Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **17** (2022), No. 4, pp. 417-438
DOI: 10.21915/BIMAS.2022404

IMPULSIVE STOCHASTIC DIFFERENTIAL EQUATIONS INVOLVING HILFER FRACTIONAL DERIVATIVES

RAHMA YASMINA MOULAY HACHEMI[1,a] AND
TOUFIK GUENDOUZI[1,b]

[1]Loboratory of Stochastic Models, Statistic and Applications, University of Saida- Dr. Moulay Tahar, P.O.Box 138 En-Nasr Saida 20000, Algeria.

[a]E-mail: yasmin.moulayhachemi@yahoo.com

[b]E-mail: tf.guendouzi@gmail.com (Corresponding Author)

Abstract

In this paper, we study the existence of mild solutions of Hilfer fractional stochastic differential equation with impulses driven by sub-fractional Brownian motion. The results are obtained by using Burton-Kirk's fixed point theorem. In the end, an example is given to illustrate the obtained results.

1. Introduction

Differential equations and inclusions with fractional derivatives have recently proved to be strong tools in the modeling of many phenomena in various fields of engineering, economics, physics, biology, ecology, aerodynamics and fluid dynamic traffic models [6, 26, 28, 30]. For some fundamental results in the theory of differential equations involving Caputo and Riemann-Liouville fractional derivatives, please see [1, 2, 24, 32, 33, 34, 41] and the references therein. Since Hilfer [18] proposed the generalized Riemann-Liouville fractional derivative, there has been some interest in studying differential equations involving Hilfer fractional derivatives (see [10, 11, 20] and the references therein). Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [5, 35, 36]. Jaiswal and

Received May 14, 2022.

AMS Subject Classification: 60H10, 34F05, 60H15, 35R60, 60H20, 60H30, 60H05.

Key words and phrases: Hilfer fractional derivatives, fixed point, impulsive stochastic differential equation, sub-fractional Brownian motion, mild solutions.

Bahuguna [19] studied the existence of mild solutions for an abstract Cauchy problem with the Hilfer fractional derivative and an almost sectorial operator. Karthikeyan et al. [21] investigated a class of nonlocal integro-differential equations involving Hilfer fractional derivatives and almost sectorial operators. Zhou et al. [40] obtained new sufficient conditions of the existence of mild solutions for Hilfer fractional evolution equations in the cases that the semigroup associated with an almost sectorial operator is compact as well as noncompact. On the other hand, stochastic perturbation is unavoidable in nature and hence it is important and necessary to consider stochastic effect into the investigation of fractional differential equations. There are many interesting results on the theory and applications of stochastic differential equations involving fractional derivatives (see [9, 13, 15, 16, 17] and references therein). Guendouzi and Bousmaha [14] investigated the existence of mild solutions for a class of fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic α-resolvent operators in Hilbert spaces. Guendouzi and Benzatout [12] studied the existence of mild solutions for a class of impulsive fractional stochastic differential inclusions with state-dependent delay. However, to the best of our knowledge, it seems that there are very few contributions regarding stochastic differential equations with Hilfer fractional derivatives in the literature [3, 37]. Ahmed and El-Borai [4] investigated the existence of mild solutions of Hilfer fractional stochastic integro-differential equations with nonlocal conditions. Kasinathan et al. [22] investigated the existence of mild solution of Hilfer fractional neutral stochastic differential equations with non-instantaneous impluses.

Strongly inspired by the above-mentioned papers, we offer to study the existence of integral solutions for impulsive stochastic differential equations driven by sub-fractional Brownian motion with Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta} X\left(t, x_t\right) = A\left(t\right) X\left(t, x_t\right) + f(t, x_t) \\ \qquad\qquad + \sigma\left(t, x_t\right) \frac{dS_Q^H(t)}{dt}, & \text{for } t \in [s_k, t_{k+1}], k = 0, \ldots, m, \\ x\left(t\right) = h_k\left(t, x_t\right), & \text{for } t \in (t_k, s_k], k = 1, 2, \ldots, m, \\ (I_0^{1-\gamma} x)(t)|_{t=0} = \phi \in \mathcal{D}_{\mathcal{F}_0}((-\infty, 0], U], \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha,\beta}$ is the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$, $\beta \in [0,1]$, $x(\cdot)$ takes value in a real separable Hilbert space U with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$. S_Q^H is an Q-sub-fBm with Hurst parameter $H \in (\frac{1}{2},1)$, and $I_0^{1-\gamma}$ is the fractional integral of orders $1-\gamma$ ($\gamma = \alpha+\beta-\alpha\beta$). The impulses times satisfy $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq \cdots < t_m \leq s_m < t_{m+1} = T$, for $t > 0$. x_t means a segment solution which is defined by $x(\cdot,\cdot) : (-\infty,T] \times \Omega \to U$, and for any $t \geq 0$, $x_t(\cdot,\cdot) : (-\infty,0) \times \Omega \to U$ is given by $x_t(\theta,\omega) = x(t+\theta,\omega)$, $\theta \in (-\infty,0]$, $\omega \in \Omega$. $\mathcal{D}_{\mathcal{F}_T}^{\gamma}$ is defined as

$$\mathcal{D}_{\mathcal{F}_T}^{\gamma} = \left\{x : (-\infty,T] \times \Omega \to U;\ x\mid_{J_k} \in C(J_k;U),\ t^{1-\gamma}x(t) \in \mathcal{D}_{\mathcal{F}_T},\ \right\} \quad k = 1,\ldots,m,$$

with the norm

$$\| x \|_{\mathcal{D}_{\mathcal{F}_T}^{\gamma}} = \| \phi \|_{\mathcal{D}_{\mathcal{F}_0}} + \left(\sup_{0\leq t\leq T} \mathbb{E} \| t^{1-\gamma}x(t) \|^2 \right)^{\frac{1}{2}},$$

and $\phi \in \mathcal{D}_{\mathcal{F}_0}$, where $J_k = (s_k, t_{k+1}]$, $k = 1,\ldots,m$.

If the space $\mathcal{D}_{\mathcal{F}_t}$ is the space formed by all $\mathcal{F}_t$-adapted measurable square integrable $\mathcal{H}$-valued stochastic process $\{x(t) : t \in [0,T]\}$ with the norm $\|x\|^2_{\mathcal{D}_{\mathcal{F}_t}} = \sup_{t\in[0,T]} \mathbb{E}\|x(t)\|^2$, then $(\mathcal{D}_{\mathcal{F}_t}, \|.\|_{\mathcal{D}_{\mathcal{F}_t}})$ is a Banach space.
$\mathcal{D}_{\mathcal{F}_0}$ denotes the family of all almost surely bounded $\mathcal{F}_0$-measurable, and $\tilde{D}$-valued random variables. $\tilde{D} = D((-\infty,0],U)$ denotes the family of all right piecewise continuous functions with left-hand limit ϕ from $(-\infty,0]$ to U, with the norm

$$\| \phi \|_t = \sup_{-\infty<\theta\leq t} \| \phi(\theta) \| .$$

We assume in the sequel that $X(t,x_t) : J\times U \longrightarrow U$, such that $X(t,x_t) = \phi(0) - g(t,x_t)$, $g : J\times\mathcal{D}_{\mathcal{F}_T}^{\gamma} \to U$ and $f : J\times\mathcal{D}_{\mathcal{F}_T}^{\gamma} \to U$, $h_k \in (t_k,s_k]\times\mathcal{D}_{\mathcal{F}_T}^{\gamma} \longrightarrow U$ for all $k = 1,\ldots,m$. $\sigma : J\times\mathcal{D}_{\mathcal{F}_T}^{\gamma} \to L_Q^0(K,H)$.

