$ and $\forall\, y' \in \mathcal{C}_{\mathbb{R}}$:

$$|f^x(y) - f^{x'}(y')| \leq C\left(|\delta(x,x')|^{b_1} + |y-y'|^{b_2}\right).$$

(ii) $f^x(\cdot)$ is a bounded function, twice differentiable and its second derivative $f^{x(2)}(\cdot)$ is continuous on a neighborhood of $\Theta(x)$, $f^{x(1)}(\Theta(x)) = 0$ and $f^{x(2)}(\Theta(x)) < 0$.

(iii)

$$\int |y| f^x(y) \leq \infty.$$

(H.3) The function ρ satisfies the following condition:
$\forall\, w \in \mathcal{F}$, $C|\delta\,(x,w)\,| \leq |\rho\,(x,w)\,| \leq C'|\delta\,(x,w)\,|$.

(H.4) (i) K is a nonnegative bounded kernel supported on $[-1,1]$.

(ii) J is a positive kernel, bounded and Lipschitzian continuous function, such that:

$$\int |v|^{b_2} J(v)dv < \infty \quad \text{and} \quad \int J^2(v)dv < \infty.$$

(iii) $\mathbb{E}\left(J(\frac{y-Y_j}{h_J})|\mathcal{G}_{j-1}\right) = \mathbb{E}\left(J(\frac{y-Y_j}{h_J})|X_j\right)$.

(H.5) The bandwidths h_K and h_J satisfy:

$$\lim_{n\to\infty} h_K = 0,\ \lim_{n\to\infty} n^{\alpha} h_J^{\gamma} = \infty \,\text{with}\, \gamma = 1,2,\,\text{and}\, \alpha > 1,$$

$$\lim_{n\to\infty}\frac{\varphi_x\left(h_K\right)\log n}{n^2 h_J\phi_x^2(h_K)}=0,\ \text{where}\,\varphi_x\left(h_K\right)=\sum_{i=1}^{n}\phi_{i,x}(h_K)$$

and

$$h_K\int_{B(x,h_K)}\rho\left(u,x\right)dP\left(u\right)=o\Big(\int_{B(x,h_K)}\rho^2\left(u,x\right)dP\left(u\right)\Big),$$

where $dP(x)$ is cumulative distribution of X.

Remarks on the hypotheses

Firstly, hypotheses (H.1)(i-iii) are the same (H.1) in [1]; whereas, assumption (H.1)(iv) is very standard in nonparametric censoring estimation. Indeed, the independence between C and (X,Y) allows us to deduce that $\mathbb{P}(Y\le C|(X,Y))=\mathbb{P}(Y\le C|Y)$. Hypotheses (H.2) and (H.3) are mild regularity hypotheses on the conditional density function. Finally, conditions (H.4) and (H.5) are technical assumptions for the brevity of proofs.

Theorem 1. *Under the assumptions* (H.1), (H.2) (i), (H.3)–(H.5), *we have*

$$\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}^x\left(y\right)-f^x\left(y\right)|=O\left(h_K^{b_1}\right)+O\left(h_J^{b_2}\right)+O\left(\sqrt{\frac{\varphi_x\left(h_K\right)\log n}{n^2h_J\phi_x^2\left(h_K\right)}}\right),\quad a.co.$$

***Proof of Theorem* 1.** In order to prove Theorem 1, we introduce the following decomposition:

$$\widehat{f}^x\left(y\right)-f^x\left(y\right)=\widehat{f}^x\left(y\right)-\tilde{f}^x\left(y\right)+\tilde{f}^x\left(y\right)-f^x\left(y\right).\tag{3}$$

Next, for a sake of simplicity the following notation is needed:

$$\overline{\widetilde{f}}_l^x\left(y\right)=\frac{1}{nh_J^l\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^{n}\mathbb{E}\left(\delta_j^l\bar{G}^{-l}(T_j)\Gamma_jK_jJ_j^l|\mathfrak{F}_{j-1}\right),\qquad\text{with}\quad l=0,1.$$

Then, we can write:

$$\begin{aligned}&\tilde{f}^x\left(y\right)-f^x\left(y\right)\\&\quad=\Big(\frac{\overline{\widetilde{f}}_1^x\left(y\right)}{\overline{\widetilde{f}}_0^x\left(y\right)}-f^x\left(y\right)\Big)+\frac{1}{\widehat{f}_0^x\left(y\right)}\left[\left(\frac{\overline{\widetilde{f}}_1^x\left(y\right)}{\overline{\widetilde{f}}_0^x\left(y\right)}-f^x\left(y\right)\right)\left(\overline{\widetilde{f}}_0^x\left(y\right)-\widehat{f}_0^x\left(y\right)\right)\right.\\&\qquad\left.+\left(\left(\widehat{f}_1^x\left(y\right)-\overline{\widetilde{f}}_1^x\left(y\right)\right)-f^x\left(y\right)\left(\widehat{f}_0^x\left(y\right)-\overline{\widetilde{f}}_0^x\left(y\right)\right)\right)\right].\end{aligned}\tag{4}$$

Thus, the proof of Theorem 1 is a direct consequence of Lemma 1 of Louani and Laib [29], Lemmas 3 and 5 of Ayad et al. and the following auxiliary results which play a main role and for which proofs are given in the appendix.

