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FROM FAMILIES IN WEYL GROUPS TO SPRINGER REPRESENTATIONS

GEORGE LUSZTIG

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA.
E-mail: gyuri@math.mit.edu

Abstract

Let G be a simple reductive group over the complex numbers. Let W be the Weyl group of G. We propose a description of the Springer representations of W associated to various unipotent classes of G by a purely algebraic method involving the families of various reflection subgroups of W and which is suitable to computer calculations.

1. The function $c : \operatorname{Irr} W \to \mathbf{N}$

1.1. Let G be an almost simple, simply connected algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Let W be the Weyl group of G. In this section and until the end of 3.3 we assume that p is 0 or a good prime for G.

For any unipotent class C of G let E_C be the representation of W defined by Springer [13] on the top l-adic cohomology of the Springer fibre $\mathcal{B}_u$ at an element $u \in C$. (Here l is fixed prime number $\neq p$.) For any finite group Γ let $\operatorname{Irr}\Gamma$ be a set of representatives for the irreducible representations of Γ over the l-adic numbers and let R_Γ be the free abelian group with basis $\operatorname{Irr}\Gamma$. We can view E_C as an element of R_W. According to Springer, there is a unique partition $\operatorname{Irr} W = \sqcup_C \operatorname{Irr}_C W$ (the "Springer partition") where C runs over the unipotent classes of G such that $\operatorname{Irr}_C W$ consists of all $E \in \operatorname{Irr} W$ which appear in E_C with coefficient > 0. We define $\gamma : \operatorname{Irr} W \to \mathbf{N}$ by $\gamma(E) = \dim \mathcal{B}_u$ where $E \in \operatorname{Irr}_C W$ and $u \in C$. The purpose of this paper is to propose

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a definition of the Springer partition, of the collection of representations E_C and of the function $\gamma : \mathrm{Irr}\, W \to \mathbf{N}$ which is purely algebraic (without use of geometry) and which is suitable for computer calculations. The same objects can be obtained (without use of geometry) from the approach of [11]; but that approach is more complicated than the present one and seems to be unsuitable for computer calculations.

1.2. Let W' be a Weyl group. For $E' \in \mathrm{Irr}\, W'$ the generic degree of the irreducible Hecke algebra representation E'_q corresponding to E' is of the form $(1/m_{E'})q^{a_{E'}}+$ terms of strictly higher degree in q. (Here $m_{E'}$ is an integer ≥ 1 independent of q.) In [4] we have defined a partition of $\mathrm{Irr}\, W'$ into subsets called *families*. The definition is purely algebraic; it involves induction from parabolic subgroups, tensoring by the sign representation and the knowledge of the function $\mathrm{Irr}\, W' \to \mathbf{N}$, $E' \mapsto a_{E'}$. It is known that $E' \mapsto a_{E'}$ is constant on each family. Let $\mathrm{Irr}_{sp} W'$ be the subset of $\mathrm{Irr}\, W'$ consisting of the special representations, see [3]. If $E' \in \mathrm{Irr}\, W'$ then there is a unique $E'_0 \in \mathrm{Irr}_{sp} W'$ in the same family as E' and $n_{E'} = m_{E'_0}/m_{E'}$ is an integer ≥ 1.

1.3. Let W_{af} be the affine Weyl group associated to the reductive group over $\mathbf{C}$ of type dual to that of G. Let S_{af} be the set of simple reflections of W_{af}. We can identify W with the quotient of W_{af} by the group of translations. For any $K \subsetneqq S_{af}$, the subgroup of W_{af} generated by K is a finite Weyl group; we will identify it with its image W_K under the canonical surjection $W_{af} \to W$.

For $E \in \mathrm{Irr}\, W$ let $\Gamma(E)$ be the set of all pairs (K, E') where $K \subsetneqq S_{af}$ and $E' \in \mathrm{Irr}\, W_K$ is such that the multiplicity $(E : \mathrm{Ind}_{W_K}^{W}(E'))$ is nonzero. We have $\Gamma(E) \neq \emptyset$. Let $c_E = \max\{a_{E'}; (K, E') \in \Gamma(E)\}$; here $a_{E'}$ is computed in terms of W_K. Let $\Gamma_M(E) = \{(K, E') \in \Gamma(E); a_{E'} = c_E\}$. We have $\Gamma_M(E) \neq \emptyset$. Let $N_E = \max\{n_{E'}; (K, E') \in \Gamma_M(E)\}$. Let $\sim$ be the equivalence relation on $\mathrm{Irr}\, W$ generated by the relation $E_1 \sim E_2$ if there exist $(K_1, E'_1) \in \Gamma_M(E_1), (K_2, E'_2) \in \Gamma_M(E_2)$ such that $K_1 = K_2$ and E'_1, E'_2 are in the same family of $W_{K_1} = W_{K_2}$. The equivalence classes on $\mathrm{Irr}\, W$ for $\sim$ are called c-families. Note that the function $E \mapsto c_E$ is constant on c-families. For any c-family $\mathfrak{f} \subset \mathrm{Irr}\, W$ we set $\mathbf{e}_{\mathfrak{f}} = \sum_{E \in \mathfrak{f}} N_E E \in R_W$.

In the definition of $\Gamma(E)$ one could add the condition that $\sharp(K) = \sharp(S_{af}) - 1$. This would not affect the validity of the following result.

Proposition 1.4.