This paper is structured as follows. In Section 2 we introduce some notations, definitions and preliminary facts about sub-fractional Brownian motion and fractional calculus which are useful throughout the paper. In

Section 3 we prove the existence of $\mathcal{D}^{\gamma}_{\mathcal{F}_T}$-mild solutions for problem (1). Finally an example is given to illustrate our result in Section 5.

2. Preliminaries

In this section, we give some basic definitions, notations, lemmas and some basic facts about sub-fractional Brownian motion and fractional calculus.

Definition 2.1 (*Cylindrical Sub-Fractional Brownian Motion*)**.** Let $\mathcal{K}$ be a separable Hilbert space. A continuous, zero mean, $\mathcal{K}$-valued Gaussian process $\left(S^H_I(t), t \geq 0\right)$ is said to be cylindrical sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ if its covariance is given by

$$E\left\langle k, S^H_I(s)\right\rangle \left\langle k', S^H_I(t)\right\rangle = \left\langle k, k^{'}\right\rangle \left[s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right]$$
$$\text{for all } \; s,t \in \mathbb{R}^{+} \text{and } \; k, k^{'} \in \mathcal{K}.$$

Definition 2.2. Let Q be a non-negative, self-adjoint bounded linear operator that is not nuclear. Then, a cylindrical sub-fractional Brownian motion is defined by the formal series

$$S^H_I(t) = \sum_{n=1}^{\infty} S^H_n(t) e_n \quad t \geq 0;$$

where $\{S^H_n(t)\}_{n=1}^{\infty}$ is a sequence of independent, real valued standard sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $\{e_n\}_{n=1}^{\infty}$ being a complete orthonormal basis in the Hilbert space $\mathcal{K}$.

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}}, (\cdot,\cdot)_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}}, (\cdot,\cdot)_{\mathcal{K}})$ being the separable Hilbert spaces. The notation $\mathcal{C}(J,\mathcal{H})$ stands for the Banach space of continuous functions from J to $\mathcal{H}$ with supermum norm, i.e., $\| x \|_J = \sup_{t\in J} \| x(t) \|$ and $L^1(J,\mathcal{H})$ denotes the Banach space of functions $x : J \to \mathcal{H}$ which are Bochner integrable normed by $\| x \|_{L^1} = \int_0^b \| x(t) \| \, dt$, for all $x \in L^1(J,\mathcal{H})$. A measurable function $x : J \to \mathcal{H}$ is Bochner integrable if and only if $\| x \|$ is Lebesgue integrable.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

Definition 2.3. The sub-fractional Brownian motion (sub-fBm in short) with Hurst parameter $H \in (0,1)$ is a mean zero Gaussian process $S^H = \left\{S_t^H : t \geq 0\right\}$ with $S_H^0 = 0$ and the covariance

$$C_H(t,s) = \mathbb{E}\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + \mid t-s \mid^{2H}\right], \quad (2)$$

for all $s, t \geq 0$.

For $H = \frac{1}{2}$, S^H coincides with the standard Brownian motion B. S^H is neither semimartingale nor a Markov process when $H \neq \frac{1}{2}$. The sub-fBm S^H has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), but it does not have stationary increments. For more details on sub-fBm, we refer to [7, 31, 29].

The sub-fractional Brownian motion satisfies the following estimates:

$$\left[(2-2^{2H-1}) \wedge 1\right] \mid t-s \mid^{2H} \mathbb{E} \mid S^H(t) - S^H(s) \mid^2 \leqslant \left[(2-2^{2H-1}) \wedge 1\right] \mid t-s \mid^{2H}. \quad (3)$$

Thus, Kolmogorov's continuity criterion implies that sub-fBm is Hölder continuous of order γ for any $\gamma < H$ on any finite interval. Therefore, if y is a stochastic process with Hölder continuous trajectories of order $\beta > 1-H$ then the pathwise Riemann-Stieltjes integral $\int_0^b y_t(\omega) dS^H(t)(\omega)$ exists for all $b \geq 0$. In particular, if $H > \frac{1}{2}$, the pathwise integral $\int_0^b f'(S_t^H) dS_t^H$ exists for all $f \in \mathcal{C}^2(\mathbb{R})$, and

$$f(S_b^H) - f(0) = \int_0^b f'(S_t^H) dS_t^H. \quad (4)$$

However, when $H < \frac{1}{2}$ the pathwise Riemann-Stieltjes integral $\int_0^b f'(S_t^H) dS_t^H(\omega)$ does not exist. For more details, we refer the reader to [29][38][39].

Now we aim at introducing the Wiener integral with respect to one dimensional sub-fBm S^H. Fix a time interval $[0, b]$. We denote by Λ the linear space of $\mathbb{R}$-valued step functions on $[0, b]$, that is, $y \in \Lambda$ if

$$y(t) = \sum_{i=1}^{n-1} x_i 1_{[t_i, t_{i+1}](t)},$$

where $t \in [0, b]$, $x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots < t_n = b$. For $y \in \Lambda$ we define its Wiener integral with respect to S^H as

$$\int_0^b y(s) dS_Q^H(s) = \sum_{i=1}^{n-1} x_i (S_{t_i+1}^H - S_{t_i}^H).$$

Let $\mathcal{H}_{\mathcal{S}^H}$ be the canonical Hilbert space associated to the sub-fBm S^H, that is, $\mathcal{H}_{\mathcal{S}^H}$ is the closure of the linear span Λ with respect to the scalar product

$$\left(1_{[0,t]}, 1_{[0,s]}\right)_{\mathcal{H}_{\mathcal{S}^H}} = C_H(t, s).$$

We know that the covariance of sub-fBm can be written as

$$\mathbb{E}\left[S_t^H S_s^H\right] = \int_0^t \int_0^s \eta_H(u, v) du dv = C_H(t, s), \tag{5}$$

where $\eta_H(u, v) = H(2H-1)\left(\mid u - v \mid^{2H-2} - (u+v)^{2H-2}\right)$.