Lemma 1. *Assume that* (H.1), (H.2)(i), (H.3) *and* (H.4) *are satisfied. Then, we have*

$$\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\left(\frac{\widetilde{\bar{f}}_1^x(y)}{\widetilde{\bar{f}}_0^x(y)}-f^x(y)\right)\right|=O\left(h_K^{b_1}\right)+O\left(h_J^{b_2}\right).$$

Lemma 2. *Under the assumptions* (H.1), (H.3)–(H.5), *we have*

$$\widehat{f}_0^x(y)-\widetilde{\bar{f}}_0^x(y)=O_{a.co}\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2\phi_x^2(h_K)}}\right), \tag{5}$$

and

$$\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\left(\widehat{f}_1^x(y)-\widetilde{\bar{f}}_1^x(y)\right)\right|=O_{a.co}\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right). \tag{6}$$

Lemma 3. *Under the conditions* (H.1), (H.2)(i), (H.3)–(H.5), *we have*

$$\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\widehat{f}^x(y)-\tilde{f}^x(y)\right|=O_{a.co}\left(\sqrt{\frac{\log\log n}{n}}\right).$$

3.1.1. Application to conditional mode

In addition to the assumptions introduced along the previous section, we need the following conditions to establish the consistency of the conditional mode estimator:

(H.6) There exists $\Theta(x)\in\mathcal{C}_{\mathbb{R}}$, such that $f^x(y)<f^x(\Theta(x))$, for all $y\neq\Theta(x),\ y\in\mathcal{C}_{\mathbb{R}}$.

Theorem 2. *Assume that* (H.1)–(H.6) *hold, we have*

$$|\widehat{\Theta}(x)-\Theta(x)|=O\Big(h_K^{\frac{b_1}{2}}\Big)+O\Big(h_J^{\frac{b_2}{2}}\Big)+O\left(\left(\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}\right)^{\frac{1}{4}}\right),\quad a.co.$$

***Proof of Theorem* 2.** Before starting the proof of this last theorem, the following lemma is necessary:

Lemma 4.

$$\lim_{n\to\infty} |\widehat{\Theta}(x) - \Theta(x)| = 0. \qquad a.co.$$

Proof. Since $f^x(\cdot)$ is uniformly continuous on $\mathcal{C}_{\mathbb{R}}$, it is easy to see that (H.6) implies that:

$$\forall \varepsilon > 0, \exists\ \eta(\varepsilon) > 0, |f^x(y) - f^x(\Theta(x)| \leq \eta(\varepsilon) \Rightarrow |y - \Theta(x)| \leq \varepsilon.$$

Which implies that:

$$\forall \varepsilon > 0, \exists\ \eta(\varepsilon) > 0, \mathbb{P}\left(|\widehat{\Theta}(x) - \Theta(x)| > \varepsilon\right) \leq \mathbb{P}\left(|f^x(\widehat{\Theta}(x)) - f^x(\Theta(x))| > \eta(\varepsilon)\right). \tag{7}$$

Then, by a simple algebra, we get

$$|f^x(\widehat{\Theta}(x)) - f^x(\Theta(x))| \leq 2 \sup_{y\in\mathcal{C}_{\mathbb{R}}} |\widehat{f^x}(y) - f^x(y)|. \tag{8}$$

Finally, the almost complete convergence of $\widehat{\Theta}(x)$ to $\Theta(x)$ can be deduced from the latter together with (7) and Theorem 1. □

Next, since $f^{x(1)}(\Theta(x)) = f^{x(1)}(\widehat{\Theta}(x)) = 0$ and by a Taylor expansion of the function f^x, we have:

$$f^x(\widehat{\Theta}(x)) = f^x(\Theta(x)) + \frac{1}{2} f^{x(2)}(\Theta^*(x)) \left(\widehat{\Theta}(x) - \Theta(x)\right)^2, \tag{9}$$

where $\Theta^*(x)$ is between $\Theta(x)$ and $\widehat{\Theta}(x)$. So, by using (H.2)(ii), we obtain

$$\exists\ c > 0, \sum_{n=1}^{\infty} \mathbb{P}\left(|f^{x(2)}(\Theta^*(x))| < c\right) < \infty. \tag{10}$$

Therefore, by combining the statements (8), (9) and (10) we obtain:

$$|\widehat{\Theta}(x) - \Theta(x)|^2 = O\Big(\sup_{t\in\mathcal{C}_{\mathbb{R}}} |\widehat{f^x}\,(y) - f^x\,(y)\,|\Big), \quad a.co.$$

Thus, the proof of Theorem 2 can be deduced from Theorem 1.

3.2. Convergence in distribution

The following deals with the asymptotic distribution of $\widehat{f^x}(y)$. For this, we need the following hypotheses:

(A.1) The hypothesis (H.1) holds and there exists a function $\Psi_x(\cdot)$ such that:

$$\forall\, t \in [-1,1],\ \lim_{h_K \to 0} \frac{\phi_x(-h_K, th_K)}{\phi_x(h_K)} = \Psi_x(t).$$

(A.2) The hypothesis (H.2) holds and for all $(x_1, x_2, y_1, y_2) \in \mathcal{N}_x \times \mathcal{N}_x \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$:

(i)

$$\begin{cases} f : \mathcal{F} \times \mathbb{R} \longrightarrow \mathbb{R},\ \lim\limits_{|\rho(x_1,x_2)| \to 0} f^{x_1}(y) = f^{x_2}(y), \\ \text{and} \\ \lim\limits_{|y_1 - y_2| \to 0} f^x(y_1) = f^x(y_2). \end{cases}$$

(A.3) The hypothesis (H.3) holds and

$$\sup_{u \in B(x,r)} |\rho(u,x) - \delta(x,u)| = o(r).$$

(A.4) The hypothesis (H.4) holds and the first derivative K' of the kernel K satisfies:

$$K^2(1) - \int_{-1}^{1} (K^2(u))' \Psi_x(u) du > 0.$$

(A.5) The hypothesis (H.5) holds and $\lim\limits_{n \to \infty} (n-1)^k h_K^l \phi(h_K) = 0$, for $k = 1, 2$ and $l = 4, 5$.

