Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **17** (2022), No. 4, pp. 439-475
DOI: 10.21915/BIMAS.2022405

SOLVABILITY AND BLOW-UP OF THE WEAK SOLUTION FOR A SEMI-LINEAR BESSEL PROBLEM WITH NEUMANN INTEGRAL CONDITION

DEHILIS SOFIANE[1,a], TAKI EDDINE OUSSAEIF[1,b]
AND CHEBANA ZAINOUBA[1,c]

[1]Department of Mathematics and Informatics, Oum El Bouaghi University, Dynamic and Control Systems Laboratory, Oum El Bouaghi, Algeria.

[a]E-mail: dehilissofiane@yahoo.fr

[b]E-mail: takieddine.oussaeif@gmail.com

[c]E-mail: chebanazainouba@yahoo.com

Abstract

Under the consideration that the non-local condition has a spatial impact to the study of the boundary values problems, we present a study of the existence and uniqueness of weak solution for nonlinear parabolic Bessel problem with Neumann integral conditions, in addition to part devoted to the proof of the finite time blow up solutions. Actually, in the case $p \geq 1$, sufficient conditions of blow up of solutions can be established by Kaplan's method backed by the numerical results.

1. Introduction

Since the last century, many mathematicians have been to a meeting point for practitioners interested in real-world problems and striving to be part of the international scientific community, breaking down the traditional models of isolation and partial differential equations formed a fertile ground to describe evolution, describing a variety of realities. They appear in all branches of science, such as the vibration of solids, the flow of liquids, the diffusion of chemicals, the propagation of heat, the structure of molecules,

Received July 29, 2022.

AMS Subject Classification: 35K58, 35K61, 35B45, 35A01, 35A02.

Key words and phrases: Bessel operator, energy inequality, parabolic equation, integral condition, blow up solutions, numerical simulation.

the interaction of photons and electrons, and the emission of electromagnetic waves. Principal scientists of partial differential equations include Euler, d'Alembert, Lagrange and Laplace,.... One part of them worked on the parabolic equations which are an important class of EDPs with different boundary conditions, covering the different types of classical (Dirichlet, Neumann, ...), non-classical (non-local condition), linear, non-linear .... Thus, integral type boundary conditions can be used as a central tool to describe the domain in cases where the quantity cannot be measured directly on the boundary whose overall or mean value is known. We have an interest in phenomena modeled by parabolic equations. One amongst the foremost remarkable properties that distinguish nonlinear evolution problems from the linear ones is that the possibility of the eventual occurrence of singularities ranging from perfectly smooth data, more specifically, from classes of information that a theory of existence, uniqueness, and continuous dependence may be established for tiny time intervals, so-called well-posedness within the small. On the other hand, we are interested in the solutions that become infinite in finite time due to the cumulative effect of the nonlinearities we call an explosion phenomenon which attracted firstly special attention to the early developments by the Russian school. But the researchers drew attention to the subject after the fundamental work of Fujita who proved that the Cauchy problem with $f = u^P$ has no global positive nontrivial solutions if $1 < p < 1 + 2/N$. Motivated by this in this work we are interested in studying the solvability and finite time blow-up for nonlinear parabolic problems with Bessel operator with integral boundary conditions.

2. Formulation of the Nonlinear Problem

Let $Q = \left\{(x,t) \in \mathbb{R}^2,\ x \in \Omega =]0,1[\text{ and } 0 < t < T\right\}$. This work is devoted to the study of a solution $u(x,t)$ satisfying the following parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = f\left(x,t,u\right), & \forall\ (x,t) \in Q \\ u\left(x,0\right) = \varphi\left(x\right) & \forall\, x \in (0,1) \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^1 xu(x,t)dx & \forall\, t \in (0,T) \\ \dfrac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 xu(x,t)dx & \forall\, t \in (0,T)\,. \end{cases} \tag{P}$$

For bounded domain Ω of $\mathbb{R}$ with smooth boundary $\partial\Omega$. Also, f, φ and the weight function k_2 are known functions such that k_2 is continuous on $[0,1]$, and we have the following conditions:

The function f is Lipschitzian, which means that there exists a positive constant k such that :

$$\|f(x,t,u_1) - f(x,t,u_2)\|_{L^2(Q)} \le k(\|u_1 - u_2\|_{L^2(Q)}),$$
$$\forall\, u_1, u_2 \in L^2(Q). \tag{1}$$

We will denote u_t and u_x to the partial derivative with respect to t, x respectively, and

$u = u(x,t); \forall\,(x,t) \in (0,1) \times (0,T).$

3. Study of the Linear Problem

3.1. Position of the problem

In the domain $Q = \{(x,t) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } 0 < t < T\}$, consider the following linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = f(x,t) & \forall\,(x,t) \in Q \\ u(x,0) = \varphi(x) & \forall\, x \in (0,1) \\ u_x(0,t) = \displaystyle\int_0^1 xu(x,t)dx, & \forall\, t \in (0,T) \\ u_x(1,t) = \displaystyle\int_0^1 xu(x,t)dx, & \forall\, t \in (0,T), \end{cases} \tag{P_1}$$

where the functions f, φ and k_2 are known functions, whose parabolic equation is given as follows:

$$Lu = \frac{\partial u}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = f(x,t)$$

With the initial condition:

$$u(x,0) = \varphi(x), \qquad \forall\, x \in (0,1),$$

And the integral conditions of the second type

$$u_x(0,t) = \int_0^1 xu(x,t)dx, \quad t \in (0,T), \tag{2}$$

$$u_x(1,t) = \int_0^1 k_2(x)u(x,t)dx, \quad t \in (0,T), \tag{3}$$

we divide the main linear problem to two other linear problems which are:

$$\begin{cases} v_t - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial v}{\partial x}\right) = 0, & \forall\ (x,t) \in Q. \\ v\,(x,0) = \varphi\,(x), & \forall\, x \in (0,1). \\ v_x(0,t) = \displaystyle\int_0^1 xv(x,t)dx, & \forall\, t \in (0,T). \\ v_x(1,t) = \displaystyle\int_0^1 k_2(x)v(x,t)dx, & \forall\, t \in (0,T). \end{cases} \tag{P_2}$$

and

$$\begin{cases} w_t - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial w}{\partial x}\right) = f\,(x,t), & \forall\ (x,t) \in Q. \\ w\,(x,0) = 0, & \forall\, x \in (0,1). \\ w_x(0,t) = w_x(1,t) = 0, & \forall\, t \in (0,T). \end{cases} \tag{P_3}$$

3.2. Solving the problem (P_2)

To solve the homogeneous problem P_2, we use the variable separation method:

We pose

$$v(x,t) = X(x)T(t). \tag{4}$$

Replacing (4) in P_2, we get the problem:

$$\begin{cases} T^{'}X - \dfrac{a}{x}TX^{'} - aTX^{''} = 0. \\ X(x)T(0) = \varphi(x). \\ X'(0)T(t) = \displaystyle\int_0^1 xX(x)T(t)dx. \\ X'(1)T(t) = \displaystyle\int_0^1 xX(x)T(t)dx. \end{cases}$$

For $\lambda = \omega^2 > 0$, we get:

$$\frac{a}{x}\frac{X'}{X} + a\frac{X''}{X} = \frac{T^{'}}{T} = -\omega^2. \tag{5}$$

►Find $X(x)$: The equality (5) gives the following Sturm-Liouville problem :

$$\begin{cases} ax^2X^{''}(x) + xX(x)^{'} + \omega^2 X = 0 \\ X'(0) = \displaystyle\int_0^1 xX(x)dx \\ X'(1) = \displaystyle\int_0^1 xX(x)dx \end{cases}$$

So the solution of this problem is given by

$$X_n(x) = AJ + By.$$

►Find $T(t)$: According to the superposition principle, we put:

$$v(x,t) = \sum_{n\geq 0} X_n(x) \cdot T_n(t),$$

which implies

$$\begin{aligned} v(x,0) &= \sum_{n\geq 0} X_n(x)T_n(0) \\ &= \varphi(x) \\ &= \sum_{n\geq 0} \varphi_n \cdot X_n(x)dx. \end{aligned}$$

Then

$$\varphi_n = \int_0^1 \varphi(x) \cdot X_n(x)dx,$$

so

$$\begin{aligned} T_n\left(0\right) &= \varphi_n, \\ T_n\left(t\right) &= \varphi_n e^{-w_n^2 t}. \end{aligned}$$

3.4. Solvability of the problem (P_3) by the energy inequality method

$$\begin{cases} w_t - \frac{a}{x}(xw_x)_x = f\,(x,t), & \forall\,(x,t) \in Q, \\ w\,(x,0) = 0, & \forall\, x \in (0,1), \\ w_x(0,t) = w_x(1,t) = 0, & \forall\, t \in (0,T), \end{cases} \tag{P_3}$$

where f is a known function and $\forall\, a \gneqq 0$, whose parabolic equation is given as follows :