Equation (5) implies that

$$(y, z)_{\mathcal{H}_{\mathcal{S}^H}} = \int_0^t \int_0^s y_u z_v \eta_H(u, v) du dv, \tag{6}$$

for any pair step functions y and z on $[0, b]$. Consider the kernel

$$K_H(t, s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H - \frac{1}{2})} s^{3/2-H} \left(\int_0^t (x^2 - s^2)^{H-3/2} ds\right) 1_{[0,t]}(s). \tag{7}$$

By Dzhaparidze and Van Zanten [8], we have

$$C_H(t, s) = c_H^2 \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du \tag{8}$$

where

$$c_H^2 = \frac{\Gamma(1+2H)sin(\pi H)}{\pi}.$$

Then, (8) implies that $C_H(s,t)$ is non-negative definite. Consider the linear operator $K_H^* : \Lambda \to L^2([0,b])$ defined by

$$\left(K_H^* y\right)(s) = c_H \int_s^r y_r \frac{\partial K_H}{\partial r}(r,s)dr.$$

Using (6) and (8) we have

$$\begin{aligned}
&(K_H^* y, K_H^* z)_{L^2([0,b])} \\
&= c_H^2 \int_0^b \left(\int_s^b y_r \frac{\partial K_H}{\partial r}(r,s)dr \right) \left(\int_s^b z_u \frac{\partial K_H}{\partial u}(u,s)du \right) ds \\
&= c_H^2 \int_0^b \int_0^b \left(\int_0^{r\wedge u} \frac{\partial K_H}{\partial r}(r,s) \frac{\partial K_H}{\partial u}(u,s)ds \right) y_r z_u dr du \\
&= c_H^2 \int_0^b \int_0^b \frac{\partial^2 K_H}{\partial r \partial u}(u,s) y_r z_u dr du \\
&= H(2H-1) \int_0^b \int_0^b \left(\mid u-r \mid^{2H-2} - (u+r)^{2H-2} \right) y_r z_u dr du \\
&= (y,z)_{\mathcal{H}_{\mathcal{S}^H}}. \qquad (9)
\end{aligned}$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert space $\mathcal{H}_{\mathcal{S}^H}$ and $L^2([0,b])$. Hence, the process W defined by $W(t) := S^H((K_H^*)^{-1}(1_{[0,t]}))$ is a Wiener process, and S^H has the following Wiener integral representation:

$$S^H(t) = c_H \int_0^t K_H(t,s)dW(s)$$

because $(K_H^*)(1_{[0,t]})(s) = c_H K_H(t,s)$. By [8], we have

$$W(t) = \int_0^t Z_H(t,s)dS^H(s),$$

where

$$Z_H(t,s) = \frac{s^{H-1/2}}{\Gamma(3/2-H)} \Big[t^{H-3/2}(t^2-s^2)^{1/2-H}$$

$$- (H - 3/2) \int_s^t (x^2 - s^2)^{1/2-H} x^{H-3/2} dx \Big] (1_{[0,t]})(s).$$

In addition, for any $y \in \mathcal{H}_{\mathcal{S}^H}$,

$$\int_0^b y(s) dS^H(s) = \int_0^b (K_H^* y)(t) dW(t)$$

if and only if $K_H^* y \in L^2([0,b])$.

Also, denote $L^2_{\mathcal{H}_{S^H}}([0,b]) = \big\{ y \in \mathcal{H}_{S^H}, K_H^* y \in L^2([0,b]) \big\}$. Since $H > \frac{1}{2}$, we have by (9) and Lemma 2.1 of [25],

$$L^2([0,b]) \subset L^{\frac{1}{H}}([0,b]) \subset L^2_{\mathcal{H}_{S^H}([0,b])}. \tag{10}$$

Lemma 2.1 ([27]). *For $y \in L^{\frac{1}{H}}([0,b])$,*

$$H(2H-1) \int_0^b \int_0^b | y_r || y_u || u - r |^{2H-2} drdu \le C_H \parallel y \parallel_{L^{\frac{1}{H}}([0,b])},$$

where $C_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{1/2}$, with β denoting the beta function.

Let $L(\mathcal{K}, \mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ with the usual norm $\|.\|_{L(\mathcal{K},\mathcal{H})}$. Let $Q \in L(\mathcal{K}, \mathcal{H})$ be a non-negative self-adjoint operator. Denoted by $L_Q^0(\mathcal{K}, \mathcal{H})$ the space of all $\xi \in L(\mathcal{K}, \mathcal{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L_Q^0(\mathcal{K},\mathcal{H})} = \|\xi Q^{\frac{1}{2}}\|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from $\mathcal{K}$ to $\mathcal{H}$. Let $\{S_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensionnal standard sub-fractional Brownian motions mutually independent over $(\Omega, \mathcal{F}, \mathbb{P})$.

Set

$$S_Q^H(t) = \sum_{n=1}^{\infty} S_n^H(t) Q^{\frac{1}{2}} e_n, \qquad t \ge 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathcal{K}$.

If Q is a non-negative self-adjoint trace class operator, then the above $\mathcal{K}$-valued stochastic process $S_Q^H(t)$ is called Q-cylindrical sub-fractional Brownian motion with covariance operator Q.

Lemma 2.2 ([27])**.** *For any* $y : [0, b] \longrightarrow L_Q^0(\mathcal{K}, \mathcal{H})$ *such that*

$$\sum_{n=1}^{\infty} \|yQ^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,b],\mathcal{H})} < \infty$$

holds, and for any $u, v \in [0, b]$ *with* $u > v$,

$$\mathbb{E}\Big\|\int_v^u y(s)dS_Q^H(s)\Big\|_{\mathcal{H}}^2 \leq C_H(u-v)^{2H-1}\sum_{n=1}^{\infty}\int_v^u \|y(s)Q^{\frac{1}{2}}e_n\|_{\mathcal{H}}^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|y(s)Q^{\frac{1}{2}}e_n\|_{\mathcal{H}}^2 \quad \text{is uniformly convergent for } t \in [0, b],$$

then

$$\mathbb{E}\Big\|\int_v^u y(s)dS_Q^H(s)\Big\|_{\mathcal{H}}^2 \leq C_H(u-v)^{2H-1}\int_v^u \|y(s)\|_{L_Q^0(\mathcal{K},\mathcal{H})}^2 ds.$$

We suppose that $\mathcal{F}_t = \sigma\{S_Q^H; 0 \leq s \leq t\}$ is the σ-algebra generated by the $\mathcal{K}$-valued Q-cylindrical sub-fractional Brownian motion, $\mathcal{F}_b = \mathcal{F}$.

Definition 2.4 ([23])**.** The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.5. The Riemann-Liouville fractional derivative of order $\alpha > 0$ $n-1 < \alpha < n, n \in \mathbb{N}$, is defined as

$$^{(R-L)}D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-1-\alpha} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.6 ([18]). The Hilfer fractional derivative of order $0 \leq \alpha \leq 1$ and $0 < \beta < 1$ for a function f is defined by

$$D_{0^+}^{\alpha,\beta} f(t) = I_{0^+}^{\alpha(1-\beta)} \frac{d}{dt} I_{0^+}^{(1-\alpha)(1-\beta)} f(t).$$

Remark 2.1. When $\alpha = 0$, $0 < \beta < 1$, the Hilfer fractional derivative coincides with the classical Riemann-Liouville farctional derivative

$$D_{0^+}^{0,\beta} f(t) = \frac{d}{dt} I_{0^+}^{1-\beta} f(t) =^L D_{0^+}^{\beta} f(t).$$

When $\alpha = 1$, $0 < \beta < 1$, the Hilfer fractional derivative coincides with the classical Caputo fractional derivative

$$D_{0^+}^{1,\beta} f(t) = I_{0^+}^{1-\beta} \frac{d}{dt} f(t) =^c D_{0^+}^{\beta} f(t).$$

Next we mention an axiomatic definition of the phase space $\mathcal{D}_{\mathcal{F}_0}$ introduced by Hale and Kato.