In addition, for technical reasons, we need to introduce the following quantity which will appear in the computation of $\mathbb{E}(K_1^j)$

$$M_j = K^j(1) - \int_{-1}^{1} (K^j(u))' \Psi_x(u) du \text{ where } j = 1, 2.$$

Theorem 3. *Under Assumptions* (A.1)–(A.5) *and if the smoothing parameters* h_K *and* h_J *satisfy* $(nh_J\phi(h_K))^{1/2}\left(h_K^{b_1} + h_J^{b_2}\right) \to 0$ *as* $n \to \infty$, *We* *have,*

$$(nh_J\phi_x(h_K))^{1/2}\left(\widehat{f^x}(y)-f^x(y)\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2_{J,K}(x,y)\right),\qquad as\qquad n\longrightarrow+\infty$$

where

$$\sigma^2_{J,K}(x,y)=\frac{M_2}{M_1^2}\frac{f^x(y)}{\bar{G}(y)}\int_{\mathbb{R}}J^2(t)dt \tag{11}$$

and $\xrightarrow{\mathcal{D}}$ *denotes the convergence in distribution,* $\mathcal{N}(\cdot,\cdot)$ *the normal distribution.*

Remark 1. The most interesting special case of this result is obtained by letting $\bar{G}(\cdot)=1$ (without censorship data), which has been established by Ayad et al. [1].

Proof of Theorem **3.** Consider the following decomposition

$$\begin{aligned}\widehat{f^x}(y)-f^x(y)&=\left(\widehat{f^x}(y)-\tilde{f}^x(y)\right)+\left(\tilde{f}^x(y)-f^x(y)\right)\\&=\mathcal{D}_{1,n}+\mathcal{D}_{2,n}\end{aligned}$$

First, by Lemma 3, the term $\mathcal{D}_{1,n}$ converges almost surely to zero when n goes to infinity, where $\mathcal{D}_{1,n}=\mathrm{O}_{a.s.}\left(\sqrt{\log\log n/n}\right)=\mathrm{o}_{a.s.}(1)$. Furthermore, consider that

$$\mathcal{D}_{2,n}=A_n(x,y)+\frac{C_n(x,y)+B_n(x,y)}{\widehat{f_0^x}(y)},$$

where $A_n(x,y)=\frac{\overline{\widetilde{f}}_1^x(y)}{\overline{\widetilde{f}}_0^x(y)}-f^x(y)$, $\quad C_n(x,y)=A_n(x,y)\left(\overline{\widetilde{f}}_0^x(y)-\widehat{f_0^x}(y)\right)$ and $B_n(x,y)=\left(\widehat{f_1^x}(y)-\overline{\widetilde{f}}_1^x(y)\right)-f^x(y)\left(\widehat{f_0^x}(y)-\overline{\widetilde{f}}_0^x(y)\right)$.

The quantity $A_n(x,y)$ converges almost surely to zero when n goes to infinity, using Lemma 1. Then, according to the first part of Lemma 2 , C_n converges almost surely to zero when n goes to infinity. Moreover, the asymptotic normality is given by the proof the following lemmas. □

Lemma 5. *Under the assumptions of Theorem* 3*, we have*

$$(nh_J\phi_x(h_K))^{1/2}B_n(x,y)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2_{J,K}(x,y)\right),\qquad as\ n\to\infty.$$

Lemma 6. *Under the assumptions* (H.1), (H.3), (H.4)(i) *and* (H.5), *we have*

$$\widehat{f}_0^x(y) - 1 = o_p(1).$$

3.2.1. Application to conditional mode

Theorem 4. *Under Assumptions* (A.1)–(A.5) *and if the smoothing parameters* h_K *and* h_J *satisfy* $(nh_J^3\phi_x(h_K))^{1/2}\left(h_K^{b_1} + h_J^{b_2}\right) \to 0$ *as* $n \to \infty$, *we have,*

$$(nh_J^3\phi_x(h_K))^{1/2}\left(\widehat{\Theta}(x) - \Theta(x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_1^2(x, \Theta(x))\right), \quad as \quad n \longrightarrow +\infty$$

where

$$\sigma_1^2(x, \Theta(x)) = \frac{M_2 f^x(\Theta(x))}{M_1^2 \bar{G}(\Theta(x))\left(f''^x(\Theta(x))\right)^2}\int_{\mathbb{R}}\left(J'(t)\right)^2 dt \tag{12}$$

and $\xrightarrow{\mathcal{D}}$ *denotes the convergence in distribution,* $\mathcal{N}(\cdot,\cdot)$ *the normal distribution.*

***Proof of Theorem* 4.** Based on the Taylor expansion of $\widehat{f}'^x(\cdot)$ in the neighborhood of $\Theta(x)$ and since $\widehat{f}'^x(\widehat{\Theta}(x)) = 0$, we have

$$(nh_J^3\phi_x(h_K))^{1/2}\left(\widehat{\Theta}(x) - \Theta(x)\right) = -\frac{(nh_J^3\phi_x(h_K))^{1/2}\widehat{f}'^x(\Theta(x))}{\widehat{f}''^x(\Theta^*(x))},$$

where $\Theta^*(x)$ is between $\Theta(x)$ and $\widehat{\Theta}(x)$.