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) = f\,(x,t)\,. \tag{6}$$

With the initial condition

$$w(x,0) = 0, \qquad \forall\, x \in (0,1)\,,$$

and the integral conditions of the second type

$$\begin{aligned} w_x(0,t) = 0, \quad t \in (0,T)\,, \\ w_x(1,t) = 0, \quad t \in (0,T)\,, \end{aligned} \tag{7}$$

we obtain uniqueness of the solution of the problem (P_3) :

Theorem 1. *For any function $w \in C(0,T,\ L^2_{\sqrt{x}}(0,1))$, we obtain the estimate :*

$$\|w\|_E \leq k\,\|Lw\|_F\,, \tag{8}$$

where k is a positive constant independent of w, such that :

$$k = \sqrt{\frac{e^T}{\min\{1, 2a\}}},$$

where E is the Banach space with finite norm

$$\|w\|^2_E = \|w\|^2_{C(0,T,\ L^2_{\sqrt{x}}(0,1))} + \|w_x\|^2_{L^2(0,T,\ L^2_{\sqrt{x}}(0,1))}$$

and F is a Hilbert space with the finite norm

$$\|w\|^2_F = \|f\,\|^2_{L^2(Q)}\,.$$

Proof. We first multiply the equation (6) by the following multiplier Mw:

$$Mw = xw(x,t).$$

We get

$$\Big[w_t - \frac{a}{x}(xw_x)_x\Big] \cdot Mw(x,t) = f(x,t) \cdot Mw(x,t).$$

Integrating both sides of this identity over $Q_\tau = (0,1) \times (0,\tau)$, where $\tau \in [0,T]$, gives us:

$$\begin{aligned}
&\int_{Q_\tau} \Big[w_t - \frac{a}{x}(xw_x)_x\Big] \cdot Mw(x,t)dxdt \\
&= \int_{Q_\tau} \Big[w_t - \frac{a}{x}(xw_x)_x\Big] \cdot xw(x,t)dxdt \\
&= \int_{Q_\tau} w_t(x,t) \cdot xw(x,t)dxdt - \int_{Q_\tau} \Big(\frac{a}{x}(xw_x)_x\Big) \cdot xw(x,t)dxdt \\
&= \int_{Q_\tau} f(x,t) \cdot xw(x,t)dxdt,
\end{aligned}$$

where w_x, w_t indicate the partial derivative with respect to x, t respectively, such that $w_x = w_x(x,t)$ and $w_t = w_t(x,t)$.

Let us use an integration by parts for each term. By taking account of the initial condition and the boundary conditions, we find :

$$\frac{1}{2}\int_\Omega xw^2(x,\tau)dx + a\int_{Q_\tau} xw_x^2 dxdt = \int_{Q_\tau} f \cdot xwdxdt.$$

Thus, we apply the Cauchy inequality, and it becomes:

$$\frac{1}{2}\int_\Omega xw^2(x,\tau)dx + a\int_{Q_\tau} xw_x^2 dxdt \le \frac{1}{2}\int_Q f^2 dxdt + \frac{1}{2}\int_Q (xw)^2\, dxdt.$$

Applying Gronwall's lemma, we find :

$$\int_\Omega xw^2(x,\tau)dx + 2a\int_{Q_\tau} xw^2 dxdt \le \Big(\int_Q f(x,t)^2 dxdt\Big) e^{\int_0^T dt}$$

which implies that

$$\max_{0<t<T} \int_{\Omega} x w^2(x,\tau)dxdt + \int_{Q_\tau} x w_x^2(x,t)dxdt,$$
$$\leq \frac{e^T}{\min\{1,2a\}} \Big(\int_Q f(x,t)^2 dxdt \Big),$$

Therefore, we obtain

$$\|w\|^2_{C(0,T,\ L^2_{\sqrt{x}}(0,1))} + \|w_x\|^2_{L^2(0,T,\ L^2_{\sqrt{x}}(0,1))} \leq C \|f\ \|^2_{L^2(Q)},$$

where

$$C = \frac{e^T}{\min\{1,2a\}}$$

finally, it follows that

$$\|w\|_E \leq k \|\mathcal{F}\|_F, \text{ where } \ k = \sqrt{C}.$$

This completes the proof. □

Corollary 1. *If for any function $w \in D(L)$, we have the following estimate:*

$$\|w\|_E \leq k \|\mathcal{F}\|_F,$$

then if the solution of the problem (P_3) exists, it is unique.

3.3.1. Existence of the solution of the problem (P_3) :

In this part, we shall establish the existence of solutions for the second linear problem. Specifically, we shall prove the following items:

(1) The operator

$$L = (\mathcal{L}, \ell) : E \longrightarrow F$$

is closable.

(2) $R(L)$ is dense in F for any $w \in E$ and for any arbitrary $\mathcal{F} = (f, \varphi) \in F$.

Proposition 1. *The operator L of E in F is closable.*

Proof. Let $\{w_n\} \in D(L)$ be a sequence such that :

$$w_n \longrightarrow 0 \ \text{ in } E,$$

and

$$Lw_n \longrightarrow (f;0) \ \text{ in } F. \tag{9}$$

We must prove that

$$f \equiv 0 \ .$$

The convergence of w_n towards 0 in E implies:

$$w_n \longrightarrow 0 \ \text{ in } D'(Q) \,. \tag{10}$$

From the continuity of the derivation of $D'(Q)$ in $D'(Q)$, the relation (10) implies:

$$\mathcal{L}w_n \longrightarrow 0 \ \text{ in } D'(Q) \,, \tag{11}$$

Moreover, the convergence of $\mathcal{L}_n$ towards f in $L^2(Q)$ generates:

$$\mathcal{L}w_n \longrightarrow f \ \text{ in } D'(Q) \,. \tag{12}$$

By vertue of the uniqueness of the limit in $D'(Q)$, we conclude from (11) and (12) that

$$f = 0,$$

which is the result. □

Let $\overline{L}$ be the closure of L, and $D(\overline{L})$ the domain of definition of $\overline{L}$.

Theorem 2. *If for $\vartheta \in L^2(Q)$ and for any $w \in C(0, T, L^2_{\sqrt{x}}(0,1))$, we have*

$$\int_Q \mathcal{L}w \cdot \vartheta dxdt = 0, \tag{13}$$

then ϑ vanishes almost everywhere in Q.

Proof. The scalar product of F is defined by:

$$(Lw, W)_F = \int_Q \mathcal{L}w \cdot \omega dxdt,$$

where $W = (\omega, \omega_0)$. The equality (13) can be written as follows:

$$\int_Q \frac{\partial w}{\partial t} \cdot \omega dxdt - \int_Q \frac{a}{x} \frac{\partial}{\partial x} \left(x \frac{\partial w}{\partial x} \right) \cdot \omega dxdt = 0, \tag{14}$$

which implies

$$\int_Q \frac{\partial w}{\partial t} \cdot \omega dxdt = \int_Q \frac{a}{x} \frac{\partial}{\partial x} \left(x \frac{\partial w}{\partial x} \right) \cdot \omega dxdt, \tag{15}$$

where w, $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial x} \in L^2(Q)$, with w satisfying the boundary conditions (7). We put

$$w(x,t) = \int_0^t z(x,\tau) d\tau = \Im_t z \tag{16}$$

By replacing (16) in (15) we get

$$\int_Q z \cdot \omega dxdt = a \int_Q \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \Im_t z}{\partial x} \right) \omega dxdt. \tag{17}$$

During the establishment of the function ω,and from this last equality, we give the function ω in terms of the function z as follows:

$$\omega = x \Im_t z$$

Since z satisfies the same conditions as the function w in (7), z, $\frac{\partial z}{\partial x} \in L^2(Q)$, so $\omega \in L^2(Q)$.

Now replacing ω in (17), we obtain :

$$\int_Q xz \Im_t z dxdt = a \int_Q \Im_t z \cdot \frac{\partial}{\partial x} \left(x \frac{\partial \Im_t z}{\partial x} \right) dxdt.$$

According to integration by parts and using the boundary conditions of Neumann, we get :

$$\int_0^1 \frac{x}{2} (\Im_t z)^2 \Big|_{\tau=0}^{\tau=T} dx = -a \int_Q x \left(\frac{\partial \Im_t z}{\partial x} \right)^2 dxdt \leqslant 0;$$

which gives

$$\int_Q a(x,t) \left(\Im_t z \right)^2 dxdt = 0.$$

So

$$(\Im_t z) = 0.$$

Therefore, it becomes $w = 0$ in Q, which gives $\omega = 0$ in Q. Finally, we have

$$\overline{R(L)} = F.$$

This was demonstrated. □

4. The Uniqueness of the Linear Problem

In this section we will study the uniqueness of the linear problem (P_1).