(a) *For any $E \in \mathrm{Irr}\, W$ we have $c_E = \gamma(E)$.*

(b) *The Springer partition of $\mathrm{Irr}\, W$ coincides with the partition of $\mathrm{Irr}\, W$ into c-families.*

(c) *The collection of Springer representations E_C of W for various unipotent classes C coincides with the collection of elements $\mathbf{e}_{\mathfrak{f}} \in R_W$ for various c-families $\mathfrak{f}$ in $\mathrm{Irr}\, W$.*

1.5. For any $K \subsetneqq S_{af}$ we define a homomorphism $\mathcal{J}_{W_K}^W : R_{W_K} \to R_W$ by setting for any $E' \in \mathrm{Irr}\, W_K$

$$\mathcal{J}_{W_K}^W(E') = \sum_{E \in \mathrm{Irr}\, W; c_E = a_{E'}} (E : \mathrm{Ind}_{W_K}^W(E'))E \in R_W.$$

Let ϕ be a family of W_K and let $\mathfrak{f}$ be a c-family. We say that (K, ϕ) is *adapted* to $\mathfrak{f}$ if there exists a subset ϕ_* of ϕ such that

(a) $\mathcal{J}_{W_K}^W(E') \in \mathfrak{f}$ (and in particular is in $\mathrm{Irr}\, W$) for any $E' \in \phi_*$;

(b) $E' \mapsto E := \mathcal{J}_{W_K}^W(E')$ is a bijection $\phi_* \xrightarrow{\sim} \mathfrak{f}$;

(c) if E', E are as in (b) then $n_{E'} = N_E$;

(d) if $E' \in \phi - \phi_*$ and $E \in \mathrm{Irr}\, W$ appears in $\mathrm{Ind}_{W_K}^W(E')$, then $c_E > a_{E'}$.

Note that ϕ_* above is unique (if it exists). Moreover the value of the a-function on ϕ must be equal to the value of the c-function on $\mathfrak{f}$. For (K, ϕ) adapted to $\mathfrak{f}$ we set $E'_{\phi_*} = \sum_{E' \in \phi_*} n_{E'} E' \in R_{W_K}$. We then have $\mathcal{J}_{W_K}^W(E'_{\phi_*}) = \mathbf{e}_{\mathfrak{f}}$. Note that ϕ_* is not determined by the Coxeter group W_K (it depends on $\mathfrak{f}$).

Proposition 1.6. *Let $\mathfrak{f} \subset \mathrm{Irr}\, W$ be a c-family. There exist $K \subsetneqq S_{af}$ and a family ϕ of W_K such that (K, ϕ) is adapted to $\mathfrak{f}$.*

1.7. Let W' be a Weyl group; let S' be the set of simple reflections in W'. For any $H \subset S'$ let W'_H be the subgroup of W' generated by H. Let $E' \in \mathrm{Irr}_{sp} W'$. We say that E' is non-rigid if there exists $H \subsetneqq S'$ and $E'' \in \mathrm{Irr}_{sp} W'_H$ such that E' appears in $\mathrm{Ind}_{W_H}^W(E'')$ and $a_{E''} = a_{E'}$, $m_{E''} = m_{E'}$. (Here $a_{E''}, m_{E''}$

are defined relative to W'_H.) We say that $E' \in \mathrm{Irr}_{sp}W'$ is rigid if it is not non-rigid. For any $E' \in \mathrm{Irr}_{sp}W'$ one can find $H \subset S'$ and $E'' \in \mathrm{Irr}_{sp}W'_H$ such that E'' is rigid, E' appears in $\mathrm{Ind}_{W_H}^{W}(E'')$ and $a_{E''} = a_{E'}$, $m_{E''} = m_{E'}$. Moreover, (H, E'') is uniquely determined by E' up to conjugation by an element of W'. (A statement close to this appears in [5, 13.1].) A family of W' is said to be rigid if the unique special representation in it is rigid. If W' is a product of two Weyl groups W'_1, W'_2, the rigid special representations of W' are precisely the external tensor products of a rigid special representation of W'_1 with one of W'_2.

We denote by ϵ the sign representation of W'. It is rigid special. If W' is of type A_n, ϵ is the only rigid special representation of W'.

In the next subsection we list the rigid special representation for W' irreducible of low rank. We use the following notation. If W' is of type E_6, E_7 or E_8, an element $E' \in \mathrm{Irr}\, W'$ can be represented uniquely in the form δ_x where $\delta = \dim(E')$ and x is the smallest integer ≥ 0 such that E' appears in the x-th symmetric power of the reflection representation of W'. The same notation can be used in type F_4, G_2 but in these cases there may be two elements $E' \in \mathrm{Irr}\, W'$ with the same δ_x. (This ambiguity does not appear for rigid special representations.) If W' is of type $D_n, n \geq 4$ we represent an $E' \in \mathrm{Irr}\, W'$ as a sequence $a_1a_2 \cdots a_{2k}$ where $[a_{2k} < a_{2k-2} < \cdots < a_2, a_{2k-1} < a_{2k-3} < \cdots < a_1]$ is the symbol representing E' in [5]. If W' is of type $B_n, n \geq 2$ we represent an $E' \in \mathrm{Irr}\, W'$ as a sequence $a_0a_1a_2 \cdots a_{2k}$ where $[a_{2k} < a_{2k-2} < \cdots < a_0, a_{2k-1} < a_{2k-3} < \cdots < a_1]$ is the symbol representing E' in [5]. When $n = 2$ this notation depends on the order of S'; the representations $201, 120$ are interchanged when the order of S' is reversed; again this ambiguity does not appear for rigid special representations.