Definition 2.7. $\mathcal{D}_{\mathcal{F}_0}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into U endowed with a norm $\| \cdot \|_{\mathcal{D}_{\mathcal{F}_0}}$, which satisfies the following axioms:

(A-1) If $x : (-\infty, T] \to U$, $T > 0$ is such that $y_0 \in \mathcal{D}_{\mathcal{F}_0}$, then for every $t \in [0, T)$ the following conditions hold

(i) $y_t \in \mathcal{D}_{\mathcal{F}_0}$.

(ii) $\| y(t) \| \leq \mathcal{L} \| y_t \|_{\mathcal{D}_{\mathcal{F}_0}}$.

(iii) $\| y_t \|_{\mathcal{D}_{\mathcal{F}_0}} \leq K(t) \sup \{\| y(s) \| : 0 \leq s \leq t\} + N(t) \| y(0) \|_{\mathcal{D}_{\mathcal{F}_0}}$, where $\mathcal{L} > 0$ is a constant; $K, N : [0, \infty) \longrightarrow [0, \infty)$, K is continuous, N is locally bounded and K, N are independent of $y(\cdot)$.

(A-2) : For the function $y(\cdot)$ in (A-1), y_t is a $\mathcal{D}_{\mathcal{F}_0}$-valued function for $t \in [0, T)$.

(A-3) : The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.

Denote

$$\tilde{K} = sup \{K(t) : \ t \in J\} \ and \ \tilde{N} = sup \{N(t) : \ t \in J\}.$$

Theorem 2.8 (Banach's Fixed Point Theorem)**.** *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a contraction on X. Then T has a unique fixed point $x \in X$ (such that $T(x) = x$).*

Let us define the operators $\{S_{\alpha,\beta}(t) : t \geq 0\}$ and $\{P_\beta(t) : t \geq 0\}$ by

$$\begin{aligned} S_{\alpha,\beta}(t) &= I_{0^+}^{\alpha(1-\beta)} P_\beta(t), \\ P_\beta(t) &= t^{\beta-1} T_\beta(t), \\ T_\beta(t) &= \int_0^\infty \beta\theta \Psi_\beta(\theta) T(t^\beta \theta) d\theta; \end{aligned}$$

where

$$\Psi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-n\beta)}, 0 < \beta < 1, \theta \in (0, \infty)$$

is a function of Wright type which satisfies

$$\int_0^\infty \theta^\xi \Psi_\beta(\theta) d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\beta\xi)}, \quad \xi \in (-1, \infty).$$

Lemma 2.3 ([11])**.** *The operators $S_{\alpha,\beta}$ and P_β have the following properties*

i) *For any fixed $t \geq 0$, $S_{\alpha,\beta}(t)$ and $P_\beta(t)$ are bounded linear operators, and*

$$\begin{aligned} \|P_\beta(t)x\|^2 &\leq M \frac{t^{2(\beta-1)}}{(\Gamma(\beta))^2} \|x\|^2 \quad \textit{and} \\ \|S_{\alpha,\beta}(t)x\|^2 &\leq M \frac{t^{2(\alpha-1)(1-\beta)}}{(\Gamma(\alpha(1-\beta)+\beta))^2} \|x\|^2. \end{aligned}$$

ii) *$\{P_\beta(t) : t \geq 0\}$ is compact if $\{T(t) : t \geq 0\}$ is compact.*

Remark 2.2. $D_{0^+}^{\alpha(1-\beta)} S_{\alpha,\beta}(t) = P_\beta(t)$.

3. Existence of Mild Solution

In this section, we first establish the existence of mild solutions to stochastic differential equations with non-instantaneous impulses driven by a Q-sub-fractional Brownian motion (1). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (1). In order to establish the results, we make the following hypotheses.

(H1) The operator A is the infinitesimal generator of a strongly continuous of bounded linear operators $\{S(t)\}_{t\geq 0}$ which is compact for $t>0$ in $\mathcal{H}$ such that $\| S(t) \|^2 \leq M$ for each $t \in J$, where $J=[0,T]$.

(H2) The operators $S_{\alpha,\beta}, P_\beta \in D(A)$.

(H3) The function $f: J\times \mathcal{D}^\gamma_{\mathcal{F}_T} \to U$ satisfies that:
$\mathbb{E} \| f(t,\phi_1) - f(t,\phi_2) \|^2 \leq L_f \| \phi_1-\phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$,
for all $\phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T}$, $t\in (s_k, t_{k+1}]$ and $k=1,\dots,m$.

(H4) The function $g: J\times \mathcal{D}^\gamma_{\mathcal{F}_T} \to U$ and there exists a positive number K_g. For $t\in J$, we have
$\mathbb{E} \| g(t,\phi_1) - g(t,\phi_2) \|^2 \leq K_g \| \phi_1-\phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$, for all $\phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T}, t\in J$.

(H5) The function $\sigma: J\times \mathcal{D}^\gamma_{\mathcal{F}_T} \to L^0_Q$ satisfies that there exists a positive constant L_σ such that
$\mathbb{E} \| \sigma(t,\phi_1) - \sigma(t,\phi_2) \|^2_{L^0_2} \leq L_\sigma \| \phi_1-\phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$, for all $\phi_1,\phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T}$, $t \in (s_k, t_{k+1}]$ and $k=1,\dots,m$.

(H6) There exist constants $L_{h_k}>0$, for all $\phi_1,\ \phi_2 \in \mathcal{D}^\gamma_{\mathcal{F}_T}$, $t\in (t_k,s_k]$ and $k=1,\dots,m$ such that

$$\mathbb{E} \| h_k(t,\phi_1) - h_k(t,\phi_2) \|^2 \leq L_{h_k} \| \phi_1-\phi_2 \|^2_{\mathcal{D}^\gamma_{\mathcal{F}_T}}$$

and $h_k \in C((t_k,s_k]\times \mathcal{D}^\gamma_{\mathcal{F}_T}, U)$, for all $k=1,\dots,m$.

Now, we give the definition of mild solutions to our problem.

Definition 2.9. An $\mathcal{F}_t$-adapted stochastic process $x:(-\infty,T]\to U$ is said to be an mild solution of (1) if $x_0=\phi\in\mathcal{D}_{\mathcal{F}_0}$ and

(i) $\{x_t,\ t\in J\} \in \mathcal{D}^\gamma_{\mathcal{F}_T}$,

(ii) $\int_0^t [x_s + g(s,x_s)]\, ds \in D(A)$ $,t\in[0,T]$.

(iii) for each $t > 0$

$$x(t)=\begin{cases} S_{\alpha,\beta}(t)\left[\phi(0)-g(0,\phi)\right]+g(t,x_t) & \\ +\int_0^t P_\beta(t-s)f(s,x(s))ds & \\ +\int_0^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s), & \text{for } t\in[0,t_1],\\ h_k(t,x_t), & \text{for } t\in(t_k,s_k];\\ & k=1,\dots,m.\\ S_{\alpha,\beta}(t-s_k)h_k(s_k,x_{s_k})+g(s_k,x_{s_k}) & \\ +\int_{s_k}^t P_\beta(t-s)f(s,x_s)ds & \text{for } t\in[s_k,t_{k+1}];\\ +\int_{s_k}^t P_\beta(t-s)\sigma(s,x_s)dS_Q^H(s) & k=1,\dots,m. \end{cases} \tag{11}$$

To establish the existence and uniqueness theorem of the mild solution for system (1), we use a Banach fixed point to investigate the existence and uniqueness of solutions for impulsive stochastic differential equations.