Theorem 3. *For any function $u \in D(L)$, we have the estimate :*

$$\|u\|_E \leq R\,\|Lu\|_F$$

where k is a positive constant independent of u, such that :

$$R = \frac{\max\left\{1, \frac{1}{\varepsilon}\right\}}{\min\{1, 2a(1-\delta)\}} e^{2a\delta T + \frac{a\beta^2}{\delta}T + \varepsilon T}.$$

Proof. Assuming that a solution of the problem exists, multiplying the equation of the problem (P_1) by the following multiplicator Mu:

$$Mu = xu(x,t)\ ,$$

and by integrating on the domain $Q_\tau = (0,1) \times (0,\tau)$, where $\tau \in [0,T]$, we obtain :

$$\begin{aligned}
&\int_{Q_\tau} \mathcal{L}u \cdot Mu(x,t)dxdt \\
&= \int_{Q_\tau} \left[\partial_t u(x,t) - \frac{a}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})\right] \cdot xu(x,t)dxdt \\
&= \int_{Q_\tau} \partial_t u(x,t) \cdot xu(x,t)dxdt - a\int_{Q_\tau} \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}) \cdot xu(x,t)dxdt \\
&= \int_{Q_\tau} f(x,t) \cdot xu(x,t)dxdt.
\end{aligned}$$

After an integration by parts and using the boundary conditions, we find

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx-\frac{1}{2}\int_0^1 x\varphi^2(x)dx-a\int_0^\tau u(1,t)u_x(1,t)dt$$
$$+a\int_{Q_\tau} xu_x^2(x,t)dxdt=\int_{Q_\tau}\sqrt{x}f(x,t)\cdot\sqrt{x}u(x,t)dxdt.$$

Using the Cauchy with $\varepsilon-$inequality, we obtain

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx+a\int_{Q_\tau} xu_x^2(x,t)dxdt-a\int_0^\tau u(1,t)u_x(1,t)dt,$$
$$\leq\frac{1}{2\varepsilon}\int_{Q_\tau} xf^2(x,t)dxdt+\frac{\varepsilon}{2}\int_{Q_\tau} xu^2dxdt+\frac{1}{2}\int_0^1 x\varphi^2(x)dx. \tag{18}$$

To find our estimate, we must give an estimate for the third part of the left hand side in the inequality (18). By using integral conditions (3), (3) and the Cauchy with δ inequality, we obtain:

$$\int_0^\tau u(1,t)u_x(1,t)dt\leq\frac{\delta}{2}\int_0^\tau u^2(1,t)dt+\frac{1}{2\delta}\int_0^\tau u_x^2(1,t)dt,$$
$$=\frac{\delta}{2}\int_0^\tau u^2(1,t)dt+\frac{1}{2\delta}\int_0^\tau(\int_0^1 xu(x,t)dx)^2dt,$$
$$\leq\frac{\delta}{2}\int_0^\tau u^2(1,t)dt+\frac{1}{2\delta}\int_0^\tau(\int_0^1 xu(x,t)dx)^2dt,$$
$$\leq\frac{\delta}{2}\int_0^\tau u^2(1,t)dt+\frac{1}{2\delta}\int_0^\tau\int_0^1(\sqrt{x}u(x,t))^2dxdt. \tag{19}$$

Now, we must find the estimate of the first part of the right hand side of (19). Let's put:

$$u(1,t)=\int_x^1\frac{\partial}{\partial\xi}(\sqrt{x}u(\xi,t))d\xi+\sqrt{x}u(x,t).$$

Then, using the inequality $|a+b|^2\leq 2a^2+2b^2$, we find:

$$u^2(1,t)=\Big(\int_x^1\frac{\partial}{\partial\xi}(\sqrt{x}u(\xi,t))d\xi+\sqrt{x}u(x,t)\Big)^2,$$
$$\leq 2\Big(\int_x^1\frac{\partial}{\partial\xi}(\sqrt{x}u(\xi,t))d\xi\Big)^2+2xu^2(x,t).$$

By applying Hölder's inequality, we get

$$u^2(1,t) \leq 2\int_x^1 1^2 d\xi \cdot \int_x^1 (\frac{\partial}{\partial \xi}(\sqrt{x}u(\xi,t)))^2 d\xi + 2xu^2(x,t).$$

Integrating over $(0,\tau)$, we find

$$\int_0^\tau u^2(1,t) \leq 2\int_0^\tau \int_0^1 (\frac{\partial}{\partial x}(\sqrt{x}u(x,t)))^2 dxdt + 2\int_0^\tau xu^2(x,t)dt.$$

So

$$\begin{aligned}\frac{\delta}{2}\int_0^\tau u^2(1,t) &\leq \delta\int_0^\tau \int_0^1 (x(\frac{\partial}{\partial x}u(x,t))^2 dxdt + \delta\int_0^\tau xu^2(x,t)dt,\\ &\leq \delta\int_{Q^\tau} x(\frac{\partial u}{\partial x}(x,t))^2 dxdt + \delta\int_0^\tau xu^2(x,t)dt. \qquad (20)\end{aligned}$$

Under the previous inequalities (19) and (20), the inequality (18) becomes:

$$\begin{aligned}&\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,\\ &\quad \leq a\delta\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt + a\delta\int_0^\tau xu^2 dt + \frac{a}{2\delta}\int_{Q^\tau} xu^2 dxdt,\\ &\qquad + \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{\varepsilon}{2}\int_{Q_\tau} xu^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx.\end{aligned}$$

Integrating one more time over $(0,1)$, we get:

$$\begin{aligned}&\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,\\ &\quad \leq a\delta\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt + a\delta\int_0^1\int_0^\tau xu^2 dtdx + \frac{a\beta^2}{2\delta}\int_{Q^\tau} xu^2 dxdt,\\ &\qquad + \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{\varepsilon}{2}\int_{Q_\tau} xu^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx.\end{aligned}$$

Then, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a(1-\delta)\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,\\ &\quad \leq (a\delta + \frac{a}{2\delta} + \frac{\varepsilon}{2})\int_{Q_\tau} xu^2 dxdt + \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx.\end{aligned}$$

By applying Gronwall's lemma, we get

$$\begin{aligned}
&\int_0^1 xu^2(x,\tau)dx + \int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt, \\
&\quad \le \frac{\max\left\{1, \frac{1}{\varepsilon}\right\}}{\min\left\{1, 2a(1-\delta)\right\}} e^{2a\delta T + \frac{a}{\delta}T + \varepsilon T}\Big(\int_{Q_\tau} xf^2 dxdt + \int_0^1 x\varphi^2(x)dx\Big).
\end{aligned}$$

Finally we put

$$\begin{aligned}
&\|u\|^2_{L^\infty(0,T;L^2_{\sqrt{x}}(\Omega))} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))} \\
&\le R(\|f\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))} + \|\varphi\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))}
\end{aligned}$$

where

$$R = \frac{\max\left\{1, \frac{1}{\varepsilon}\right\}}{\min\left\{1, 2a(1-\delta)\right\}} e^{2a\delta T + \frac{a}{\delta}T + \varepsilon T}$$

which complete the proof. □

5. Solvability of the Weak Solution of the Nonlinear Problem

Based on the last section, this section is mainly devoted to nonlinear parabolic problem and the proof the existence and the uniqueness by using the the linearization method :

$$\begin{cases}
u_t - \dfrac{a}{x}(xu_x)_x = f\,(x,t,u), & \forall\ (x,t) \in Q; \\
u\,(x,0) = \varphi\,(x), & \forall\, x \in (0,1); \\
u_x(0,t) = \displaystyle\int_0^1 xu(x,t)dx, & \forall\, t \in (0,T); \\
u_x(1,t) = \displaystyle\int_0^1 xu(x,t)dx, & \forall\, t \in (0,T)\,.
\end{cases} \qquad (P)$$

Putting

$$u = y + \theta,$$

such that w is a solution to the following problem:

$$\begin{cases} \theta_t - \frac{a}{x}(x\theta_x)_x = 0, & \forall\ (x,t) \in Q, \\ \theta\,(x,0) = \varphi\,(x), & \forall\, x \in (0,1), \\ \theta_x(0,t) = \displaystyle\int_0^1 x\theta(x,t)dx, & \forall\, t \in (0,T), \\ \theta_x(1,t) = \displaystyle\int_0^1 x\theta(x,t)dx, & \forall\, t \in (0,T), \end{cases} \tag{P_4}$$

and the solution

$$y = u - w$$

satisfies the following problem

$$\mathcal{L}y = y_t - \frac{a}{x}(xy_x)_x = G\,(x,t,y)\,, \tag{21}$$

$$y(x,0) = 0, \qquad \forall\, x \in (0,1)\,, \tag{22}$$

$$y_x(0,t) = y_x\,(1,t) = 0, \qquad \forall\, t \in (0,t)\,, \tag{23}$$

where

$$G\,(x,t,y) = f\,(x,t,y+\theta)\,.$$

As the function f, the function G is also Lipschitzian, so there is a positive constant k such that:

$$\parallel G\,(x,t,u_1) - G\,(x,t,u_2)\parallel_{L^2(Q)} \leq k\left(\|u_1 - u_2\|_{L^2(0,T,H^1(0,1))}\right). \tag{24}$$

From the result of the previous section, we deduce that the problem (P_4) has a unique solution which depends continuously on the data. So it remains to prove that the problem (21)-(23) admits a unique weak solution. First, we propose the concept of studied solution.