The proof of the statement below is analogous to that of Theorem 3.

$$-(nh_J^3\phi_x(h_K))^{1/2}\widehat{f}'^x(\Theta(x)) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \varrho_1^2(x, \Theta(x))\right), \quad \text{as} \quad n \longrightarrow +\infty$$

with $\varrho_1^2(x, \Theta(x)) = \frac{M_2 f^x(\Theta(x))}{M_1^2 \bar{G}(\Theta(x))}\int_{\mathbb{R}}\left(J'(t)\right)^2 dt$.

Then, where $\widehat{f}''^x(\Theta(x)) \longrightarrow f''^x(\Theta(x))$ as $n \longrightarrow +\infty$, and the fact that $\Theta^*(x)$ is lying between $\Theta(x)$ and $\hat{\Theta}^*(x)$, which gives

$$\widehat{f}''^x(\Theta^*(x)) \longrightarrow f''^x(\Theta(x)), \text{ as } \quad n \longrightarrow +\infty.$$

4. Appendix

4.1. Proof of Lemma 1. Firstly, remind that

$$\frac{\widetilde{\overline{f}}_1^x(y)}{\widetilde{\overline{f}}_0^x(y)} - f^x(y) = \frac{\widetilde{\overline{f}}_1^x(y) - \widetilde{\overline{f}}_0^x(y)\, f^x(y)}{\widetilde{\overline{f}}_0^x(y)}$$

(H.4)(iii) combined with the property of the conditional expectation with respect to the σ-fields $\mathcal{G}_{j-1}$ and Y_j with the fact that $1\!\!1_{\{Y_j<C_j\}}\psi(T_j) = 1\!\!1_{\{Y_j<C_j\}}\psi(Y_j)$, we obtain

$$\begin{aligned}
\overline{\widetilde{f}}_1^x(y) &= \frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n \mathbb{E}\left(\delta_j\bar{G}^{-1}(T_j)\Gamma_jK_jJ_j|\mathfrak{F}_{j-1}\right)\\
&= \frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n \mathbb{E}\left(\Gamma_jK_j\mathbb{E}\left(\delta_j\bar{G}^{-1}(T_j)J_j|\mathcal{G}_{j-1},Y_j\right)|\mathfrak{F}_{j-1}\right)\\
&= \frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n \mathbb{E}\left(\Gamma_jK_j\bar{G}^{-1}(T_j)J_j\mathbb{E}\left(1\!\!1_{\{Y_j<C_j\}}|X_j,Y_j\right)|\mathfrak{F}_{j-1}\right)\\
&= \frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n \mathbb{E}\left(\Gamma_jK_jJ_j|\mathfrak{F}_{j-1}\right) \qquad (13)
\end{aligned}$$

Furthermore, a double conditioning with respect to $\mathcal{G}_{j-1}$ leads to

$$\begin{aligned}
&\overline{\widetilde{f}}_1^x(y) - \overline{\widetilde{f}}_0^x(y)\, f^x(y)\\
&\quad= \frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)\,\widetilde{\overline{f}}_0^x(y)}\sum_{j=1}^n \mathbb{E}\left(\Gamma_jK_j\left|\mathbb{E}[J_j|X_j] - h_Jf^x(y)\right||\mathfrak{F}_{j-1}\right).
\end{aligned}$$

Then, using an integration by parts followed by a change of variable, permits to get

$$\mathbb{E}(J_j|X_j) = h_J\int_{\mathbb{R}} J(u)\, f^x(y-h_Ju)\,du, \qquad (14)$$

thus, we have

$$\left|\mathbb{E}[J_j|X_j] - h_Jf^x(y)\right| \le h_J\int_{\mathbb{R}} J(u)\left|f^x(y-h_Ju) - f^x(y)\right|du.$$

Using (H.2)(i) permits us to find:

$$\mathbb{1}_{B(x,h_K)}(X_j)\left|\mathbb{E}\left[J_j|X_j\right]-h_J f^x(y)\right|\leq h_J\int_{\mathbb{R}} J(u)\left(h_K^{b_1}+|y|^{b_2}h_J^{b_2}\right)du.$$

Hence, by assumption (H.4) (ii) and Lemma 6 of [1], we can obtain

$$\begin{aligned}\overline{\widetilde{f}}_1^x(y)-\overline{\widetilde{f}}_0^x(y)f^x(y)&=\left(O\left(h_K^{b_1}\right)+O\left(h_J^{b_2}\right)\right)\times\frac{1}{n\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^n\mathbb{E}\left(\Gamma_jK_j|\mathfrak{F}_{j-1}\right)\\&=O\left(h_K^{b_1}\right)+O\left(h_J^{b_2}\right).\end{aligned}$$

□

4.2. Proof of Lemma 2. For all $l=0,1$, we have

$$\begin{aligned}&\widehat{f}_l^x(y)-\overline{\widetilde{f}}_l^x(y)\\&=\frac{1}{nh_J^l\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^n\delta_j^l\bar{G}^{-l}(T_j)\Gamma_jK_jJ_j^l(y)-\mathbb{E}\left(\delta_j^l\bar{G}^{-l}(T_j)\Gamma_jK_jJ_j^l(y)|\mathfrak{F}_{j-1}\right)\\&=\frac{1}{nh_J^l\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^n L_j(x,y),\end{aligned}$$

where $L_j(x,y)$ is a triangular array of martingale differences according to the σ-fields $\left(\mathfrak{F}_{j-1}\right)_j$.