1.8. Here are the rigid special representations for W' assumed to be irreducible of low rank.

Type B_2: 210, $22110 = \epsilon$ (with $a = 1, 4$ respectively);

Type B_3: 32110, $3322110 = \epsilon$ (with $a = 4, 9$ respectively);

Type B_4: 32210, 4322110, $443322110 = \epsilon$ (with $a = 6, 9, 16$ respectively);

Type D_4: 3210, 332110, $43322110 = \epsilon$ (with $a = 3, 7, 12$ respectively);

Type D_5: 432110, 44322110, $5443322110 = \epsilon$ (with $a = 7, 13, 20$ respectively);

Type D_6: 432210, 54322110, 44332110, 5543322110, 655443322110 $= \epsilon$ (with $a = 10$, 13, 16, 21, 30 respectively);

Type D_7: 433210, 54332110, 6543322110, 5544322110, 665443322110, 76655443322110 $= \epsilon$ (with $a = 12, 16, 21, 24, 29, 42$ respectively);

Type D_8: 443210, 54432110, 54332210, 6544322110, 5544332110, 765443322110, 665543322110, 6655443322110, 77655443322110, 8776655443322110 $= \epsilon$ (with $a = 13$, 18, 21, 24, 29, 31, 34, 42, 43, 56 respectively);

Type E_6: $80_7, 30_{15}, 6_{25}, 1_{36} = \epsilon$;

Type E_7: $512_{11}, 315_{16}, 120_{25}, 56_{30}, 27_{37}, 7_{46}, 1_{63} = \epsilon$;

Type E_8: $4480_{16}, 4200_{24}, 4096_{26}, 2240_{28}, 1400_{32}, 1400_{37},\ 700_{42},\ 560_{47},\ 210_{52}, 112_{63}, 35_{74}, 8_{91},\ 1_{120} = \epsilon$; Type F_4: $12_4, 9_{10}, 4_{13}, 1_{24} = \epsilon$;

Type G_2: $2_1, 1_6 = \epsilon$.

We now state a refinement of Proposition 1.6.

Proposition 1.9. *Let $\mathfrak{f} \subset \operatorname{Irr} W$ be a c-family. There exist $K \subsetneq S_{af}$ and a rigid family ϕ of W_K such that (K, ϕ) is adapted to $\mathfrak{f}$.*

2. Tables

2.1. In the tables below, for W of type E_8, E_7, E_6, F_4, G_2 we make a list of the elements $\mathbf{e}_{\mathfrak{f}}$ (see 1.3) for each c-family $\mathfrak{f}$ of W. (Note that $\mathfrak{f}$ is determined by $\mathbf{e}_{\mathfrak{f}}$.) In each case we give the value of the c-function on $\mathfrak{f}$ and we specify the pairs (K, ϕ) with ϕ a rigid family of W_K such that (K, ϕ) is adapted to $\mathfrak{f}$. When there are several such pairs (K, ϕ) which are obtained one from another by conjugating by an element of W or by applying an automorphism of the affine diagram of W_{af}, we list only one pair out of these. In each case there remain one or two pairs to be listed. In each case we specify K by the type of the corresponding subdiagram, as a name or as a picture (with the elements of K marked by $\bullet$). The irreducible representations which enter in ϕ are denoted using the conventions in 1.7. In type F_4 the notation δ_x in 1.7 is ambiguous; when it represents two objects, we distinguish them by denoting them by δ'_x, δ''_x (in a way compatible with the conventions in [2]); the table in 2.5 could be taken as definition of δ'_x, δ''_x as well as a definition of $201, 120$ for type B_2. In type G_2 there are two objects represented by 1_3; we write them as $1_3, \tilde{1}_3$ (they are defined by the table in 2.6).

2.2. Table for type E_8

$c = 0;\ 1_0 = \mathcal{J}_{W_K}^W(\epsilon), K = \emptyset;$

$c = 1;\ 8_1 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\,\cdot\,\underset{\cdot}{\cdot}\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot);$

$c = 2;\ 35_2 = \mathcal{J}_{W_K}^W(\epsilon), K = (\cdot\,\bullet\,\underset{\cdot}{\cdot}\,\bullet\,\cdot\,\cdot\,\cdot\,\cdot);$

$c = 3;\ 112_3 + 28_8 = \mathcal{J}_{W_K}^W(3210 + 3201),\ K = (\cdot\,\bullet\,\underset{\bullet}{\bullet}\,\bullet\,\cdot\,\cdot\,\cdot\,\cdot);$

$c = 4;\ 84_4 = \mathcal{J}_{W_K}^W(\epsilon), K = (\cdot\,\cdot\,\underset{\bullet}{\cdot}\,\bullet\,\cdot\,\bullet\,\cdot\,\bullet);$

$c = 4;\ 210_4 + 160_7 = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon),\ K = (\bullet\,\bullet\,\underset{\bullet}{\bullet}\,\bullet\,\cdot\,\bullet\,\cdot\,\cdot);$

$c = 5;\ 560_5 + 50_8 = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon),\ K = D_4 \times (A_1 \times A_1);$

$c = 6;\ 567_6 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\,\bullet\,\underset{\cdot}{\bullet}\,\cdot\,\cdot\,\cdot\,\cdot\,\cdot);$

$c = 6;\ 700_6 + 300_8 = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon), K = D_4 \times A_2;$

$c = 7;\ 400_7 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\,\bullet\,\underset{\bullet}{\cdot}\,\bullet\,\cdot\,\bullet\,\cdot\,\bullet);$

$c = 7;\ 1400_7 + 2 \times 1008_9 + 56_{19} = \mathcal{J}_{W_K}^W(80_7 + 2 \times 90_8 + 20_{10}), K = E_6;$

$c = 8;\ 1344_8 = \mathcal{J}_{W_K}^W(\epsilon), K = (\cdot\,\cdot\,\underset{\bullet}{\cdot}\,\bullet\,\cdot\,\bullet\,\bullet\,\bullet);$

$c = 8;\ 1400_8 + 2 \times 1575_{10} + 350_{14} = \mathcal{J}_{W_K}^W(80_7 \boxtimes \epsilon + 2 \times 90_8 \boxtimes \epsilon + 20_{10} \boxtimes \epsilon),$
$K = E_6 \times A_1;$

$c = 9;\ 3240_9 + 1050_{10} = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3120 \boxtimes \epsilon) = \mathcal{J}_{W_{K'}}^W(432110 \boxtimes \epsilon +$
$423110 \boxtimes \epsilon), K = D_4 \times A_3, K' = D_5 \times (A_1 \times A_1);$