Let $v = v(x;t)$ be any function of $L^2(0;T;H^1(0,1))$. Then, multiplying (21) by xv and integrating both sides over $Q = (0,1) \times (0,T)$ give us:

$$\int_Q \frac{\partial y}{\partial t}\,(x,t) \cdot xv(x,t)dxdt - a\int_Q \frac{\partial}{\partial x}(x\frac{\partial y}{\partial x}) \cdot v\,(x,t)\,dxdt$$
$$= \int_Q G(x,t) \cdot xv\,(x,t)\,dxdt.$$

Then by using integration by parts and the conditions (22) and (23) we find:

$$\int_Q \frac{\partial y}{\partial t}(x,t)\cdot v(x,t)dxdt + a\int_Q x\frac{\partial y}{\partial x}\frac{\partial v}{\partial x} = \int_Q G(x,t,u)\cdot xv(x,t)dxdt. \quad (25)$$

It follows from (25) that:

$$A(y,v) = \int_Q G(x,t,u)\cdot xv(x,t)dxdt \quad (26)$$

where

$$A(y,v) = \int_Q \frac{\partial y}{\partial t}(x,t)\cdot v(x,t)dxdt + a\int_Q x\frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dxdt.$$

Definition 4. A function $y \in L^2\left(0,T;\ H^1(0,1)\right)$ is said to be a weak solution of the problem (21) – (23) if (26) and (23) are fulfilled.

We build a recurring sequence starting with $y^{(0)} = 0$. The sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows : given the element $y^{(n-1)}$, then for $n = 1,2,3,\ldots$, we will solve the following problem:

$$\begin{cases} \frac{\partial y^{(n)}}{\partial t} - \frac{a}{x}(xy_x^{(n)})_x = G\left(x,t,y^{(n-1)}\right), \\ y^{(n)}(x,0) = 0, \\ y_x^{(n)}(0,t) = y_x^{(n)}(1,t) = 0. \end{cases} \quad (P_5)$$

According to the study of the previous linear problem each time we fix the n, the problem (P_5) admits a unique solution $y^{(n)}(x,t)$. Now we suppose

$$z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t),$$

so we get a new problem which is:

$$\begin{cases} \frac{\partial z^{(n)}}{\partial t} - \frac{a}{x}(xz_x^{(n)})_x = p^{(n-1)}(x,t), \\ z^{(n)}(x,0) = 0, \\ z_x^{(n)}(0,t) = z_x^{(n)}(1,t) = 0, \end{cases} \quad (P_6)$$

where

$$p^{(n-1)}(x,t) = G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right).$$

Lemma 1. *Assume that the condition* (24) *is satisfied. So we have the following estimate*

$$\| z^{(n)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq C \| z^{(n-1)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}$$

where

$$C = \frac{k^2 e^T}{\min\{1, 2a\}}$$

Proof. Multiplying

$$\frac{\partial z^{(n)}}{\partial t} - \frac{a}{x}(xz^{(n)}_x)_x = p^{(n-1)}(x,t)$$

by $xz^{(n)}$ and integrating it on Q_τ, we get:

$$\int_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}(x,t) \cdot xz^{(n)}(x,t)dxdt - a\int_{Q_\tau} (xz^{(n)}_x)_x \cdot z^{(n)}(x,t)\, dxdt$$
$$= \int_{Q_\tau} p^{(n-1)}(x,t) \cdot xz^{(n)}(x,t)\, dxdt.$$

If we use integration by parts for each term, taking into consideration the initial and boundary conditions, we get:

$$\frac{1}{2}\int_0^1 x(z^{(n)}(x,\tau))^2 dx + a\int_{Q_\tau} x\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$= \int_{Q_\tau} p^{(n-1)}(x,t) \cdot xz^{(n)}(x,t)\, dxdt.$$

When the Cauchy inequality is applied to the second part of the equation, the following result is obtained:

$$\int_{Q_\tau} p^{(n-1)}(x,t) \cdot xz^{(n)}(x,t)\, dxdt$$
$$= \int_{Q_\tau} \sqrt{x}p^{(n-1)}(x,t) \cdot \sqrt{x}z^{(n)}(x,t)\, dxdt$$

$$\leqslant \frac{1}{2}\int_{Q^\tau} x\left(p^{(n-1)}(x,t)\right)^2 dxdt + \frac{1}{2}\int_{Q^\tau} x\left(z^{(n)}(x,t)\right)^2 dxdt.$$

On the other hand, we have

$$\left|p^{(n-1)}(x,t)\right|^2 = \left|G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right)\right|^2.$$

Like G function is Lipschitzian, we find :

$$\begin{aligned}\left|p^{(n-1)}(x,t)\right|^2 &\leq k^2\left|y^{(n)} - y^{(n-1)}\right|^2\\ &= k^2\left|z^{(n-1)}\right|^2.\end{aligned}$$

Multiplying by x and integrating over Q, we find:

$$\int_Q x\left|p^{(n-1)}(x,t)\right|^2 dxdt \leq k^2\int_Q x\left|z^{(n-1)}\right|^2 dxdt.$$

Then

$$\left\|p^{(n-1)}\right\|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq k^2\left\|z^{(n-1)}\right\|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}. \tag{27}$$

This result gives us the following inequality

$$\begin{aligned}&\frac{1}{2}\int_0^1 x(z^{(n)}(x,\tau))^2 dx + a\int_{Q_\tau} x\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt,\\ &\leq \frac{1}{2}\int_{Q^\tau} x\left(p^{(n-1)}(x,t)\right)^2 dxdt + \frac{1}{2}\int_{Q^\tau} x\left(z^{(n)}(x,t)\right)^2 dxdt,\\ &\leq \frac{k^2}{2}\parallel z^{(n-1)}\parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} + \frac{1}{2}\parallel z^{(n)}\parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}.\end{aligned}$$

Thus, it is easy to get that

$$\begin{aligned}&\parallel z^{(n)}\parallel^2_{L^2_{\sqrt{x}}(0,1)} + 2a\parallel \partial_x z^{(n)}\parallel^2_{L^2_{\sqrt{x}}(Q^\tau)},\\ &\leq k^2\parallel z^{(n-1)}\parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} + \parallel z^{(n)}\parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}.\end{aligned}$$

Now we shall apply Gronwall's lemma. We get:

$$\parallel z^{(n)}\parallel^2_{L^2_{\sqrt{x}}(0,1)} + 2a\parallel \partial_x z^{(n)}\parallel^2_{L^2_{\sqrt{x}}(Q^\tau)},$$

$$\leqslant k^2 e^{\int_0^T dt} \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} .$$

The result enables us to pass to the maximum in the right part of the last inequality, and we obtain:

$$\begin{aligned} &\parallel z^{(n)} \parallel^2_{L^\infty(0,T;L^2_{\sqrt{x}}(0,1))} + \parallel \partial_x z^{(n)} \parallel^2_{L^2_{\sqrt{x}}(Q^\tau)}, \\ &\leqslant \frac{k^2 e^T}{\min\{1,2a\}} \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} . \end{aligned}$$

Finally, we get :

$$\parallel z^{(n)} \parallel_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq C \parallel z^{(n-1)} \parallel_{L^2(0,T;H^1_{\sqrt{x}}(0,1))},$$

where

$$C = \sqrt{\frac{k^2 e^T}{\min\{1,2a\}}}.$$

According to the convergence criterion of the series, the series $\sum\limits_{n=1}^{\infty} z^{(n)}$ converges if $|C| < 1$, which implies:

$$\begin{aligned} \sqrt{\frac{k^2 e^T}{\min\{1,2a\}}} &< 1, \\ k^2 &< \frac{\min\{1,2a\}}{e^T}, \\ k &< \sqrt{\frac{\min\{1,2a\}}{e^T}}. \end{aligned}$$