Similar to the proof of Lemma 2 of [1] and under (H.1), (H.3) and (H.4), we can write

$$\mathbb{E}\left(L_j^2(x,y)|\mathfrak{F}_{j-1}\right)\ \leq\ 2C'n^2h_J^lh_K^4\phi_{j,x}\left(h_K\right).$$

Then, we apply the exponential inequality of Lemma 1 in [29] (with $d_j^2=C'n^2h_J^lh_K^4\phi_{j,x}(h_K)$ to obtain for $\varepsilon>0$:

$$\begin{aligned}&\mathbb{P}\left(|\widehat{f}_l^x(y)-\overline{\widetilde{f}}_l^x(y)|>\varepsilon\sqrt{\frac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2(h_K)}}\right)\\&\quad\leq\mathbb{P}\left(|L_j(x,y)|>n\varepsilon h_J^l\mathbb{E}\left(\Gamma_1K_1\right)\sqrt{\frac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2(h_K)}}\right)\end{aligned}$$

$$\leq 2\exp\left\{-\frac{n^2h_J^{2l}\left(\mathbb{E}\left(\Gamma_1K_1\right)\right)^2\varepsilon^2\dfrac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2(h_K)}}{2\left(D_n+Cnh_J^l\mathbb{E}\left(\Gamma_1K_1\right)\varepsilon\sqrt{\dfrac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2(h_K)}}\right)}\right\}.$$

Then, using Lemma 5(iii) of [1] leads:

$$\mathbb{P}\left(|\widehat{f}_l^x\left(y\right)-\overline{\widetilde{f}}_l^x\left(y\right)|>\varepsilon\sqrt{\frac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2(h_K)}}\right)\leq 2n^{-C'\varepsilon^2}.$$

Therefore, by using Borel-Cantelli's Lemma and by choosing ε large enough, we find:

$$\widehat{f}_l^x\left(y\right)-\overline{\widetilde{f}}_l^x\left(y\right)=O_{a.co}\left(\sqrt{\frac{\varphi_x\left(h_K\right)\log n}{n^2h_J^l\phi_x^2\left(h_K\right)}}\right). \tag{15}$$

Finally, the proof of the first part of Lemma 2 can be deduced by replacing l by 0 in Equation (15).

Now, if we use the compactness of $\mathcal{C}_{\mathbb{R}}$, we can write $\mathcal{C}_{\mathbb{R}}\subset\bigcup_{k=1}^{d_n}\mathcal{C}_k$, where

$$\mathcal{C}_k=(y_k-l_n,y_k+l_n),\ \text{with}\ l_n=n^{-1-\alpha}\ \text{and}\ l_nd_n=O(1).$$

Thus, we obtain

$$\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}_1^x\left(y\right)-\overline{\widetilde{f}}_1^x\left(y\right)|\leq\underbrace{\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}_1^x\left(y\right)-\widehat{f}_1^x\left(z\right)|}_{\mathscr{C}_1}+\underbrace{\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}_1^x\left(z\right)-\overline{\widetilde{f}}_1^x\left(z\right)|}_{\mathscr{C}_2}$$
$$+\underbrace{\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\overline{\widetilde{f}}_1^x\left(z\right)-\overline{\widetilde{f}}_1^x\left(y\right)|}_{\mathscr{C}_3}.$$

For the term $\mathscr{C}_1$, by using assumptions (H.4)(ii) and (H.5), we obtain:

$$\begin{aligned}\mathscr{C}_1&\leq\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\frac{1}{nh_J\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^n\delta_j\bar{G}^{-1}(T_j)\Gamma_jK_j|J_j(y)-J_j(z)|\right|,\\&\leq\sup_{y\in\mathcal{C}_{\mathbb{R}}}\frac{C|y-z|}{h_J}\left(\left|\frac{1}{nh_J\mathbb{E}\left(\Gamma_1K_1\right)}\sum_{j=1}^n\bar{G}^{-1}(T_j)\Gamma_jK_j\right|\right),\end{aligned}$$

$$\leq \quad C\frac{l_n}{h_J^2}|\widehat{f}_0^x(y)|.$$

Next, using Lemma 3 of [1] allows to have:

$$\mathscr{C}_1 \leq C\frac{l_n}{h_J^2}.$$

Since $l_n = n^{-1-\alpha}$ and by using the first part of (H.5), we obtain:

$$\frac{l_n}{h_J^2} = o\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}}\right).$$

So, for n large enough, we have

$$\mathscr{C}_1 = O_{a.co}\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}}\right). \tag{16}$$

Similarly, for $\mathscr{C}_3$, by using Formula (13) we obtain:

$$\mathscr{C}_3 \leq C\frac{l_n}{h_J^2}|\bar{f}_0^x(y)|.$$

Therefore, by using Lemma 6 of [1], we get:

$$\mathscr{C}_3 \leq C\frac{l_n}{h_J^2}.$$

Using similar arguments as $\mathscr{C}_1$, we can obtain for n large enough:

$$\mathscr{C}_3 = O_{a.co}\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}}\right). \tag{17}$$

Concerning $\mathscr{C}_2$, by using (15) for $l = 1,$ we get for $\varepsilon_0 > 0$ and for all $z \in \mathcal{C}_k$:

$$\mathbb{P}\left(|\widehat{f}_1^x(z) - \bar{\bar{f}}_1^x(z)| > \varepsilon_0\sqrt{\frac{\varphi_x(h_K)\log n}{n^2 h_J \phi_x^2(h_K)}}\right) \leq C' n^{-C_1\varepsilon_0^2}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}_1^x(z)-\overline{\widetilde{f}}_1^x(z)|>\varepsilon_0\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right)\\
&\quad\leq\mathbb{P}\left(\max_{z\in\mathcal{C}_k}|\widehat{f}_1^x(z)-\overline{\widetilde{f}}_1^x(z)|>\varepsilon_0\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right)\\
&\quad\leq 2d_n\max_{z\in\mathcal{C}_k}\mathbb{P}\left(|\widehat{f}_1^x(z)-\overline{\widetilde{f}}_1^x(z)|>\varepsilon_0\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right)\\
&\quad\leq C'n^{-C_1\varepsilon_0^2+1+\alpha}.
\end{aligned}$$