Theorem 2.10. *Let* (H1)-(H6) *hold with* $\phi(0)-g(0,\phi)\in\overline{D(A)}$, *and*

$$L_0=\max(\mu_1,\ \mu_2,\ \mu_3)<1,$$

where

$$\mu_1=3t_1^{2(\alpha\beta+1)}\left(t_1^{1-\alpha-\beta}K_g+\frac{t_1^{-\alpha}L_fM}{(\Gamma(\beta))^2}+\frac{ML_\sigma t_1^{H-\alpha}}{(\Gamma(\beta))^2}\right),$$

$$\mu_2=\max_{k=1,\dots,m}2L_{h_k}T^{2(1-\gamma)},$$

$$\mu_3=\max_{k=1,\dots,m}\left[\frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2}+4t^{2(1-\gamma)}K_g+\frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}L_f}{(\Gamma(\beta))^2}+\frac{C_HML_\sigma t^{2(H-\gamma)+1}(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2}\right].$$

Then for every initial function $\phi\in\mathcal{D}_{\mathcal{F}_0}$ *there exists a unique associated mild solution* $x\in\mathcal{D}^\gamma_{\mathcal{F}_T}$ *of the problem (1).*

Proof. The proof is given in several steps. Consider the problem (1)

$$\begin{cases} D_{0^+}^{\alpha,\beta}X\left(t,x_t\right)=A\left(t\right)X\left(t,x_t\right)+f(t,x_t) & \text{for } t\in[s_k,t_{k+1}],\\ +\sigma\left(t,x_t\right)\frac{dS_Q^H(t)}{dt}, & k=0,\dots,m,\\ x\left(t\right)=h_k\left(t,x_t\right), & \text{for } t\in(t_k,s_k],\ k=1,2,\dots,m,\\ (I_0^{1-\gamma}x)(t)|_{t=0}=\phi\in\mathcal{D}_{\mathcal{F}_0}((-\infty,0],U]. & \end{cases}$$

We transform the problem into a fixed point one. Consider the operator $\Phi : \mathcal{D}^{\gamma}_{\mathcal{F}_T} \longrightarrow \mathcal{D}^{\gamma}_{\mathcal{F}_T}$ defined by

$$\Phi(x)(t) = \left\{ \begin{array}{ll} \phi(t); & t \in (-\infty, 0], \\ S_{\alpha,\beta}(t)\left[\phi(0) - g(0,\phi)\right] + g(t,x_t) & \\ + \int_0^t P_\beta(t-s) f(s,x_s) ds & \\ + \int_0^t P_\beta(t-s)\sigma(s,x_s) dS^H_Q(s), & \text{if } t \in (0,t_1], \\ h_k(t,x_t), & \text{if } t \in (t_k, s_{k+1}], \\ & k = 1,\ldots,m, \\ S_{\alpha,\beta}(t-s_k) h_k(s_k, x_{s_k}) + g(s_k, x_{s_k}) & \\ + \int_{s_k}^t P_\beta(t-s) f(s,x_s) ds & \text{if } t \in (s_k, t_{k+1}], \\ + \int_{s_k}^t P_\beta(t-s)\sigma(s,x_s) dS^H_Q(s), & k = 1,\ldots,m. \end{array} \right\}$$

For $\phi \in \mathcal{D}_{\mathcal{F}_0}$, we define $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \left\{ \begin{array}{ll} \phi(t), & t \in (-\infty, 0], \\ S_{\alpha,\beta}(t)\left[\phi(0) - g(0,\phi)\right], & t \in (0, t_1]. \end{array} \right\}$$

It is clear that $\tilde{\phi} \in \mathcal{D}^{\gamma}_{\mathcal{F}_T}$. Let $x(t) = z(t) + \tilde{\phi}(t);\ t \in (-\infty, T]$, $z(t)$ satisfies that

$$z(t) = \left\{ \begin{array}{ll} 0, & \text{for } t \in (-\infty, 0], \\ g(t, z_t + \tilde{\phi}_t) + \int_0^t S_{\alpha,\beta}(t-s) f(s, z_s + \tilde{\phi}_s) ds & \\ + \int_0^t P_\beta(t-s)\sigma(s, z_s + \tilde{\phi}_s) dS^H_Q(s), & \text{for } t \in (0, t_1], \\ h_k(t, z_t + \tilde{\phi}_t), & \text{for } t \in (t_k, s_k], \\ S_{\alpha,\beta}(t-s_k) h_k(s_k, z_{s_k} + \tilde{\phi}_{s_k}) + g(s_k, z_{s_k} + \tilde{\phi}_{s_k}) & \\ + \int_{s_k}^t P_\beta(t-s) f(s, z_s + \tilde{\phi}_s) ds & \text{for } t \in (s_k, t_{k+1}], \\ + \int_{s_k}^t P_\beta(t-s)\sigma(s, z_s + \tilde{\phi}_s) dS^H_Q(s), & k = 1,\ldots,m. \end{array} \right\}$$

So, for any $z \in \mathcal{D}'_{\mathcal{F}_T}$, we have $\mathcal{D}'{}_{\mathcal{F}_T} = \{z \in \mathcal{D}^{\gamma}_{\mathcal{F}_T}$, such that $z(0) = 0\}$, then $(\mathcal{D}'{}_{\mathcal{F}_T}, \|\cdot\|_{\mathcal{F}_T})$ is a Banach space.

Let the operator $\tilde{\Phi} : {\mathcal{D}'}_{\mathcal{F}_T} \longrightarrow {\mathcal{D}'}_{\mathcal{F}_T}$ be defined by

$$\tilde{\Phi}(x)(t)=\begin{cases} 0, & \text{for } \ t\in(-\infty,0], \\ g(t,z_t+\tilde{\phi}_t) \\ +\int_0^t P_\beta(t-s)f(s,z_s+\tilde{\phi}_s)ds \\ +\int_0^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s), & \text{for } t\in(0,t_1], \\ h_k(t,z_t+\tilde{\phi}_t), & \text{for } t\in(t_k,s_k], \\ S_{\alpha,\beta}(t-s_k)h_k(s_k,z_{s_k} \\ +\tilde{\phi}_{s_k})+g(s_k,z_{s_k}+\tilde{\phi}_{s_k}) & \text{for } t\in(s_k,t_{k+1}], \\ +\int_{s_k}^t P_\beta(t-s)\sigma(s,z_s+\tilde{\phi}_s)dS_Q^H(s), & k=1,\dots,m. \end{cases}$$

From the assumptions, it is clear that $\tilde{\Phi}$ is well defined. Now we need only to show that $\tilde{\Phi}$ is a contraction mapping.