As

$$z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t),$$

$$y^{(n)} = \sum_{i=1}^{n-1} z^{(i)}.$$

Then $y^{(n)}$ converges to an element $y \in L^2(0,T;H^1_{\sqrt{x}}(0,1))$. Now, we will prove that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of the problem (P_5) by showing that y satisfies

$$A(y, v) = \int_Q G(x, tt) \cdot xv(x, t) dxdt. \tag{28}$$

Therefore we consider the weak formulation of the problem (P_5)

$$A(y^{(n)}, v) = \int_Q \frac{\partial y^{(n)}}{\partial t}(x, t) \cdot xv(x, t) dxdt + a \int_Q x \frac{\partial y^{(n)}}{\partial x} \frac{\partial v}{\partial x} dxdt.$$

From the linearity of A we have

$$\begin{aligned} A\left(y^{(n)}, v\right) &= A\left(y^{(n)} - y, v\right) + A(y, v), \\ &= \int_Q xvdxdt + a \int_Q x \frac{\partial (y^{(n)} - y)}{\partial x} \frac{\partial v}{\partial x} dxdt + A(y, v). \end{aligned} \tag{29}$$

When the Cauchy-Schwartz inequality is applied to $A(y^{(n)} - y, v)$, we get

$$A\left(y^{(n)}, v\right) \leq \|v\|_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \left\| (y^{(n)} - y)_t \right\|$$

On the other hand,

$$y^{(n)} \longrightarrow y \qquad \text{in } L^2\left(0, T;\ H^1_{\sqrt{x}}(0,1)\right),$$

so

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2\left(0, T;\ L^2_{\sqrt{x}}(0,1)\right), \\ y^{(n)}_t &\longrightarrow y_t && \text{in } L^2\left(0, T;\ L^2_{\sqrt{x}}(0,1)\right), \\ y^{(n)}_x &\longrightarrow y_x && \text{in } L^2\left(0, T;\ L^2_{\sqrt{x}}(0,1)\right). \end{aligned}$$

Let us pass to the limit when $n \longrightarrow +\infty$. We find

$$\lim_{n \longrightarrow +\infty} A\left(y^{(n)} - y, v\right) = 0. \tag{30}$$

According to (30) and by passing to the limit in (29) we obtain

$$\lim_{n \longrightarrow +\infty} A\left(y^{(n)}, v\right) = A(y, v).$$

Thus, we have proved the following result : $\square$

Theorem 5. *If the condition* (24) *is satisfied, and*

$$k < \sqrt{\frac{\min\{1, 2a\}}{e^T}}.$$

Then the problem (21) – (23) *admits a weak solution belonging to* $L^2\left(0, T;\ L^2_{\sqrt{x}}(0,1)\right)$.

Now, we will show that the solution of the problem (21) – (23) is unique.

Theorem 6. *If the condition* (24) *is verified, then the solution is unique.*

Proof. Let y_1, y_2 be two solutions of $L^2\left(0, T\ ; H^1_{\sqrt{x}}(0,1)\right)$ (21) – (23), and then

$$y = y_1 - y_2,$$

is also a solution in $L^2\left(0, T\ ; H^1_{\sqrt{x}}(0,1)\right)$ and we check

$$\frac{\partial y}{\partial t} - a\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = G(x, t, y), \tag{31}$$

$$y(x, 0) = 0, \tag{32}$$

$$\frac{\partial y}{\partial x}(0, t) = \frac{\partial y}{\partial x}(1, t) = 0, \tag{33}$$

$$\frac{\partial y}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = \Psi(x,t), \qquad \forall\ (x,t) \in Q$$

$$y(x, 0) = 0,$$

$$\frac{\partial y}{\partial x}(0, t) = \frac{\partial y}{\partial x}(1, t) = 0$$

and

$$\Psi(x,t) = G(x, t, y_1) - G(x, t, y_2).$$

Using the Lemma 1, we can conclude that

$$\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)} \le C\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)},$$

from which

$$(1 - C)\,\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)} \le 0,$$

and as $C \leq 1$, then we get

$$\|y\|_{L^2(0,T,H^1(0,1))} = 0,$$

from which

$$y_1 = y_2,$$

This contributes to the solutions uniqueness. □

6. Finite Time Blow-Up for Nonlinear Problem by using Kaplan's First Eigenvalue Method

In this section we are interested in the study of the blow-up phenomena for the nonlinear problem where the diffusion term is given in the following manner $f(x,t,u) = u^p$.

6.1. Statement of the problem

Let $T > 0$, $\Omega = (0,1)$ and $Q = \Omega \times (0,T) = \left\{ (x,t) \in \mathbb{R}^2,\ x \in \Omega \text{ and } 0 < t < T \right\}$.

Consider the following nonlinear problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = u^p, & \forall\ (x,t) \in Q; \\ u\left(x,0\right) = \varphi\left(x\right), & \forall\, x \in \Omega; \\ \frac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^1 k_1(x)u(x,t)dx, & \forall\, t \in \Omega; \\ \frac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 k_2(x)u(x,t)dx, & \forall\, t \in (0,T)\,. \end{cases} \tag{P_7}$$

Assuming that $f \in L^2\left(Q\right)$, k_1, k_2 are known functions such that $k_2(x) > \omega > 0$ and $p > 1$, whose parabolic equation is given as follows

$$\frac{\partial u}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = u^p,\ x \in \Omega \text{ and } 0 < t \leq T, \tag{34}$$

with the initial condition

$$u(x,0) = \varphi(x),\ 0 < x < 1, \tag{35}$$

and the nonlocal boundary conditions

$$u(0,t) = \int_0^1 k_1(x)u(x,t)dx, \qquad 0 < t \leq T, \tag{36}$$

$$u(1,t) = \int_0^1 k_2(x)u(x,t)dx, \qquad 0 < t \leq T. \tag{37}$$

6.2. Determining the finite-time blow up solution by using Kaplan's first eigenvalue method

Let $\psi(x)$ be the normalized eigenfunction corresponding to the eigenvalue λ of the following Sturm-Liouville problem:

$$\begin{cases} -\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) = \lambda\psi, \\ \psi(0) = 0, \\ \psi_x(1) = 0. \end{cases} \tag{P$_8$}$$

Let's find the solution $\psi(x)$:

By multiplying the equation in the problem (P$_8$) by x,we get

$$-\frac{\partial \psi}{\partial x} - x\frac{\partial^2 \psi}{\partial x^2} - \lambda x\psi = 0,$$

which is the Bissel's function, with the following general solution

$$\psi(x) = c_1 J_0(\sqrt{\lambda}x) + c_2 Y_0(\sqrt{\lambda}x),$$

where c_1,c_2 are constants and we have

$$J_0(\sqrt{\lambda}x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(k!)^2}(\frac{\sqrt{\lambda}x}{2})^{2k},$$

and

$$Y_0(\sqrt{\lambda}x) = \lim_{\alpha\to 0} \frac{\cos(\alpha\pi)J_0(\sqrt{\lambda}x) - J_0(\sqrt{\lambda}x)}{\sin(\alpha\pi)}.$$

Since Y_0 is not bounded as $x \to 0^+$, then we must have $c_2 = 0$ where the solution becomes

$$\psi(x) = c_1 J_0(\sqrt{\lambda}x).$$

Let λ_1 be the first eigenvalue of the problem (P_8) where $J_0'(\sqrt{\lambda_1}) = 0$.

Theorem 7. *For $p > 1$ and $\forall\ (x,t) \in Q$, the solution of the problem (P_7) blows up in a finite time T^*such that:*

$$T^* = \frac{1}{K}\ln\left(\frac{\frac{r(1-p)}{K}}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)}\right), \textit{where } \Pi(0) = \left(B + \frac{r(1-p)}{K}\right)^{\frac{1}{1-p}}.$$

Proof. We based this proof on a sufficiently large initial data, for the study of one of the most profiles importantant of explosion phenomenon for the solutions of problem (P_7).