By choosing ε_0 such that $C_0\varepsilon_0^2=2+2\alpha$, we find

$$\mathbb{P}\left(\sup_{y\in\mathcal{C}_{\mathbb{R}}}|\widehat{f}_1^x(z)-\overline{\widetilde{f}}_1^x(z)|>\eta\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right)\leq C'n^{-1-\alpha}.$$

Then, by Borel-Cantelli's Lemma, we get

$$\mathscr{C}_2=O_{a.co}\left(\sqrt{\frac{\varphi_x(h_K)\log n}{n^2h_J\phi_x^2(h_K)}}\right). \tag{18}$$

Finally, the second part of Lemma 2 can be deduced directly from the results (16), (17) and (18). □

4.3. Proof of Lemma 3. From the explicit formulas (1) and (2) and by using Lemma 2, we have

$$\begin{aligned}
&\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\widehat{f}^x(y)-\tilde{f}^x(y)\right|\\
&\quad\leq\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n\delta_j\Gamma_jK_jJ_j\left(\frac{1}{\bar{G}_n(T_j)}-\frac{1}{\bar{G}(T_j)}\right)\right|\\
&\quad\leq\frac{\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\bar{G}_n(y)-\bar{G}(y)\right|}{\bar{G}_n(\tau)}\left|\frac{1}{nh_J\mathbb{E}(\Gamma_1K_1)}\sum_{j=1}^n\delta_j\Gamma_jK_jJ_j\bar{G}^{-1}(T_j)\right|\\
&\quad=\frac{\sup_{y\in\mathcal{C}_{\mathbb{R}}}\left|\bar{G}_n(y)-\bar{G}(y)\right|}{\bar{G}_n(\tau)}\left|\tilde{f}(x,y)\right|
\end{aligned}$$

Since $\bar{G}(\tau) > 0$, in conjunction with the strong law of large numbers and the law of the iterated logarithm on the censoring law (see formula (4.28) in Deheuvels and Einmahl [14], 2000), the result is an immediate consequence of Lemmas 1 and 2. □

4.4. Proof of Lemma 5. It is easily seen that, for all $j = 1, \dots, n$,

$$(nh_J\phi_x(h_K))^{1/2}\, B_n(x,y) = \sum_{j=1}^{n} \xi_j, \tag{19}$$

where $\xi_j = \eta_j - \mathbb{E}\left(\eta_j \mid \mathfrak{F}_{j-1}\right)$, with

$$\eta_j = \left(\frac{\phi_x\left(h_K\right)}{nh_J}\right)^{1/2} \left(\frac{\delta_j}{\bar{G}\left(T_j\right)} J_j - h_J f^x(y)\right) \frac{\Gamma_j K_j}{\mathbb{E}\left(\Gamma_1 K_1\right)}.$$

The summands in (19) form a triangular array of stationary martingale differences with respect to σ-field $\mathfrak{F}_{j-1}$. The asymptotic normality of $B_n(x,y)$ is assembled by applying the central limit theorem for discrete-time arrays of real-valued martingales. Nevertheless, the establishing of the following statements is a must :

(i) $\sum_{j=1}^{n} \mathbb{E}\left[\xi_j^2 \mid \mathfrak{F}_{j-1}\right] \xrightarrow{\mathbb{P}} \sigma^2_{J,K}(x,y)$,

(ii) $n\mathbb{E}\left[\xi_j^2 \mathbb{1}_{[|\xi_j| > \epsilon]}\right] = o(1)$ holds for any $\epsilon > 0$ (Lindeberg's condition).

Proof of part (i).

$$\sum_{j=1}^{n} \mathbb{E}\left(\xi_j^2 | \mathfrak{F}_{j-1}\right) = \sum_{j=1}^{n} \mathbb{E}\left[\eta_j^2 | \mathfrak{F}_{j-1}\right] - \sum_{i=1}^{n} \left(\mathbb{E}\left[\eta_j | \mathfrak{F}_{i-1}\right]\right)^2.$$

By always using a double conditioning with respect to $\mathcal{G}_{j-1}$ and Lemma 5 of of [1], we obtain

$$\begin{aligned}
&\sum_{j=1}^{n} \mathbb{E}\left[\eta_j \mid \mathfrak{F}_{j-1}\right] \\
&= \left(\frac{\phi_x\left(h_K\right)}{nh_J}\right)^{1/2} \frac{1}{\mathbb{E}\left(\Gamma_1 K_1\right)} \mathbb{E}\left(\Gamma_j K_j \left(\frac{\delta_j}{\bar{G}\left(Y_j\right)} J_j - h_J f^x(y)\right) \mid \mathfrak{F}_{j-1}\right)
\end{aligned}$$

$$= \left(\frac{\phi_x(h_K)}{nh_J}\right)^{1/2} \frac{1}{\mathbb{E}(\Gamma_1 K_1)} \mathbb{E}\left(\Gamma_j K_j \left(J_j - h_J f^x(y)\right) \mid \mathfrak{F}_{j-1}\right)$$
$$\leq C\left(nh_J\phi_x(h_K)\right)^{1/2}\left(h_K^{b_1} + h_J^{b_2}\right)\frac{1}{n\phi_{j,x}(h_K)}\phi_j(h_K).$$

Thus, by using (H1) (iii), we find

$$\sum_{j=1}^{n}\Big(\mathbb{E}(\eta_j|\mathfrak{F}_{j-1})\Big)^2 = O_{a.co}\Big(nh_J\phi_x(h_K)\left(h_K^{b_1}+h_J^{b_2}\right)^2\Big).$$