$c = 9;\ 448_9 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\,\bullet\,\underset{\cdot}{\cdot}\,\bullet\,\bullet\,\cdot\,\bullet\,\bullet);$

$c = 10;\ 2240_{10} + 2 \times 175_{12} + 840_{13} = \mathcal{J}_{W_K}^W(80_7 \boxtimes \epsilon + 2 \times 10_9 \boxtimes \epsilon + 20_{10} \boxtimes \epsilon), K =$
$E_6 A_2;$

$c = 10;\ 2268_{10} + 1296_{13} = \mathcal{J}_{W_K}^W(432210 + 432201), K = D_6;$

$c = 11;\ 4096_{11} + 4096_{12} = \mathcal{J}_{W_K}^W(512_{11} + 512_{12}) = \mathcal{J}_{W_{K'}}^W(432210 \boxtimes \epsilon + 432201 \boxtimes$
$\epsilon),\ K = E_7, K' = D_6 A_1;$

$c = 11;\ 1400_{11} = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\,\bullet\,\underset{\bullet}{\cdot}\,\bullet\,\cdot\,\bullet\,\bullet\,\bullet);$

$c = 12;\ 525_{12} = \mathcal{J}_{W_K}^K(\epsilon), K = D_4;$

$c = 12;\ 972_{12} = \mathcal{J}_{W_K}^K(\epsilon), K = (\bullet\,\bullet\,\underset{\cdot}{\bullet}\,\cdot\,\bullet\,\bullet\,\bullet\,\cdot);$

$c = 12$; $4200_{12} + 3360_{13} = \mathcal{J}_{W_K}^{W}(433210 + 433201) = \mathcal{J}_{W_{K'}}^{W}(512_{11} \boxtimes \epsilon + 512_{12} \boxtimes \epsilon)$, $K = D_7, K' = E_7 \times A_1$;

$c = 13$; $4536_{13} + 840_{14} = \mathcal{J}_{W_K}^{W}(432110 \boxtimes \epsilon + 431201 \boxtimes \epsilon) = \mathcal{J}_{W_{K'}}^{W}(443210 + 443201)$, $K = D_5 \times A_3, K' = D_8$;

$c = 13$; $2800_{13} + 2100_{16} = \mathcal{J}_{W_K}^{W}(54322110 + 54231201), K = D_6$;

$c = 14$; $6075_{14} + 700_{16} = \mathcal{J}_{W_K}^{W}(54322110 \boxtimes \epsilon + 53422110 \boxtimes \epsilon), K = D_6 \times A_1$;

$c = 14$; $2835_{14} = \mathcal{J}_{W_K}^{W}(\epsilon), K = (\bullet \cdot \underset{\bullet}{\bullet} \cdot \bullet \bullet \bullet \bullet)$;

$c = 15$; $4200_{15} = \mathcal{J}_{W_K}^{W}(\epsilon), K = (\cdot \bullet \underset{\bullet}{\bullet} \bullet \cdot \bullet \bullet \cdot)$;

$c = 15$; $5600_{15} + 2400_{17} = \mathcal{J}_{W_K}^{W}(30_{15} + 15_{17}), K = E_6$;

$c = 16$; $4480_{16} + 5 \times 4536_{18} + 4 \times 5670_{18} + 5 \times 1400_{20} + 6 \times 1680_{22} + 4 \times 70_{32} = \mathcal{J}_{W_K}^{W}(4480_{16} + 5 \times 4536_{18} + 4 \times 5670_{18} + 5 \times 1400_{20} + 6 \times 1680_{22} + 4 \times 70_{32}), K = E_8$;

$c = 16$; $3200_{16} = c_{W_K}^{W}(\epsilon), K = A_5 \times A_1$;

$c = 17$; $7168_{17} + 5600_{19} + 448_{25} = \mathcal{J}_{W_K}^{W}(315_{16} \boxtimes \epsilon + 2 \times 280_{18} \boxtimes \epsilon + 35_{22} \boxtimes \epsilon)$, $K = E_7 \times A_1$;

$c = 18$; $3150_{18} + 1134_{20} = \mathcal{J}_{W_K}^{W}(30_{15} \boxtimes \epsilon + 15_{17} \boxtimes \epsilon), K = E_6 \times A_2$;

$c = 18$; $4200_{18} + 2688_{20} = \mathcal{J}_{W_K}^{W}(54432110 + 54431201), K = D_8$;

$c = 19$; $1344_{19} = \mathcal{J}_{W_K}^{W}(44322110 \boxtimes \epsilon), K = D_5 \times A_3$;

$c = 19$; $2016_{19} = \mathcal{J}_{W_K}^{W}(\epsilon), K = A_5 \times A_2 \times A_1$;

$c = 20$; $420_{20} = \mathcal{J}_{W_K}^{W}(\epsilon), K = A_4 \times A_4$;

$c = 20$; $2100_{20} = \mathcal{J}_{W_K}^{W}(\epsilon), K = (\bullet \bullet \underset{\bullet}{\bullet} \bullet \cdot \cdot \cdot \cdot)$;

$c = 21$; $4200_{21} + 168_{24} = \mathcal{J}_{W_K}^{W}(54332210 + 53423120), K = D_8$;

$c = 21$; $5600_{21} + 2400_{23} = \mathcal{J}_{W_K}^{W}(6543322110 + 6534231201), K = D_7$;

$c = 22$; $2835_{22} = \mathcal{J}_{W_K}^{W}(\epsilon), K = (\bullet \cdot \underset{\cdot}{\bullet} \bullet \bullet \bullet \bullet \bullet)$;

$c = 22$; $3200_{22} = \mathcal{J}_{W_K}^{W}(\epsilon), K = D_5 \times (A_1 \times A_1)$;

$c = 22$; $6075_{22} = \mathcal{J}_{W_K}^{W}(5543322110 \boxtimes \epsilon), K = D_6 \times A_1$;