To estimate the finite time blow up of the main problem we use the Kaplans method by multiplying the equation in (34) by $(x\psi)$, and integrating by parts over the domain $\Omega = (0,1)$, we get

$$\int_0^1 x\psi \cdot \frac{\partial u}{\partial t}dx - a\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\cdot \psi dx = \int_0^1 x\psi \cdot u^p dx.$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u(x,t)dx - a\left[x\psi(x)u_x|_0^1 - \int_0^1 x\psi_x \cdot u_x\right] = \int_0^1 x\psi \cdot u^p dx.$$

After using the boundary conditions and integration by parts one more time the last inequality becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u(x,t)dx - a\left[\psi(1)u_x(1,t) + \int_0^1 u\frac{\partial}{\partial x}(x\psi_x)dx\right] = \int_0^1 x\psi \cdot u^p dx.$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u(x,t)dx - ac_1 J_0(\sqrt{\lambda})\int_0^1 k_2(x)udx - a\int_0^1 u\frac{\partial}{\partial x}(x\psi_x)dx$$

$$= \int_0^1 x\psi \cdot u^p dx. \tag{38}$$

On the other hand, we have

$$\int_0^1 u \frac{\partial}{\partial x}(x\psi_x)dx = -\lambda \int_0^1 xu\psi dx,$$

So, the equality (38) becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u(x,t)dx - ac_1 J_0(\sqrt{\lambda}) \int_0^1 k_2(x)udx + a\lambda \int_0^1 x\psi \cdot u(x,t)dx$$
$$= \int_0^1 x\psi \cdot u^p dx.$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda \int_0^1 x\psi \cdot udx = ac_1 J_0(\sqrt{\lambda}) \int_0^1 k_2(x)udx + \int_0^1 x\psi \cdot u^p dx.$$

Now, by applying Jensen inequality, we find

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda \int_0^1 x\psi \cdot udx$$
$$\geq a\omega c_1 J_0(\sqrt{\lambda}) \int_0^1 udx + \Big(\int_0^1 x\psi dx\Big)^{1-p}\Big(\int_0^1 x\psi u\Big)^p. \tag{39}$$

On the other hand, we have

$$\int_0^1 u(x,t)dx \geq \frac{1}{c_1}\int_0^1 x\psi \cdot u(x,t)dx, \text{ where } c_1 = \|x\psi\|_{L^\infty(0,1)}.$$

Then, the inequality (39) becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda \int_0^1 x\psi \cdot udx$$
$$\geq a\omega J_0(\sqrt{\lambda}) \int_0^1 x\psi \cdot udx + \Big(\int_0^1 x\psi dx\Big)^{1-p}\Big(\int_0^1 x\psi u\Big)^p.$$

Finally, we obtain

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\Big(\lambda - \omega J_0(\sqrt{\lambda})\Big)\int_0^1 x\psi \cdot udx$$

$$\geq \Big(\int_0^1 x\psi dx\Big)^{1-p}\Big(\int_0^1 x\psi u\Big)^p.$$

By putting

$$\Pi(t) = \int_0^1 x\psi \cdot u dx,$$

we find

$$\Pi'(t) + a\left(\lambda - \omega J_0(\sqrt{\lambda})\right)\Pi(t) \geq (\int_0^1 x\psi dx)^{1-p}\Pi^p(t). \tag{40}$$

By putting

$$r = (\int_0^1 x\psi dx)^{1-p} \text{ and } d = a\left(\lambda - \omega J_0(\sqrt{\lambda})\right),$$

the equality (40) becomes

$$\Pi'(t) + d\Pi(t) \geq r\Pi^p(t).$$

Let's solve the following Bernoulli equation

$$\Pi'(t) + d\Pi(t) - r\Pi^p(t) = 0 \tag{41}$$

By putting

$$v = \Pi^{1-p}, \tag{42}$$

and replacing (42) in (41) we find

$$\frac{1}{1-p}v'v^{\frac{p}{1-p}} + dv^{\frac{1}{1-p}} - rv^{\frac{p}{1-p}} = 0. \tag{43}$$

Multiplying the equation (43) by $(1-p)v^{\frac{-p}{1-p}}$, we get

$$v' + Kv - r(1-p) = 0; \text{where } K = (1-p)\,d. \tag{44}$$

First, we are going to solve the following homogeneous equation:

$$v' + Kv = 0,$$

which has a known solution given by:

$$v_h(t) = Be^{-Kt}$$

Now, we move on to solving the non-homogeneous equation (44) by the method of constant variation, where we put

$$v_g(t) = B(t)e^{-Kt}, \tag{45}$$

so

$$v_g'(t) = B'(t)e^{-Kt} - KB(t)e^{-Kt}. \tag{46}$$

Combining (45) and (46) with (44) where we get

$$B'(t)e^{-Kt} = r(1-p),$$

then

$$v_g(t) = \frac{r(1-p)}{K}.$$

The final solution is given by

$$\begin{aligned} v(t) &= v_h(t) + v_g(t) \\ &= Be^{-Kt} + \frac{r(1-p)}{K}, \end{aligned}$$

so

$$\Pi(t) = \left(Be^{-Kt} + \frac{r(1-p)}{K}\right)^{\frac{1}{1-p}}.$$

For $t = 0$, we get

$$B = (\Pi(0))^{1-p} - \frac{r(1-p)}{K}.$$

Finally, we get

$$\Pi(t) = \left(\frac{1}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)e^{-Kt} + \frac{r(1-p)}{K}}\right)^{\frac{1}{p-1}},$$

where $\frac{1}{p-1} > 0$, and then

$$\Pi \to \infty \text{ if } \left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right) e^{-Kt} + \frac{r(1-p)}{K} \to 0,$$

so we get

$$T^* = \frac{1}{K} \ln \left(\frac{\frac{r(1-p)}{K}}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)} \right)$$

is the finite time blow up of the problem (P_7). □

6.3. Determining the finite-time blow up solution numerically

As an important section of the chapter, we establish a numerical study for the nonlinear problem. We aspire to support the theorical results by relying on the finite difference technique. For positive integers N and M, let $\Delta x = 1/M$ and $\Delta t = T/N$ be the spatial and temporal step sizes, respectively. For $i = 0, 1, \ldots, M$ and $n = 0, 1, \ldots, N$, denote $x_i = i\Delta x$ and $t_n = n\Delta t$. The notations u_i^n are used for the finite difference approximations of $u(x_i, t_n)$.

6.3.1. The explicit scheme:

We can approximate the time derivative by the forward difference quotient, and use the centred first and second-order approximation for the spatial derivative of first and second order in (P_7) to obtain :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{a}{x_i} \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - a\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = (u_i^n)^p .$$

This scheme can be written as :

$$u_i^{n+1} = \Delta t\, (u_i^n)^p + (1 - 2ar)u_i^n + (ar + \frac{\alpha}{2i} r)u_{i+1}^n + (ar - \frac{\alpha}{2i} r)u_{i-1}^n, \quad (47)$$

for $i = 1, 2, \ldots, M - 1$, $n = 0, 1, \ldots, N$, and $r = \Delta t/\Delta x^2$.

This procedure is explicit. We still have to determine two unknowns u_0 and u_M. For this we approximate integrals in (36) and (37) numerically by trapezoidal rule.

For $u(0,t_{n+1})$:

$$u_1^{n+1}-u_0^{n+1}=\frac{(\Delta x)^2}{2}\left[k_1(0)u_0^{n+1}+k_1(1)u_M^{n+1}+2\sum_{i=1}^{M-1}k_1(x_i)u_i^{n+1}+o(\Delta x^2)\right].$$

For $u(1,t_{n+1})$:

$$u_M^{n+1}-u_{M-1}^{n+1}=\frac{(\Delta x)^2}{2}\left[k_2(0)u_0^{n+1}+k_2(1)u_M^{n+1}+2\sum_{i=1}^{M-1}k_2(x_i)u_i^{n+1}+o(\Delta x^2)\right].$$

Thus, we can write

$$-(1+\frac{(\Delta x)^2}{2}k_1(0))u_0^{n+1}-\frac{(\Delta x)^2}{2}k_1(1)u_M^{n+1}=(\Delta x)^2\sum_{i=1}^{M-1}k_1(x_i)u_i^{n+1}-u_1^{n+1}, \tag{48}$$

$$-\frac{(\Delta x)^2}{2}k_2(0)u_0^{n+1}+(1-\frac{(\Delta x)^2}{2}k_2(1))u_M^{n+1}=\Delta x\sum_{i=1}^{M-1}k_2(x_i)u_i^{n+1}+u_{M-1}^{n+1}. \tag{49}$$

We put

$$a_1=-(1+\frac{(\Delta x)^2}{2}k_1(0)), \qquad a_2=-\frac{(\Delta x)^2}{2}k_1(1),$$

$$b_1=-\frac{(\Delta x)^2}{2}k_2(0)k_2(1), \qquad b_2=(1-\frac{(\Delta x)^2}{2}k_2(1))u_M^{n+1},$$

$$c_1=(\Delta x)^2\sum_{i=1}^{M-1}k_1(x_i)u_i^{n+1}-u_1^{n+1}, \qquad c_2=\Delta x\sum_{i=1}^{M-1}k_2(x_i)u_i^{n+1}+u_{M-1}^{n+1}.$$