Therefore, the statement of (i) follows then if we show that

$$\lim_{n\to\infty}\sum_{j=1}^{n}\mathbb{E}\left[\eta_j^2 \mid \mathfrak{F}_{j-1}\right] \xrightarrow{\mathbb{P}} \sigma_{J,K}^2(x,y). \tag{20}$$

Observe that

$$\sum_{j=1}^{n}\mathbb{E}\left[\eta_j^2 \mid \mathfrak{F}_{j-1}\right]$$
$$= \left(\frac{\phi_x(h_K)}{nh_J\mathbb{E}(\Gamma_1K_1)^2}\right)\sum_{j=1}^{n}\mathbb{E}\left[\Gamma_j^2K_j^2\left(\frac{\delta_j}{\bar{G}(T_j)}J_j - h_Jf^x(y)\right)^2 \mid \mathfrak{F}_{j-1}\right]$$
$$= \left(\frac{\phi_x(h_K)}{nh_J\mathbb{E}(\Gamma_1K_1)^2}\right)\sum_{j=1}^{n}\mathbb{E}\left[\Gamma_j^2K_j^2\mathbb{E}\left(\left(\frac{\delta_j}{\bar{G}(T_j)}J_j - h_Jf^x(y)\right)^2 \mid X_j\right) \mid \mathfrak{F}_{j-1}\right].$$

Using the definition of the conditional variance, one gets

$$\mathbb{E}\left[\left(\frac{\delta_j}{\bar{G}(T_j)}J_j - h_Jf^x(y)\right)^2 \mid X_{j-1}\right]$$
$$= \mathrm{Var}\left\{\frac{\delta_j}{\bar{G}(T_j)}J_j \mid X_j\right\} + \left\{\mathbb{E}\left(\frac{\delta_j}{\bar{G}(T_j)}J_j \mid X_i\right) - h_Jf^x(y)\right\}^2 = \mathcal{L}_{n,1} + \mathcal{L}_{n,2}.$$

For the second term, using the same argument as those used in the proof of Lemma 1, we verified that it's negligible .

For the first term, we have

$$\mathrm{Var}\left\{\frac{\delta_j}{\bar{G}(T_j)}J_j \mid X_j\right\} = \mathbb{E}\left\{\frac{\delta_j}{\bar{G}^2(T_j)}J_j^2 \mid X_j\right\} - \left[\mathbb{E}\left\{\frac{\delta_j}{\bar{G}(T_j)}J_j \mid X_j\right\}\right]^2. \tag{21}$$

By applying the change of variables and using a Taylor expansion of order one of $\bar{G}(\cdot)$, leads to the existence of some y^* is between y and $y-th_J$ such that:

$$
\begin{aligned}
&\mathbb{E}\left\{\frac{\delta_j}{\bar{G}^2\left(T_j\right)} J_j^2 \mid X_j\right\} \\
&=\mathbb{E}\left[\mathbb{E}\left\{\frac{\delta_j}{\bar{G}^2\left(T_j\right)} J^2\left(\frac{y-T_j}{h_J}\right) \mid X_j \mid Y_j\right\}\right] \\
&=\mathbb{E}\left\{\frac{1}{\bar{G}\left(Y_j\right)} J^2\left(\frac{y-Y_j}{h_J}\right) \mid X_j\right\} \\
&=\int \frac{1}{\bar{G}\left(z\right)} J^2\left(\frac{y-z}{h_J}\right) f^{X_j}(z) dz. \\
&=\int \frac{h_J J^2(t)}{\bar{G}(y-th_J)} f^{X_j}(y-th_J) dt \\
&=\int \frac{h_J}{\bar{G}(y)} J^2(t) f^{X_j}(y-th_J) dt+\frac{h_J^2}{\bar{G}^2(y)} \int t J^2(t) \bar{G}'(y^*) f^{X_j}(y-th_J) dt+o(h_J), \\
&=\beta_1+\beta_2,
\end{aligned}
$$

Under assumption (H.1)(iv) and (H.2)(iii), we have:

$$
\beta_2 \leq C \frac{h_J^2}{\bar{G}^2(\tau_L)} \int J^2(t)\; y\, f^{X_j}(y-th_J) dt+o(1)=O(h_J^2).
$$

On the other hand, with integration by parts we get:

$$
\begin{aligned}
\beta_1 &= \int_{\mathbb{R}} \frac{h_J}{\bar{G}(y)} J^2(t) f^{X_j}(y-th_J) dt, \\
&= \frac{h_J}{\bar{G}\left(y\right)} \int_{\mathbb{R}} J^2(t)\left(f^{X_j}(y-th_J)\right)-f^x(y)\, dt+\frac{h_J}{\bar{G}\left(y\right)} \int_{\mathbb{R}} J(t)^2 f^x(y) dt. \\
&\leq \frac{h_J}{\bar{G}\left(y\right)}\left(C \int_{\mathbb{R}} J^2(t)\left(h_K^{b_1}+|t|^{b_2} h_J^{b_2}\right) du+f^x(y) \int_{\mathbb{R}} J^2(t) dt\right) \\
&= O\left(h_K^{b_1}+h_J^{b_2}\right)+\frac{h_J}{\bar{G}\left(z\right)} f^x(y) \int_{\mathbb{R}} J^2(t) dt.
\end{aligned}
$$