$c = 23$; $4536_{23} = \mathcal{J}_{W_K}^{W}(\epsilon), K = (\bullet \bullet \underset{\bullet}{\bullet} \bullet \cdot \cdot \bullet \bullet)$;

$c = 24$; $4200_{24} + 3360_{25} = \mathcal{J}_{W_K}^{W}(6544322110 + 6544231201) = \mathcal{J}_{W_{K'}}^{W}(4200_{24} + 3360_{25})$, $K = D_8, K' = E_8$;

$c=25;\ 2800_{25}+2100_{28}=\mathcal{J}_{W_K}^{W}(120_{25}+105_{26}), K=E_7;$

$c=26;\ 840_{26}=\mathcal{J}_{W_K}^{W}(\epsilon), K=D_5A_3;$

$c=26;\ 4096_{26}+4096_{27}=\mathcal{J}_{W_K}^{W}(4096_{26}+4096_{27})=\mathcal{J}_{W_{K'}}^{W}(120_{25}\boxtimes\epsilon+105_{26}\boxtimes\epsilon),\ K=E_8, K'=E_7\times A_1;$

$c=28;\ 700_{28}=\mathcal{J}_{W_K}^{W}(\epsilon)=\mathcal{J}_{W_{K'}}^{W}(\epsilon)$, $K=($  $)$, $K'=($  $)$;

$c=28;\ 2240_{28}+840_{31}=\mathcal{J}_{W_K}^{W}(2240_{28}+840_{31}), K=E_8;$

$c=29;\ 1400_{29}=\mathcal{J}_{W_K}^{W}(\epsilon)=\mathcal{J}_{W_{K'}}^{W}(5544332110)$, $K=($  $)$, $K'=D_8$;

$c=30;\ 2268_{30}+1296_{33}=\mathcal{J}_{W_K}^{W}(56_{30}+21_{33}), K=E_7;$

$c=31;\ 3240_{31}+972_{32}=\mathcal{J}_{W_K}^{W}(765443322110+756443322110)=\mathcal{J}_{W_{K'}}^{W}(56_{30}\boxtimes\epsilon+35_{31}\boxtimes\epsilon),\ K=D_8, K'=E_7\times A_1;$

$c=32;\ 1400_{32}+2\times1575_{34}+350_{38}=\mathcal{J}_{W_K}^{W}(1400_{32}+2\times1575_{34}+350_{38}), K=E_8;$

$c=34;\ 1050_{34}=\mathcal{J}_{W_K}^{W}(665543322110), K=D_8;$

$c=36;\ 175_{36}=\mathcal{J}_{W_K}^{W}(\epsilon), K=A_8;$

$c=36;\ 525_{36}=\mathcal{J}_{W_K}^{W}(\epsilon),\ K=E_6;$

$c=37;\ 1400_{37}+2\times1008_{39}+56_{49}=\mathcal{J}_{W_K}^{W}(1400_{37}+2\times1008_{39}+56_{49}), K=E_8;$

$c=38;\ 1344_{38}=\mathcal{J}_{W_K}^{W}(27_{37}\boxtimes\epsilon), K=E_7\times A_1;$

$c=39;\ 448_{39}=\mathcal{J}_{W_K}^{W}(\epsilon), K=E_6\times A_2;$

$c=42;\ 700_{42}+300_{44}=\mathcal{J}_{W_K}^{W}(700_{42}+300_{44}), K=E_8;$

$c=43;\ 400_{43}=\mathcal{J}_{W_K}^{W}(77655443322110), K=D_8;$

$c=46;\ 567_{46}=\mathcal{J}_{W_K}^{W}(7_{46}), K=E_7;$

$c=47;\ 560_{47}=\mathcal{J}_{W_K}^{W}(7_{46}\boxtimes\epsilon)=\mathcal{J}_{W_{K'}}^{W}(560_{47}), K=E_7\times A_1, K'=E_8;$

$c=52;\ 210_{52}+160_{55}=\mathcal{J}_{W_K}^{W}(210_{52}+160_{55}), K=E_8;$

$c=56;\ 50_{56}=\mathcal{J}_{W_K}^{W}(8776655443322110), K=D_8;$

$c=63;\ 112_{63}+28_{68}=\mathcal{J}_{W_K}^{W}(112_{63}+28_{68}), K=E_8;$

$c=64;\ 84_{64}=\mathcal{J}_{W_K}^{W}(\epsilon), K=E_7\times A_1;$

$c = 74;\ 35_{74} = \mathcal{J}_{W_K}^W(35_{74}), K = E_8;$

$c = 91;\ 8_{91} = \mathcal{J}_{W_K}^W(8_{91}), K = E_8;$

$c = 120;\ 1_{120} = \mathcal{J}_{W_K}^W(\epsilon), K = E_8.$

2.3. Table for type E_7

$c = 0;\ 1_0 = \mathcal{J}_{W_K}^W(\epsilon), K = \emptyset;$

$c = 1;\ 8_1 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\cdot\underset{\cdot}{\cdot}\cdot\cdot\cdot\cdot);$

$c = 2;\ 27_2 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\cdot\bullet\underset{\cdot}{\cdot}\cdot\cdot\cdot);$

$c = 3;\ 56_3 + 21_6 = \mathcal{J}_{W_K}^W(3210 + 3201), K = D_4;$

$c = 3;\ 21_3 = \mathcal{J}_{W_K}^W(\epsilon), K = (\cdot\cdot\cdot\underset{\bullet}{\cdot}\bullet\cdot\bullet);$

$c = 4;\ 35_4 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\cdot\bullet\underset{\cdot}{\cdot}\bullet\cdot\bullet);$

$c = 4;\ 120_4 + 105_5 = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon), K = D_4 \times A_1;$

$c = 5;\ 189_5 + 15_7 = \mathcal{J}_{W_K}^W(3210 \boxtimes \epsilon + 3120 \boxtimes \epsilon), K = D_4 \times (A_1 \times A_1);$