Under the condition $a_1b_2-a_2b_1\neq 0$ which is true for sufficiently small

Δx, by Cramer rule we have

$$u_0^{n+1}=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1},$$

$$u_M^{n+1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

The consistency and stability of the scheme

We shall analyze the forward-centered scheme (47) as usual, by establishing consistency and stability. Let $u_i^n = u(x_i, t_n)$ denote the restriction of the exact solution. We have the finite difference scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{a}{x_i}\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - a\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = (u_i^n)^p\,. \tag{50}$$

1-Consistency:

By using Taylor's formula, we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = u_t(x_i, t_n) + \frac{\Delta t}{2}u_{tt}(x_i, \xi_1). \tag{51}$$

Then

$$\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = u_x(x_i, t_n) + \frac{\Delta x^2}{6}u^{(3)}(\eta_1, t_n), \tag{52}$$

and thus

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = u_{xx}(x_i, t_n) + \frac{\Delta x^2}{12}u^{(4)}(\eta_2, t_n). \tag{53}$$

Combining (51), (52) and (53) with (50), we find

$$u_t(x_i, t_n) - \frac{a}{x_i}u_x(x_i, t_n) - au_{xx}(x_i, t_n) + o(\Delta t) + o(\Delta x^2) = u^p(x_i, t_n),$$

such that

$$o(\Delta t) = \frac{\Delta t}{2}u_{tt}(x_i, \xi_1) \quad \text{and} \quad o(\Delta x^2) = -\frac{a}{x_i}\frac{\Delta x^2}{6}u^{(3)}(\eta_1, t_n) - a\frac{\Delta x^2}{12}u^{(4)}(\eta_2, t_n).$$

While $o(\Delta t) + o(\Delta x^2) \to 0$, we obtain

$$u_t(x_i, t_n) - \frac{a}{x_i}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}) = u^p(x_i, t_n).$$

So the explicit schema is consistent.

2-Stability

Next we establish a stability result. Suppose that a mesh function u_i^n satisfies (50). We want to bound an appropriate norm of the mesh function u_i^n in terms of an appropriate norm of the function $(u_i^n)^p$ on the right hand side. For the norm, we use the max norm :

$$\|U\|_{L^\infty} = \max_{1\leq i\leq M} |u_i^n| .$$

Write

$$M_n = \max_{1\leq i\leq M} |u_i^n| ,\ F^n = \max_{1\leq i\leq M} |(u_i^n)^p| .$$

From (50) we get by putting $r = \frac{1}{2i}$

$$u_i^{n+1} = \Delta t\, (u_i^n)^p + (1-2ar)u_i^n + (ar + \frac{1}{2i}r)u_{i+1}^n + (ar - \frac{1}{2i}r)u_{i-1}^n,$$

and thus

$$\begin{aligned} \left|u_i^{n+1}\right| &= \left|\Delta t\, (u_i^n)^p + (1-2ar)u_i^n + (ar + \frac{\alpha}{2i}r)u_{i+1}^n + (ar - \frac{\alpha}{2i}r)u_{i-1}^n\right| \\ &\leq \Delta t \left|(u_i^n)^p\right| + |1-2ar|\, |u_i^n| + \left|ar + \frac{\alpha}{2i}r\right| \left|u_{i+1}^n\right| + \left|ar - \frac{\alpha}{2i}r\right| \left|u_{i-1}^n\right| . \end{aligned}$$

Now we make the assumption that $r \leq \frac{1}{2a}$. We get :

$$\left|u_i^{n+1}\right| \leq \Delta t F^n + \left|(1-2ar)\right| M_n + (ar + \frac{a}{2i}r)M_n + (ar - \frac{a}{2i}r)M_n,$$

so it easily follows that

$$\left|u_i^{n+1}\right| \leq M_n + \Delta t F^n. \tag{54}$$

We deduce by passing to the maximum on $\left(u_i^{n+1}\right)_{1\leq i\leq M}$, and (54) becomes

:

$$M_{n+1} \leq M_n + \Delta t F^n.$$

By recurrence

$$\begin{aligned} M_{n+1} &\leq M_n + \Delta t F^n \leq M_{n-1} + \Delta t(F^n + F^{n-1}) \leq M_{n-2} + \Delta t(F^n + F^{n-1} + F^{n-2}) \\ &\leq \cdots \leq M_0 + \Delta t(F^n + F^{n-1} + F^{n-2} + \cdots + F^0). \end{aligned}$$

Putting

$$F = \max_{0<n<N} |F^n| .$$

We obtain

$$M_{n+1} \leq M_0 + \Delta t F,$$

so

$$\|U\|_{L^\infty} \leq \left\|U^0\right\|_{L^\infty} + \|F\|_{L^\infty} ,$$

which is a stability result. We have thus shown stability under the condition that $r \leq \frac{1}{2a}$. We say that the forward-centered difference method for the heat equation is conditionally stable.

The implicit scheme:

Basically an implicit scheme contains information at the current level which requires solving of simultaneous equations where the scheme are very stable and can have much larger timesteps. So we write the scheme at the point (x_i, t_n) so that the difference equation now becomes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{a}{x_i} \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = (u_i^n)^p , \qquad (55)$$

after,

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{a}{x_i} \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} - a\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} &= (u_i^n)^p , \\ u_i^{n+1} - u_i^n - \frac{a\Delta t}{i\Delta x^2} \left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) - \frac{a\Delta t}{\Delta x^2} \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) &= \Delta t \, (u_i^n)^p . \end{aligned}$$

By putting $R = \frac{a\Delta t}{\Delta x^2}$, we obtain:

$$\begin{aligned}
u_i^{n+1} - u_i^n - \frac{R}{i}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) - R\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) &= \Delta t\left(u_i^n\right)^p, \\
R\left(\frac{1}{i} - 1\right)u_{i-1}^{n+1} + (1+2R)\,u_i^{n+1} + R\left(-\frac{1}{i} + 1\right)u_{i+1}^{n+1} &= \Delta t\left(u_i^n\right)^p - u_i^n,
\end{aligned}$$

to get the first line and the last line of the matrix:

$$\left(\frac{(\Delta x)^2}{2}k_1(0)+1\right)u_0^{n+1} + \frac{(\Delta x)^2}{2}k_1(1)u_M^{n+1} + (\Delta x)^2\sum_{i=1}^{M-1}k_1(x_i)u_i^{n+1} = u_1^{n+1},$$

$$\begin{aligned}
\left(\frac{(\Delta x)^2}{2}k_1(0)+1\right)u_0^{n+1} + \frac{(\Delta x)^2}{2}k_1(1)u_M^{n+1} + \left((\Delta x)^2\ k_1(x_1) - 1\right)u_1^{n+1} \\
+ (\Delta x)^2\sum_{i=2}^{M-1}k_1(x_i)u_i^{n+1} = 0.
\end{aligned}$$

then

$$\begin{aligned}
\frac{(\Delta x)^2}{2}k_2(0)u_0^{n+1} + \left(\frac{(\Delta x)^2}{2}k_2(1) - 1\right)u_M^{n+1} + (\Delta x)^2\sum_{i=1}^{M-2}k_2(x_i)u_i^{n+1} \\
+ \left((\Delta x)^2 k_2(x_{M-1}) + 1\right)u_{M-1}^{n+1} = 0.
\end{aligned}$$

Now when we simplify this expression, the matrix once and at each timestep perform. In particular if we write the system as

$$A^{n+1}U^{n+1} = B^{n+1} \tag{56}$$

which

$$A^{n+1} = \begin{pmatrix}
k(0) & k(1) & k(2) & \cdots & \cdots & k(M-1) & k(M) \\
0 & r & 0 & 0 & \cdots & & 0 \\
0 & \delta_2 & r & -\delta_2 & 0 & \cdots & 0 \\
\cdots & & & & & & \\
0 & & & & \delta_{M-1} & r & -\delta_{M-1} \\
p(0) & p(1) & p(2) & \cdots & \cdots & p(M-1) & p(M)
\end{pmatrix}_{(M+1)\times(M+1)}$$

with

$$\begin{cases} \delta_i = R\left(\frac{1}{i} - 1\right), \\ r = 1 + 2R, \\ k(0) = \left(\frac{(\Delta x)^2}{2} k_1(0) + 1\right), \\ k(1) = \left((\Delta x)^2 \ \ k_1(x_1) - 1\right), \\ k(i) = (\Delta x)^2\, k_1(x_i), \forall\, i = 2, \ldots, M-1, \\ k(M) = \frac{(\Delta x)^2}{2} k_1(1), \\ p(0) = \frac{(\Delta x)^2}{2} k_2(0), \\ p(i) = (\Delta x)^2\, k_2(x_i), \forall\, i = 1, \ldots, M-2, \\ p(M-2) = \left((\Delta x)^2\, k_2(x_{M-1}) + 1\right), \\ p(M) = \left(\frac{(\Delta x)^2}{2} k_2(1) - 1\right), \end{cases}$$

$$U^{n+1} = \begin{pmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{pmatrix}_{(M+1)\times 1},$$

$$B^{n+1} = \begin{pmatrix} 0 \\ \Delta t\,(u_1^n)^p - u_1^n \\ \Delta t\,(u_2^n)^p - u_2^n \\ \cdots \\ \cdots \\ \Delta t\left(u_{M-1}^n\right)^p - u_{M-1}^n \\ 0 \end{pmatrix}.$$