Case 1. For $u,v\in{\mathcal{D}'}_{\mathcal{F}_T}$, and for $t\in[0,t_1]$ we have

$$\begin{aligned}
&\mathbb{E}\parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t))-\tilde{\Phi}(v(t))\right]\parallel^2 \\
&\le 3t^{2(1-\gamma)}\mathbb{E}\parallel g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\parallel^2 \\
&\quad+3t^{2(1-\gamma)}\mathbb{E}\parallel\int_0^t P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\right]\parallel^2 ds \\
&\quad+3t^{2(1-\gamma)}\mathbb{E}\parallel\int_0^t P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s)\right]dS_I^H(s)\parallel^2 \\
&\quad\le I_1+I_2+I_3,
\end{aligned}\tag{12}$$

where

$$\begin{aligned}
I_1 :&= 3t^{2(1-\gamma)}\mathbb{E}\parallel g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\parallel^2 \\
&\le 3t^{2(1-\gamma)}K_g\parallel u-v\parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \\
I_2 :&= 3t^{2(1-\gamma)}\mathbb{E}\parallel\int_0^t P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\right]\parallel^2 ds \\
&\le 3\frac{t^{2(1-\gamma)}M}{\Gamma^2(\beta)}\mathbb{E}\int_0^t(t-s)^{2(\beta-1)}\parallel f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\parallel^2 ds \\
&\le 3\frac{t^{2\alpha(\beta-1)}L_fM}{(\Gamma(\beta))^2}\parallel u-v\parallel^2_{\mathcal{D}'_{\mathcal{F}_T}},
\end{aligned}$$

$$\begin{aligned}
I_3 :&= 3t^{2(1-\gamma)}\mathbb{E} \parallel \int_0^t P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s)\right] dS_Q^H(s) \parallel^2 \\
&\le 3t^{2(H-\gamma)+1}C_H \int_0^t \parallel P_\beta(t-s)\left[\sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s)\right] ds \parallel^2_{L_Q^0} \\
&\le \frac{3t^{2(H-\gamma)+1}C_H M}{(\Gamma(\beta))^2}\int_0^t (t-s)^{2(\beta-1)} \parallel \sigma(s,u_s+\tilde{\phi}_s)-\sigma(s,v(s)+\tilde{\phi}_s) \parallel_{L_Q^0} \\
&\le \frac{ML_\sigma t^{2(H-\alpha+\alpha\beta)}}{(\Gamma(\beta))^2} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}$$

By taking the supremum over t, we obtain

$$\begin{aligned}
&\parallel \tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t) \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} \\
&\quad = \sup_{t\in[0,t_1]} \mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t))-\tilde{\Phi}(v(t))\right] \parallel^2 \\
&\quad \le 3t_1^{2(\alpha\beta+1)}\left(t_1^{1-\alpha-\beta}K_g+\frac{t_1^{-\alpha}L_f M}{(\Gamma(\beta))^2}+\frac{ML_\sigma t_1^{H-\alpha}}{(\Gamma(\beta))^2}\right) \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}$$

Case 2. For $u,v\in \mathcal{D}'_{\mathcal{F}_T}$, $t\in(t_k,s_k],\ k=1,\dots,m$,

$$\begin{aligned}
\mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t)\right] \parallel^2 &\le L_{h_k} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} \\
&\le 2\tilde{K}t^{2(1-\gamma)}L_{h_k} \parallel u_t-v_t \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}$$

By taking the supremum over t, we obtain

$$\begin{aligned}
\parallel \tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t) \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} &= \sup_{t\in[t_k,s_k],k=1,\dots,m} \mathbb{E} \parallel t^{1-\gamma}\left[\tilde{\Phi}(u(t))-\tilde{\Phi}(v(t))\right] \parallel^2 \\
&\le 2L_{h_k}T^{2(1-\gamma)} \parallel u-v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}} .
\end{aligned}$$

Case 3. For $u,v\in \mathcal{D}'_{\mathcal{F}_T}$ and for $t\in(s_k,t_{k+1}],\ k=1,\dots,m$, we have

$$\begin{aligned}
&\mathbb{E} \parallel t^{1-\gamma}\left(\tilde{\Phi}(u)(t)-\tilde{\Phi}(v)(t)\right) \parallel^2 \\
&\le 4t^{2(1-\gamma)}\mathbb{E} \parallel S_{\alpha,\beta}(t-s_k)\left[h(s_k,u_{s_k}+\tilde{\phi}_{s_k})-h(s_k,v_{s_k}+\tilde{\phi}_{s_k})\right] \parallel^2 \\
&\quad + 4t^{2(1-\gamma)}\mathbb{E} \parallel \left(g(t,u_t+\tilde{\phi}_t)-g(t,v_t+\tilde{\phi}_t)\right) \parallel^2 \\
&\quad + 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s)\left[f(s,u_s+\tilde{\phi}_s)-f(s,v_s+\tilde{\phi}_s)\right] \parallel^2 ds
\end{aligned}$$

$$+ 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s) \Big[\sigma(s, u_s + \tilde{\phi}_s) - \sigma(s, v_s + \tilde{\phi}_s)\Big] dS_Q^H(s) \parallel^2$$
$$\leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel S_{\alpha,\beta}(t-s_k) \Big[h(s_k, u_{s_k} + \tilde{\phi}_{s_k}) - h(s_k, v_{s_k} + \tilde{\phi}_{s_k})\Big] \parallel^2 \\
&\leq 4t^{2(1-\gamma)} \frac{Mt^{2(\alpha-1)(\beta-1)}}{\Gamma(\alpha(1-\alpha)+\beta)^2} \mathbb{E} \parallel h(s_k, u_{s_k} + \tilde{\phi}_{s_k}) - h(s_k, v_{s_k} + \tilde{\phi}_{s_k}) \parallel^2 \\
&\leq \frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2} \parallel u - v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \\
I_2 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \Big(g(t, u_t + \tilde{\phi}_t) - g(t, v_t + \tilde{\phi}_t)\Big) \parallel^2 \\
&\leq 4t^{2(1-\gamma)} K_g \parallel u - v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \\
I_3 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s) \Big[f(s, u_s + \tilde{\phi}_s) - f(s, v_s + \tilde{\phi}_s)\Big] ds \parallel^2 \\
&\leq 4t^{2(1-\gamma)}\mathbb{E} \int_{s_k}^{t_{k+1}} \parallel P_\beta(t-s) \Big[f(s, u_s + \tilde{\phi}_s) - f(s, v_s + \tilde{\phi}_s)\Big] ds \parallel^2 \\
&\leq \frac{4t^{2(1-\gamma)} M (t_{k+1} - s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2} \mathbb{E} \int_{s_k}^{t_{k+1}} \parallel f(s, u_s + \tilde{\phi}_s) - f(s, v_s + \tilde{\phi}_s) \parallel^2 ds \\
&\leq \frac{4t^{2(1-\gamma)} M (t_{k+1} - s_k)^{2(\beta-1)} L_f}{(\Gamma(\beta))^2} \parallel u - v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}, \\
I_4 &= 4t^{2(1-\gamma)}\mathbb{E} \parallel \int_{s_k}^{t_{k+1}} P_\beta(t-s) \Big[\sigma(s, u_s + \tilde{\phi}_s) - \sigma(s, v_s + \tilde{\phi}_s)\Big] dS_Q^H(s) \parallel^2 \\
&\leq 3t^{2(H-\gamma)+1} C_H \mathbb{E} \int_{s_k}^{t_{k+1}} \parallel P_\beta(t-s) \Big[\sigma(s, u_s + \tilde{\phi}_s) - \sigma(s, v(s) + \tilde{\phi}_s)\Big] \parallel^2_{L_Q^0} ds \\
&\leq \frac{3t^{2(H-\gamma)+1} C_H M}{(\Gamma(\beta))^2} \mathbb{E} \int_{s_k}^{t_{k+1}} (t-s)^{2(\beta-1)} \parallel \sigma(s, u_s + \tilde{\phi}_s) - \sigma(s, v(s) + \tilde{\phi}_s) \parallel^2_{L_Q^0} \\
&\leq \frac{C_H M L_\sigma t^{2(H-\gamma)+1} (t_{k+1} - s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2} \parallel u - v \parallel^2_{\mathcal{D}'_{\mathcal{F}_T}}
\end{aligned}$$