$c = 6;\ 105_6 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\cdot\underset{\bullet}{\cdot}\bullet\cdot\bullet);$

$c = 6;\ 168_6 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\cdot\underset{\cdot}{\cdot}\cdot\bullet\bullet);$

$c = 6;\ 210_6 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\bullet\underset{\cdot}{\cdot}\cdot\cdot\cdot);$

$c = 7;\ 315_7 + 2 \times 280_9 + 35_{13} = \mathcal{J}_{W_K}^W(80_7 + 2 \times 90_9 + 20_{10}), K = E_6;$

$c = 7;\ 189_7 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\bullet\underset{\bullet}{\cdot}\cdot\cdot\cdot);$

$c = 8;\ 280_8 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\bullet\underset{\cdot}{\cdot}\bullet\cdot\bullet);$

$c = 8;\ 405_8 + 189_{10} = \mathcal{J}_{W_K}^W(432110\epsilon + 431201\epsilon), K = D_5 \times A_1;$

$c = 9;\ 70_9 = \mathcal{J}_{W_K}^W(\epsilon), K = A_2 \times A_2 \times A_2;$

$c = 9;\ 216_9 = \mathcal{J}_{W_K}^W(\epsilon), K = A_3 \times A_1 \times A_1;$

$c = 9;\ 378_9 = \mathcal{J}_{W_K}^W(\epsilon), K = (\bullet\bullet\bullet\underset{\cdot}{\cdot}\cdot\bullet\bullet);$

$c = 10;\ 420_{10} + 336_{11} = \mathcal{J}_{W_K}^W(432210 + 432201), K = D_6;$

$c = 10;\ 210_{10} = \mathcal{J}_{W_K}^W(\epsilon) = \mathcal{J}_{W_{K'}}^W(\epsilon),\ K = (\bullet\bullet\bullet\underset{\bullet}{\cdot}\cdot\bullet\bullet);$

$c = 11;\ 512_{11} + 512_{12} = \mathcal{J}_{W_K}^W(512_{11} + 512_{12}) = \mathcal{J}_{W_{K'}}^W(432210 \boxtimes \epsilon + 432210 \boxtimes \epsilon), K = E_7, K' = D_6 \times A_1;$

$c = 12;\ 84_{12} = \mathcal{J}_{W_K}^W(\epsilon), K = A_3 \times A_3;$

$c = 12;\ 105_{12} = \mathcal{J}_{W_K}^W(\epsilon), K = D_4;$

$c = 13;\ 210_{13} = \mathcal{J}_{W_K}^W(\epsilon) = \mathcal{J}_{W_{K'}}^W(\epsilon), K = A_3 \times A_3 \times A_1, K' = A_4 \times A_2;$

$c = 13;\ 420_{13} + 336_{14} = \mathcal{J}_{W_K}^W(54322110 + 54231201), K = D_6;$

$c = 14;\ 378_{14} + 84_{15} = \mathcal{J}_{W_K}^W(54322110 \boxtimes \epsilon + 53422110 \boxtimes \epsilon), K = D_6 \times A_1;$

$c = 15;\ 105_{15} = \mathcal{J}_{W_K}^W(\epsilon), K = ($  $);$

$c = 15;\ 405_{15} + 189_{17} = \mathcal{J}_{W_K}^W(30_{15} + 15_{17}), K = E_6;$

$c = 16;\ 315_{16} + 2 \times 280_{18} + 35_{22} = \mathcal{J}_{W_K}^W(315_{16} + \times 280_{18} + 35_{22}), K = E_7;$

$c = 16;\ 216_{16} = \mathcal{J}_{W_K}^W(\epsilon), K = ($  $);$

$c = 17;\ 280_{17} = \mathcal{J}_{W_K}^W(44332110 \boxtimes \epsilon), K = D_6 \times A_1;$

$c = 18;\ 70_{18} = \mathcal{J}_{W_K}^W(\epsilon), K = A_5 \times A_2;$

$c = 20;\ 189_{20} = \mathcal{J}_{W_K}^W(\epsilon), K = D_5;$

$c = 21;\ 105_{21} = \mathcal{J}_{W_K}^W(\epsilon), K = A_6;$

$c = 21;\ 168_{21} = \mathcal{J}_{W_K}^W(\epsilon), K = D_5 \times A_1;$

$c = 21;\ 210_{21} = \mathcal{J}_{W_K}^W(5543322110), K = D_6;$

$c = 22;\ 189_{22} = \mathcal{J}_{W_K}^W(5543322110 \boxtimes \epsilon) = \mathcal{J}_{W_{K'}}^W(189_{22}), K = D_6 \times A_1, K' = E_7;$

$c = 25;\ 120_{25} + 105_{26} = \mathcal{J}_{W_K}^W(120_{25} + 105_{26}), K = E_7;$

$c = 28;\ 15_{28} = \mathcal{J}_{W_K}^W(\epsilon), K = A_7;$

$c = 30;\ 56_{30} + 21_{33} = \mathcal{J}_{W_K}^W(56_{30} + 21_{33}), K = E_7;$

$c = 31;\ 35_{31} = \mathcal{J}_{W_K}^W(\epsilon), K = D_6 \times A_1;$

$c = 36;\ 21_{36} = \mathcal{J}_{W_K}^W(\epsilon), K = E_6;$

$c = 37;\ 27_{37} = \mathcal{J}_{W_K}^W(27_{37}), K = E_7;$

$c = 46;\ 7_{46} = \mathcal{J}_{W_K}^W(7_{46}), K = E_7;$

$c = 63;\ 1_{63} = \mathcal{J}_{W_K}^W(1_{63}), K = E_7.$

2.4. Table for type E_6

$c = 0;\ 1_0 = \mathcal{J}_{W_K}^W(\epsilon), K = \emptyset;$

$c = 1;\ 6_1 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 2;\ 20_2 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \cdot & \bullet & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 3;\ 30_3 + 15_5 = \mathcal{J}_{W_K}^W(3210 + 3201), K = D_4;$