6.4. Numerical examples

In this section we will use the two discrete finite equations derived (explicit and implicit). Three numerical experiments will be considered.

$$\text{Problem 1:}\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{1}{x}\dfrac{\partial}{\partial x}(x\dfrac{\partial u}{\partial x})=u^{3}, & x\in[0.1]\,,\, t>0,\\ u(0,t)=\displaystyle\int_{0}^{1}xu(x,t)dx, & \forall\, t\in(0,T)\,,\\ u(1,t)=\displaystyle\int_{0}^{1}xu(x,t)dx, & \forall\, t\in(0,T)\,,\\ u(x,0)=-\sin(x)+1, & x\in[0,1]\,. \end{cases}$$

$$\text{Problem 2:}\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{1}{x}\dfrac{\partial}{\partial x}(x\dfrac{\partial u}{\partial x})=u^{5}, & x\in[0,1]\,,\, t>0,\\ u(0,t)=\displaystyle\int_{0}^{1}xu(x,t)dx, & \forall\, t\in(0,T)\,,\\ u(1,t)=\displaystyle\int_{0}^{1}xu(x,t)dx, & \forall\, t\in(0,T)\,,\\ u(x,0)=-\sin(x)+1, & x\in[0,1]\,. \end{cases}$$

Since the analytical (exact) solutions to problems 1, 2 and 3 with the associated initial condition are known, we can only estimate numerically the blow-up times.

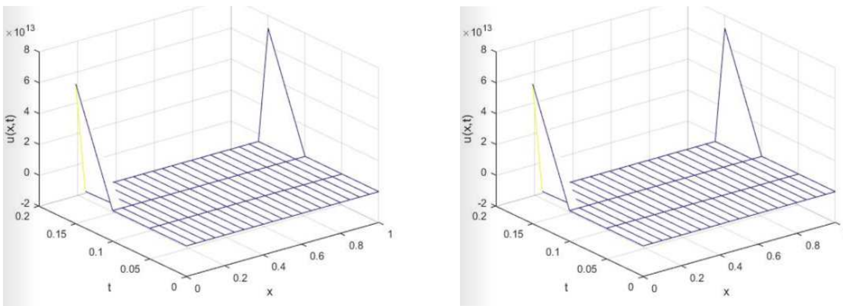

Figure 1:

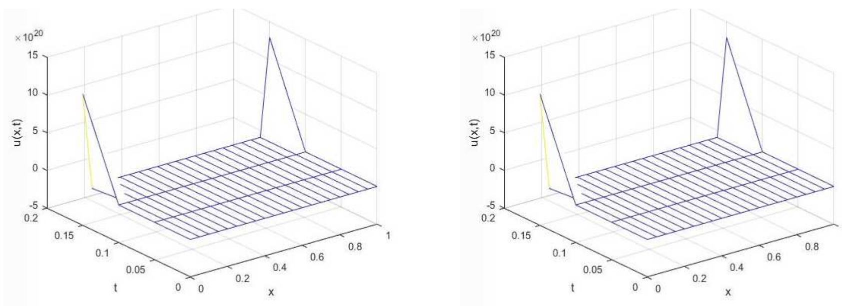

Figure 2:

References

1. A. A. Samarskii, Some problems of the theory of differential equations, *Differential'nye Uravneniya*, **16**(1980), no.11, 1925-1935.

2. S. Dhelis, A. Bouziani and T.-E. Oussaeif, Study of Solution for a Parabolic Integrodifferential Equation with the Second Kind Integral Condition, *Int. J. Anal. Appl.*, **16** (2018), no.4, 569-593.

3. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York–Toronto–London, 1955.

4. Henri Cartan, *Cours de Calcul Différentiel, Hermann Paris, Collection Méthodes, (1967), Editeurs des sciences et Arts.*

5. *J. R. Cannon, The solution of the heat equation anubject to the specification of energy, Quarterly of Applied Mathematics,* ***21*** *(1963), 155-160.*

6. *Jean-Pierre Demailly, Analyse numérique et équations différentielles, EDP science.*

7. *Song-Mu Zheng, Nonlinear Evolution Equations,* , Chapman & Hall/CRC monographs and surveys in pure and applied mathematics, 2004.

8. C. Zainouba, O. Taki-Eddine, D. Sofiane and L. Benoua, A priori estimate and numerical study of solution for a parabolic equation with nonlinear integral condition, *Bulletin of the Institute of Mathematics Academia Sinica NEW SERIES*, **17**(2022), DOI:10.21915/bimas.2022103.

9. T.-E. Oussaeif and A. Bouziani, Inverse problem of a hyperbolic equation with an integral overdetermination condition, *Electronic Journal of Differential Equations*, **2016**(2016), No.138, 1-7.

10. O. Taki-Eddine, B. Abdelfatah, A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition, *Chaos, Solitons & Fractals*, **103**(2017), 79-89.

11. A. Benaoua, O. Taki-Eddine and I. Rezzoug, Unique solvability of a Dirichlet problem for a fractional parabolic equation using energy-inequality method, *Methods Funct. Anal. Topology*, **26**(2020), no.3, 216-226.

12. A. Bouziani, T.-E. Oussaeif and L. Benaoua, A Mixed Problem with an Integral Two-Space-Variables Condition for Parabolic Equation with The Bessel Operator, *Journal of Mathematics*, **8**(2013), Article ID 457631, 8 pages.

13. J. L. Lions et E. Magenes, Problémes aux limites non homogè nes et applications, **1 et 2**(1968), Paris, Dunod.

14. T. Oussaeif and A. Bouziani, Solvability of Nonlinear Goursat Type Problem for Hyperbolic Equation with Integral Condition, *Khayyam Journal of Mathematics*, **4**(2018), no.2, 198-213. doi: 10.22034/kjm.2018.65161

15. T.-E. Oussaeif and A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, *Electron J. Differ. Equ.*, **2014** (2014), (179), 1-10.

16. R. Imad, O. Taki-Eddine and B. Abdelouahab, Solvability of a solution and controllability for nonlinear fractional differential equations, *Bulletin of the Institute of Mathematics*, **15**(2020), no.3, 237-249

17. T.-E. Oussaeif and A. Bouziani, Mixed Problem with an Integral Two-Space-Variables Condition for a Class of Hyperbolic Equations, *International Journal of Analysis*, **2013**(2013), 8 pages.

18. T.-E. Oussaeif and A. Bouziani, Mixed Problem with an Integral Two-Space-Variables Condition for a Parabolic Equation, *International Journal of Evolution Equations*, **9**(), No.2, 181-198.

19. T.-E. Oussaeif and A. Bouziani, Mixed Problem with an Integral Two-Space-Variables condition for a Third Order Parabolic Equation, *International Journal of Analysis and Applications*, **12**(2016), No.2, 98-117.

20. T.-E. Oussaeif and A. Bouziani, Solvability of nonlinear viscosity equation with a boundary integral condition, *J. Nonl. Evol. Equ. Appl.* **2015** (2015), no.3, 31-45.

21. T.-E. Oussaeif and A. Bouziani, Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions, *Electronic Journal of Differential Equations*,**2014**(2014), No. 179, 1-10.

22. Zhuoqun Wu (Jilin University, China), Jingxue Yin (Jilin University, China), Huilai Li (Jilin University, China) and Junning Zhao (Xiamen University, China), *Nonlinear Diffusion Equations*; November 2001.

23. H. Fujita, On the blowing up of solutions to the Cauchy problem for $u_t = 2206u + u^{1+3b1}$, *J. Fac. Sci. Univ. Tokyo, Sect. IA, Math.* **13**, 109-124.

24. Victtor A. Galaktionov and Juan L. Vazquez, The problem of blow up in nonlinear parabolic equation, *Discrete and Continuous Dynamical Systems*, **8**(2002), no.2, 399-433.

25. Y. P. Chen, The blow-up profile for a nonlocal nonlinear parabolic equation with a nonlocal boundary condition, *Quart. Appl. Math.*, **70** (2012), 759-772.

26. A. Bouziani, S. Bensaid and S. Dehilis, A second order accurate difference scheme for the diffusion equation with nonlocal nonlinear boundary conditions, *J. Phys. Math.*, Art. ID 2, **11**(2020), 1-7.