By taking the supremum over t, we obtain

$$\begin{aligned}
&\| \tilde{\Phi}(u)(t) - \tilde{\Phi}(v)(t) \|^2_{\mathcal{D}'_{\mathcal{F}_T}} \\
&\quad \leq \left(\frac{4ML_{h_k}}{\Gamma(\alpha(1-\alpha)+\beta)^2} + 4t^{2(1-\gamma)}K_g + \frac{4t^{2(1-\gamma)}M(t_{k+1}-s_k)^{2(\beta-1)}L_f}{(\Gamma(\beta))^2} \right. \\
&\qquad \left. + \frac{C_H M L_\sigma t^{2(H-\gamma)+1}(t_{k+1}-s_k)^{2(\beta-1)}}{(\Gamma(\beta))^2} \right) \| u - v \|^2_{\mathcal{D}'_{\mathcal{F}_T}},
\end{aligned}$$

which implies that $\tilde{\Phi}$ is a contraction and there exists a unique fixed point $z(t) \in \mathcal{D}'_{\mathcal{F}_T}$ of Φ, so $x_t \in \mathcal{D}^{\gamma}_{\mathcal{F}_T}$ is a mild solution of (1). The proof is completed. □

4. Application

$$\begin{cases}
D^{\frac{1}{2},\frac{1}{4}}_{0^+}\left[v_t(\cdot,\xi) - G(t, v_t(\cdot,\xi))\right] & \\
= \frac{\partial^2}{\partial \xi^2}\Big[v_t(\cdot,\xi) - G(t, v_t(\cdot,\xi))\Big]dt, & \text{for } 0 \leq \xi \leq \pi, t \in [s_k, t_{k+1}], \\
+F(t, v_t(\cdot,\xi) + \sigma(t, v_t(\cdot,\xi))\frac{dS^H_Q(t)}{dt}, & k = 0, \dots, m, \\
v(t,\xi) = H_k(t, v_t(\cdot,\xi)), & \text{for } t \in (t_k, s_k], k = 1, 2, \dots, m, \\
v_t(\cdot, 0) = v_t(\cdot,\pi) = 0, & \text{for } t \in [0, 2], \\
(I^{\frac{3}{8}}_0 v_t(\cdot,\xi))|_{t=0} = \phi(t,\xi), & \text{for } t \in (-\infty, 0],
\end{cases} \tag{13}$$

where $D^{\frac{1}{2},\frac{1}{4}}_{0^+}$ denotes the Hilfer fractional derivative. $S^H_Q(t)$ is an Q-sub-f.B.m with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The impulses times satisfy $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq \cdots < t_m \leq s_m < t_{m+1} = T$, for $t > 0$. v_t means a segment solution which is defined by $v(\cdot,\cdot) : (-\infty, T] \times \Omega \to U$. Then for any $t \geq 0$, $v_t(\cdot,\cdot) : (-\infty, 0) \times \Omega \to U$ is given by $v_t(\theta,\omega) = x(t+\theta,\omega)$, for $\theta \in (-\infty, 0]$, $\omega \in \Omega$ with its value in $\mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$, and $U = L^2[0,\pi]$. $F, G : [0,2] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow \mathbb{R}$ are continuous functions. $I^{\frac{3}{8}}_0$ is the fractional integral of order $\frac{3}{8} = 1 - \frac{5}{8}$, where $\gamma = \frac{5}{8} = \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$.

Now let
$y(t)(\xi) = u(t,\xi),\ t \in [0,2],\ \xi \in [0,\pi]$,
$H_k(t,\phi(\theta,\xi)) = h_k(t,\phi)(\xi),\ \theta \in (-\infty,0),\ \xi \in [0,\pi]\ k = 1,\dots,m$, and, $\phi(\theta)(\xi) = \phi(\theta,\xi)$. We need now to define the operator $Q : K \longrightarrow K$, for this we choose a sequence $\{\sigma_n\}_{n\geq 1} \in \mathbb{R}^+$ such that $Qe_n = \sigma_n e_n$ and suppose that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$.

The process $S_Q^H(s)$ will be defined by $S_Q^H(t) = \sum_{n=1}^{\infty} S_n^H(t)\sqrt{\sigma_n} e_n$, where $H \in (\frac{1}{2},1)$ and $\left\{S_n^H(t)\right\}_{n\in\mathbb{N}}$ is a sequence of one dimensional standard sub-fractional Brownian motions mutually independent over $(\omega,\mathcal{F},P)$.

Finally we assume that:

- For all $k = 0,\dots,m$, the function $f : [s_k,t_{k+1}] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow U$ defined by $f(t,v)(\cdot) = F(t,v(\cdot))$ is continuous and we impose conditions on F to verify assumption (H_3). For example we take $F(t,\phi) = t + \frac{2\phi}{1+\|\phi\|_{\mathcal{D}^{\frac{5}{8}}}}$; $t \in [s_k,t_{k+1}]$; $\phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.
- For all $k = 0,\dots,m$, the function $\sigma[s_k,t_{k+1}] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow L_Q^0(K,U)$ is continuous. We impose conditions on σ to make assumptions (H_5) hold. We put: $\sigma(t,\phi) = t^3 + sin\phi$; $t \in [s_k,t_{k+1}]$; $\phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.
- For all $k = 0,\dots,m$, the function $h_k : [t_k,s_k] \times \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T} \longrightarrow U$ defined by $h_k(t,v)(\cdot) = H_k(t,v(\cdot))$ is continuous and we impose conditions on H_k to make assumption (H_6) hold. For example we take: $H_k(t,\phi) = R_k\phi,\ \xi \in \Omega,\ t \in [s_k,t_{k+1}],\ \phi \in \mathcal{D}^{\frac{5}{8}}_{\mathcal{F}_T}$.

Thus the problem (13) can be written in the abstract form

$$\begin{cases} D_{0^+}^{\alpha,\beta} X(t,x_t) = A(t) X(t,x_t) & \\ +f(t,x_t) + \sigma(t,x_t) \frac{dS_Q^H(t)}{dt}, & \text{for } t \in [s_k,t_{k+1}],\ k = 0,\dots,m, \\ x(t) = h_k(t,x_t), & \text{for } t \in (t_k,s_k],\ k = 1,2,\dots,m, \\ (I_0^{1-\gamma}x)(t)|_{t=0} = \phi \in \mathcal{D}_{\mathcal{F}_0}((-\infty,0],U]. & \end{cases} \tag{14}$$

Thanks to these assumptions, it is easy to check that (H1)-(H6) hold and thus assumptions in Theorem 3.1 are fulfilled, ensuring that system 13 possesses a mild solution on $(-\infty,T)$.