$c = 4;\ 15_4 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \cdot & \bullet & \cdot & \bullet \\ & & \cdot & & \\ & & \bullet & & \end{pmatrix};$

$c = 4;\ 64_4 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \cdot & \bullet & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 5;\ 60_5 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \cdot & \bullet & \cdot \\ & & \bullet & & \\ & & \cdot & & \end{pmatrix};$

$c = 6;\ 24_6 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \cdot & \bullet & \bullet \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 6;\ 81_6 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \cdot & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 7;\ 80_7 + 2 \times 90_8 + 20_{10} = \mathcal{J}_{W_K}^W(80_7 + 2 \times 90_8 + 20_{10}), K = E_6;$

$c = 8;\ 60_8 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \cdot & \bullet \\ & & \cdot & & \\ & & \bullet & & \end{pmatrix};$

$c = 9;\ 10_9 = \mathcal{J}_{W_K}^W(\epsilon), K = A_2 \times A_2 \times A_2;$

$c = 10;\ 81_{10} = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \cdot \\ & & \cdot & & \\ & & \cdot & & \end{pmatrix};$

$c = 11;\ 60_{11} = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \cdot \\ & & \cdot & & \\ & & \bullet & & \end{pmatrix};$

$c = 12;\ 24_{12} = \mathcal{J}_{W_K}^W(\epsilon), K = D_4;$

$c = 13;\ 64_{13} = \mathcal{J}_{W_K}^W(44322110), K = D_5;$

$c = 15;\ 30_{15} + 15_{17} = \mathcal{J}_{W_K}^W(30_{15} + 15_{17}), K = E_6;$

$c = 16;\ 15_{16} = \mathcal{J}_{W_K}^W(\epsilon), K = A_5 \times A_1;$

$c = 20;\ 20_{20} = \mathcal{J}_{W_K}^W(\epsilon), K = D_5;$

$c = 25;\ 6_{25} = \mathcal{J}_{W_K}^W(6_{25}), K = E_6;$

$c = 36;\ 1_{36} = \mathcal{J}_{W_K}^W(1_{36}), K = E_6;$

2.5. Table for type F_4

In this table we denote by A_2 (resp. A_2') a subset $K \subsetneq S_{af}$ of type A_2 which is contained (resp. not contained) in a subset of S_{af} of type A_3.

$c = 0;\ 1_0 = \mathcal{J}_{W_K}^W(\epsilon), K = \emptyset;$

$c=1;\ 4_1+2''_4=\mathcal{J}_{W_K}^W(210+201), K=B_2;$

$c=2;\ 9_2+2'_4=\mathcal{J}_{W_K}^W(210\boxtimes\epsilon+120\boxtimes\epsilon), K=B_2\times A_1;$

$c=3;\ 8''_3=\mathcal{J}_{W_K}^W(\epsilon), K=A'_2;$

$c=3;\ 8'_3=\mathcal{J}_{W_K}^W(\epsilon)=\mathcal{J}_{W_{K'}}^W(\epsilon), K=A_2, K'=A_1\times A_1\times A_1;$

$c=4;\ 12_4+3\times 9''_6+2\times 6''_6+3\times 1''_{12}=\mathcal{J}_{W_K}^W(12_4+3\times 9''_6+2\times 6''_6+3\times 1''_{12}), K=F_4;$

$c=5;\ 16_5+4''_7=\mathcal{J}_{W_K}^W(32110\boxtimes\epsilon+31201\boxtimes\epsilon), K=C_3\times A_1;$

$c=6;\ 9'_6+4_8=\mathcal{J}_{W_K}^W(32210+23120), K=B_4;$

$c=6;\ 6'_6=\mathcal{J}_{W_K}^W(\epsilon), K=A_2\times A'_2;$

$c=7;\ 4'_7=\mathcal{J}_{W_K}^W(\epsilon), K=A_3\times A_1;$

$c=9;\ 8'_9+1_{12}=\mathcal{J}_{W_K}^W(4322110+4231201), K=B_4;$

$c=9;\ 8''_9=\mathcal{J}_{W_K}^W(\epsilon), K=C_3;$

$c=10;\ 9_{10}=\mathcal{J}_{W_K}^W(\epsilon)=\mathcal{J}_{W_{K'}}^W(9_{10}), K=C_3\times A_1, K'=F_4;$

$c=13;\ 4_{13}+2''_{16}=\mathcal{J}_{W_K}^W(4_{13}+2''_{16}), K=F_4;$

$c=16;\ 2_{16}=\mathcal{J}_{W_K}^W(\epsilon), K=B_4;$

$c=24;\ 1_{24}=\mathcal{J}_{W_K}^W(1_{24}), K=F_4.$

2.6. Table for type G_2

$c=0;\ 1_0=\mathcal{J}_{W_K}^W(\epsilon), K=\emptyset;$

$c=1;\ 2_1+2\times 1_3=\mathcal{J}_{W_K}^W(2_1+2\times 1_3), K=G_2;$

$c=2;\ 2_2=\mathcal{J}_{W_K}^W(\epsilon), K=A_1\times A_1;$

$c=3;\ \tilde{1}_3=\mathcal{J}_{W_K}^W(\epsilon), K=A_2;$

$c=6;\ 1_6=\mathcal{J}_{W_K}^W(1_6), K=G_2.$

3. Comments

3.1. The tables in §2 can be obtained by combining the induction/restriction tables in [1] with the explicit knowledge of the a-function in [5]. This is how I first tried to obtain them. (While doing that I found that, in [5], the entry $[189_c]$ on p.364, line -5, should be replaced by $[189'_c]$.) But eventually

I used instead the induction/restriction tables in the CHEVIE package [2] combined with a package to compute the a-function kindly supplied to me by Meinolf Geck. I am very grateful to Gongqin Li for doing the necessary programming.