Similarly, the second term of 21 is treated directly by evaluating its square root that is negligible,

$$
\mathbb{E}\left\{\frac{\delta_j}{\bar{G}\left(Y_j\right)} J_j \mid X_j\right\}=\mathbb{E}\left\{J\left(\frac{y-T_j}{h_J}\right) \mid X_j\right\}
$$

$$= O\left(h_K^{b_1} + h_J^{b_2}\right) + h_J f^x(y) \int_{\mathbb{R}} J(t)dt$$
$$= O\left(h_K^{b_1} + h_J^{b_2}\right) + h_J f^x(y).$$

By Lemma (A.2) (iii) of [2], we obtain

$$\lim_{n\to\infty} \left(\frac{\phi_x(h_K)}{nh_J \mathbb{E}(\Gamma_1 K_1)^2}\right) \sum_{j=1}^{n} \mathbb{E}\left[\Gamma_j^2 K_j^2 \mathcal{L}_{n,1} \mid \mathfrak{F}_{j-1}\right]$$
$$= \frac{h_J}{\bar{G}(y)} f^x(y) \int_{\mathbb{R}} J^2(t)dt \times \left(\frac{\phi_x(h_K)}{nh_J \mathbb{E}(\Gamma_1 K_1)^2}\right) \sum_{j=1}^{n} \mathbb{E}\left[\Gamma_j^2 K_j^2 \mid \mathfrak{F}_{j-1}\right]$$
$$\longrightarrow \frac{M_2}{M_1^2} \frac{f^x(y)\int_{\mathbb{R}} J^2(t)dt}{\bar{G}(y)}$$
$$= \sigma_{J,K}^2(x,y).$$

Proof of part (ii). The Lindeberg's condition results from Corollary 9.5.2 in Chow and Teicher (1998) which implies that

$$n\mathbb{E}\left[\xi_j^2 \mathbb{1}_{[|\xi_j|>\epsilon]}\right] \leq 4n\mathbb{E}\left[\eta_j^2 \mathbb{1}_{[|\eta_j|>\epsilon/2]}\right].$$

Let $a > 1$ and $b > 1$ such that $1/a + 1/b = 1$. Through Hölder's and Markov's inequalities, we write, for all $\epsilon > 0$ $\mathbb{E}\left[\eta_j^2 \mathbb{1}_{[|\eta_j|>\epsilon/2]}\right] \leq \frac{\mathbb{E}|\eta_j|^{2a}}{(\epsilon/2)^{2a/b}}$. Taking C_0 a positive constant and $2a = 2 + \varsigma$, we get that

$$4n\mathbb{E}\left[\eta_j^2 \mathbb{1}_{[|\eta_j|>\epsilon/2]}\right]$$
$$\leq C_0 \left(\frac{\phi_x(h_K)}{nh_J}\right)^{(2+\varsigma)/2} \frac{n}{(\mathbb{E}(\Gamma_1 K_1))^{2+\varsigma}} \mathbb{E}\left(\left[\Gamma_j K_j \left|\frac{\delta_j}{\bar{G}(T_j)} J_j - h_J f^x(y)\right|\right]^{2+\varsigma}\right)$$
$$\leq C_0 \left(\frac{\phi_x(h_K)}{nh_J}\right)^{(2+\varsigma)/2} \frac{n}{(\mathbb{E}(\Gamma_1 K_1))^{2+\varsigma}} \mathbb{E}\,|(\Gamma_j K_j|)^{2+\varsigma}\, \mathbb{E}\left[|J_j - h_J f^x(y)|^{2+\varsigma} \mid X_j\right].$$

By changing variable, we get

$$\mathbb{E}\left[|J_j - h_J f^x(y)|^{2+\varsigma} \mid X_j\right]$$
$$= C_0 h_J \int_{\mathbb{R}} J_j^{2+\varsigma}(w) f^{X_j}(y - h_J w)dw + h_J^{2+\varsigma}(f^x(y))^{2+\varsigma}$$
$$\leq C_0 h_J \Big(\int_{\mathbb{R}} J_j^{2+\varsigma}(w) f^{X_j}(y - h_J w)dw + h_J^{1+\varsigma}(f^x(y))^{2+\varsigma}\Big)$$

which implies that

$$\begin{aligned} 4n\mathbb{E}\left[\eta_j^2 1\!\!1_{[|\eta_j|>\epsilon/2]}\right] \leq & C_0 \left(\frac{\phi_x(h_K)}{nh_J}\right)^{(2+\varsigma)/2} \frac{nh_J}{(\mathbb{E}(\Gamma_1 K_1))^{2+\varsigma}}\mathbb{E}\,|(\Gamma_j K_j|)^{2+\varsigma} \\ & \times \left(\int_{\mathbb{R}} J_j^{2+\varsigma}(w) f^{X_j}(y-h_J w)dw + h_J^{1+\varsigma}(f^x(y))^{2+\varsigma}\right) \\ = & O\Big(nh_J\phi_x(h_K)\Big)^{\frac{-\varsigma}{2}}. \end{aligned}$$

This yields the claimed result. □

4.5. Proof of Lemma 6

Observe that

$$\widehat{f_0^x}(y) - 1 = \underbrace{\widehat{f_0^x}(y) - \overline{\widehat{f}}_0^x(y)}_{I_1} + \underbrace{\overline{\widehat{f}}_0^x(y) - 1}_{I_2}.$$

Since $I_2 \longrightarrow 0$ almost completely as $n \to \infty$ in view of (H1) (iii), it suffices to show that $I_1 = o(1)$ as $n \to \infty$. Indeed, by using the first part of Lemma 2 we obtain

$$\widehat{f_0^x}(y) - \overline{\widehat{f}}_0(y) = o(1) \text{ almost completely as } n \to \infty,$$

which completes the proof of the lemma. □

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