References

1. R. Agarwal, S. Hristova and D. O'Regan, A survey of lyapunov functions, stability and impulsive Caputo fractional differential equations, *Fract. Calc. Appl. Anal.*, **19** (2016), no.2, 290-318.

2. R. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.*, **109** (2010), no.3, 973-1033.

3. H. M. Ahmed, M. M. El-Borai and M. E. Ramadan, Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps, *Adv. Differ. Equ.*, (2019), 2019:82.

4. H. M. Ahmed and M. M. El-Borai, Hilfer fractional stochastic integro-differential equations, *Appl. Math. Comp.*, **331** (2018), 182-189.

5. S. Asawasamrit, A. Kijjathanakorn, S. K. Ntouyas and J. Tariboon, Nonlocal boundary value problems for Hilfer fractional differential equations, *Bull. Korean Math. Soc.*, **55**(2018), 1639-1657.

6. K. Balachandran, P. Balasubramaniam and J. P. Dauer, Local null controllability of nonlinear functional differential systems in Banach spaces, *J. Optim Theory Appl.*, **88**(1996), no.1, 61-75.

7. T. L. G. Bojdecki, L. G. Gorostiza and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, *Stat. Prob. Lett.*, **69** (2004), 405-419.

8. K. Dzhaparidze and H. Van Zanten, A series expansion of fractional Brownian motion, *Probab. Theory Relat. Fields*, **102** (2004), 39-55.

9. S. Farahi and T.Guendouzi, Approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions, *Results. Math.*, **65** (2014), 501-521.

10. K. M. Furati and M. D. Kassim and N. E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Computers and Mathematics with Applications*, **64** (2012), 1616-1626.

11. H. Gu and J. J. Trujilo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Applied Mathematics and Computation*, **255** (2015), 344-345.

12. T. Guendouzi and O. Benzatout, Existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay, *Ch. J. Math.*, V 2014, Art ID 981714, 13 p.

13. T. Guendouzi and L. Bousmaha, Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay, *Qual. Theory Dyn. Syst.*, **13** (2014), 89-119.

14. T. Guendouzi and L. Bousmaha, Existence of solutions for fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators, *Vietnam J. Math.*, **43** (2015), no.4, 663-675.

15. T. Guendouzi, Existence and controllability results for fractional stochastic semilinear differential inclusions, *Differ. Equ. Dyn. Syst.*, **23** (July 2015), No.3, 225-240.

16. T. Guendouzi and S. Farahi, Approximate controllability of Sobolev-type fractional functional stochastic integro-differential systems, *Bol. Soc. Mat. Mex.*, **21** (2015), 289-308.

17. T. Guendouzi and S. Farahi, Approximate controllability of semilinear fractional stochastic dynamic systems with nonlocal conditions in Hilbert spaces, *Mediterr. J. Math.*, **13** (2016), 637-656.

18. R. Hilfer, *Application of fractional calculus in physics.*, Singapore: world Scientific, 2000.

19. A. Jaiswal and D. Bahuguna, Hilfer fractional differantial equations with almoste sectorial operators, *Differ. Equ. Dyn. Syst.*, **8** (2020), 1-17.

20. Jingyun Lv1 and Xiaoyuan Yang, A class of Hilfer fractional stochastic differential equations and optimal controls, *LMIB and School of Mathematics and Systems Science*, Beihang University, Beijing, P.R. China, 2019.

21. K. Karthikeyan, A. Debbouche and D. F. M. Torres, Analysis of Hilfer fractional integro-differential equations with almost sectorial operators, *Fractal Fract.*, **5** (2021), 22.

22. R. Kasinathan, R. Kasinathan, D. Baleanu and A. Annamalai, Hilfer fractional neutral stochastic differential equations with non-instantaneous impulses, *AIMS Mathematics*, **6** (2021), no.5, 4474-4491.

23. A. A. Kilbas, H. M. Srivastava and J. J. Trujilo, Theory and applications of fractional differential equations, North Holland Mathematics studies, vol.204. Elsevier Science publishes BV. Amsterdam, 2006.

24. J. W. M. Li, Finite time stability of fractional delay differential equations, *Appl. Math. Lett.*, **64** (2017), no.3, 170-176.

25. I. Mendy, Parametric estimation for sub-fractional Ornstein- Uhlenbeck process, *J. Stat. Plan. Inference*, **143** (2013), 633-674.

26. X. Mao, Stochastic differential equations and applications, Horwood: Chixhester, 1997.

27. D. Nualart, *The Malliavin Calculus and related Topics*, 2nd edn. Springer, Berlin, 2006.

28. P. Protter, *Stochastic Integration and Differential Equations*, Applications of Mathematics, New York, Springer, Berlin, 1990.

29. G. Shen and C. Chen, Stochastic integration with respect to the sub-fractional Brownian with $H \in (0, \frac{1}{2})$, *Stat. Prob. Letters*, **82** (2012), 240-251.

30. K. Sobczyk, *Stochastic Differential Equations with Applications to Physics And Engineering*, London, Kluwer Academic, 1991.

31. C. Tudor, Some properties of the sub-fractional Brownian motion, *Stochastics: An Intenational Jourmal of Probability and Stochastic Processes*, **79** (2007), 431-448.

32. J. Wang, Approximate mild solutions of fractional stochastic evolution equations in Hilbert space, *Appl. Math. Comput.*, **256** (2015), 315-323.

33. J. Wang, X. Li, M. Feckan and Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integral via two kinds of convexity, *Appl. Anal.*, **92** (2013), no.11, 2241-2253.

34. J. Wang, Y. Zhou and Z. Lin, On a new class of impulsive fractional differential equations, *Appl. Math. Comput.*, **242** (2014), no.1, 649-657.

35. A. Wongcharoen, S. K. Ntouyas and J. Tariboon, Boundary Value Problems for Hilfer Fractional Differential Inclusions with Nonlocal Integral Boundary Conditions, *Mathematics, MDPI*, **8** (2020), 1905; doi:10.3390/math8111905.

36. A. Wongcharoen, S. K. Ntouyas and J. Tariboon, Nonlocal boundary value problems for Hilfer-type pantograph fractional differential equations and inclusions, *Adv. Differ. Equ.*, **279** (2020).

37. M. Yang, A. Alsaedi, B. Ahmad and Y. Zhou, Attractivity for Hilfer fractional stochastic evolution equations, *Adv. Differ. Equ.*, (2020), 2020:130.

38. L. Yan, G. Shen and K. He, Ito's formula for a sub-fractional Brownian motion, *Commun. Stoch. Anal.*, **5** (2011), 135-159.

39. L. C. Young, An inequality of the Hölder type connected with Stieltjes integration, *Acta Math.*, **67** (1936), 251-282.

40. M. Zhou, C. Li and Y. Zhou, Existence of mild solutions for Hilfer fractional evolution equations with almost sectorial operators, *Axioms*, **11** (2022), 144.

41. Y. Zhou, F. Zhou and J. Pecaric, Abstract Cauchy problem for fractional functional differential equations, *Topol. Meth. Nonlinear Anal.*, **42** (2013), no.1, 119-136.