3.2. Now Propositions 1.4, 1.9 (and hence 1.6) follow (in the case of exceptional groups) from the tables in §2 and the explicit determination of the Springer representations (see [12] and the references there). The case of classical groups requires additional arguments; it will be considered elsewhere. One ingredient in the proof is the interpretation [7] of the function $\gamma : \mathrm{Irr}\, W \to \mathbf{N}$ in 1.1 in terms of the a-function on W_{af}; this can be used to prove the inequality $a_{E'} \le \gamma(E)$ for any $E \in \mathrm{Irr}\, W$ and any $(K, E') \in \Gamma(E)$. In addition to this, one has to use combinatorial arguments similar to those used in [9].

3.3. Recall that in [6] an algorithm is given which determines the (ordinary) Green functions in terms of the function $\gamma : \mathrm{Irr}\, W \to \mathbf{N}$. Since by Proposition 1.4, this function is equal to the explicitly computable function $c : \mathrm{Irr}\, W \to \mathbf{N}$, we see that the Green functions can now be calculated by a computer without the input of the Springer correspondence.

3.4. We now assume that the characteristic p of the ground field $\mathbf{k}$ is a bad prime for G. Let T be a maximal torus of G and let $V = \mathbf{Q} \otimes \mathrm{Hom}(T, \mathbf{k}^*)$, $V^* = \mathbf{Q} \otimes \mathrm{Hom}(\mathbf{k}^*, T)$. Let $R \subset V$ (resp. $\check{R} \subset V^*$) be the set of roots (resp. coroots) of G. For any $(e, e') \in V \times V^*$ let $W_{e,e'}$ be the reflection subgroup of the Weyl group W (viewed as a subgroup of $GL(V)$ or of $GL(V^*)$) defined in [10, 1.1] in terms of $R, \check{R}$. Let $\mathcal{X}$ be the set of reflection subgroups of W of the form $W_{e,e'}$ for some $(e, e') \in V \times V^*$ which is isolated in the sense of [10, 1.1]. We define a subset $\mathcal{X}^p$ of $\mathcal{X}$ as follows. If G is not of type E_8 we have $\mathcal{X}^p = \mathcal{X}$. If G is of type E_8 and $p = 2$, $\mathcal{X}^p$ consists of the subgroups in $\mathcal{X}$ which are not W-conjugate to one of type $A_2A_2A_2A_2$. If G is of type E_8 and $p = 3$, $\mathcal{X}^p$ consists of the subgroups in $\mathcal{X}$ which are not W-conjugate to one of type D_6A_2, D_4D_4 or $A_3A_3A_1A_1$. It is known [10] that the irreducible representations of W attached by the Springer correspondence to unipotent classes in G with the local system $\bar{\mathbf{Q}}_l$ are exactly those obtained by truncated induction from special representations of the various subgroups in $\mathcal{X}^p$. For $E \in \mathrm{Irr}\, W$ let $\Gamma^p(E)$ be the set of all pairs (U, E') where $U \in \mathcal{X}^p$ and

$E' \in \operatorname{Irr} U$ is such that the multiplicity $(E : \operatorname{Ind}_U^W(E'))$ is nonzero. We have $\Gamma^p(E) \neq \emptyset$. Let $c_E^p = \max\{a_{E'}; (U, E') \in \Gamma^p(E)\}$; here $a_{E'}$ is computed in terms of the Weyl group U. Let $\Gamma_M^p(E) = \{(U, E') \in \Gamma(E); a_{E'} = c_E^p\}$. We have $\Gamma_M^p(E) \neq \emptyset$. For $E \in \operatorname{Irr} W$ such that the unipotent class attached to E under the Springer correspondence is $\mathfrak{c}$, we define $\gamma^p(E)$ as the dimension of the Springer fibre at an element of $\mathfrak{c}$.

Experiments suggest that the following extension of 1.4(a) holds.

(a) For any $E \in \operatorname{Irr} W$ we have $c_E^p = \gamma^p(E)$.

References

1. D. Alvis, Induce/restrict matrices for exceptional Weyl groups, math.RT.0506377.
2. M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, *Appl. Algebra Engrg. Comm. Comput.*, **7** (1996), 1175-210.
3. G. Lusztig, A class of irreducible representations of a Weyl group, *Proc. Kon. Nederl. Akad.* (A), **82** (1979), 323-335.
4. G. Lusztig, A class of irreducible representations of a Weyl group II, *Proc. Kon. Nederl. Akad.* (A), **85** (1982), 219-226.
5. G. Lusztig, Characters of Reductive Groups over a Finite Field, Ann. Math. Studies 107, Princeton U. Press, 1984.
6. G. Lusztig, Characters sheaves, V, *Adv.in Math.*, **64** (1986), 103-155.
7. G. Lusztig, Cells in affine Weyl groups IV, *J. Fac. Sci. Tokyo U.* (IA), **36** (1989), 297-328.
8. G. Lusztig, Unipotent elements in small characteristic, *Transfor. Groups*, **10** (2005), 449-487.
9. G. Lusztig, Unipotent classes and special Weyl group representations, *J. Alg.*, **321** (2009), 3418-3449.
10. G. Lusztig, On conjugacy classes in a reductive group, *Representations of Reductive Groups*, Progr.in Math. 312, Birkhäuser, (2015), 333-363.
11. G. Lusztig and Z. Yun, *Z/m*-graded Lie algebras and perverse sheaves, IV, *Repres. Th.*, **24** (2020), 360-396.
12. N. Spaltenstein, On the generalized Springer correspondence for exceptional groups, Algebraic groups and related topics, Adv. Stud. Pure Math.6, North Holland and Kinokuniya, 1985, 317-338.
13. T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, *Invent. Math.*, **36** (1976), 173-207.