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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH Ψ-HILFER FRACTIONAL DERIVATIVE WITH NONLOCAL INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, we study the existence of solutions for a boundary value problem of Ψ-Hilfer fractional derivative with nonlocal integral boundary conditions by using the measure of noncompactness combined with the Mönch's fixed point theorem. Two examples are given to illustrate our results.

1. Introduction

The study of fractional differential equations has many applications in various areas of science and engineering as in physics, chemistry, biophysics, hydrology, blood flow problems, thermodynamics, statistical mechanics and control theory, for example see [7, 11, 14, 18, 19, 24]. In 1999 Hilfer [14]

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has generalized Riemann-Liouville and Caputo fractional derivatives. The basic work on the theory of Hilfer fractional differential equations can be found in [6, 12, 25]. The boundary value problem for fractional differential equations involving Hilfer derivative has been researched in [1, 2, 22]. In [21] Sousa and Oliveira have presented the so-called Ψ-Hilfer fractional derivative with respect to another function, to combine in one fractional operator a largest number of fractional derivatives and thus, open a window for new applications.

The notion of so-called measure of noncompactness was introduced by the fundamental article of Kuratowski [16], and that has played an essential part in the theory of fixed points. In the last decades, many authors have used the technique of noncompactness measure to study existence of solutions to nonlinear integral equations of order fractional and fractional differential equations, for example see [3, 9, 10, 13, 17, 20] and the references therein.

The purpose of this paper is to study the existence of solutions for the following fractional differential equation involving Ψ-Hilfer fractional derivative with nonlocal integral boundary conditions

$$\begin{cases} {}^{H}D_{a+}^{\alpha,\beta;\Psi}x\left(t\right)=f\left(t,x\left(t\right)\right),\ t\in\left(a,b\right), \\ x\left(a\right)=0,\ I_{a+}^{2-\gamma;\Psi}x\left(b\right)=\sum\limits_{i=1}^{m}\theta_{i}I_{a+}^{\eta_{i};\Psi}x\left(\delta_{i}\right), \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha,\beta;\Psi}$ is the left sided Ψ-Hilfer fractional derivative of order $\alpha\in(1,2)$ and type $\beta\in[0,1]$, $I^{2-\gamma;\Psi}$, $I^{\eta_i;\Psi}$ are the left sided Ψ-Riemann-Liouville fractional integrals of order $2-\gamma$, $\eta_i>0$ respectively, $\gamma=\alpha+\beta\left(2-\alpha\right)\in(1,2)$, $-\infty<a<b<\infty$, $\theta_i\in\mathbb{R}$, $i=1,2,\ldots,m$, $0\le a\le\delta_1<\delta_2<\delta_3<\cdots<\delta_m\le b$, $f:[a,b]\times E\to E$ is a given continuous function satisfying some assumptions that will be specified later, and E is a Banach space with the norm $\|.\|$.

The rest of this paper is organized as follows: In Section 2, we give the basic definitions and notations. In Section 3, we investigate the existence of solutions of problem (1.1). Finally, in Section 4, we present two examples to illustrate the main results.

2. Preliminaries

Some definitions, notations and results of the fractional calculus, which will be utilized in this paper are introduced throughout this section.

Let $J = [a, b]$. By $C(J, E)$ we denote the Banach space of all continuous functions defined on J endowed with the norm

$$\|x\|_{\infty} = \sup\left\{\|x(t)\| : t \in J\right\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $x : J \to E$ that are Lebesgue integrable with norm and

$$\|x\|_{L^1} = \int_J \|x(t)\| \, dt.$$

And $C^n(J, E)$ denotes the class of all real valued functions defined on J which have a continuous nth order derivative. Moreover, for a given set V of functions $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \ t \in J,$$

and

$$V(J) = \{v(t) : v \in V, \ t \in J\}.$$

Now we're giving out some fractional calculus results and properties.

Definition 1 ([15])**.** The left-sided Ψ-Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h \in C(J, E)$ with respect to another function $\Psi : J \to \mathbb{R}$ that is an increasing differentiable function such that $\Psi'(t) \neq 0$, for all $t \in J$ is defined by

$$I_{a+}^{\alpha;\Psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \Psi'(s)\left(\Psi(t) - \Psi(s)\right)^{\alpha-1} h(s)\, ds,$$

where Γ is the gamma function.

Definition 2 ([21])**.** *Let* $\alpha > 0$, $n \in \mathbb{N}$, $h \in C^n(J, E)$, and $\Psi \in C^n(J, \mathbb{R})$ be a function such that Ψ is increasing function and $\Psi'(t) \neq 0$, for all $t \in J$. The left-sided Ψ-Riemann-Liouville fractional of a function h of order α with respect to Ψ is defined by

$$D^{\alpha;\Psi} h(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);\Psi} h(t),$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Definition 3 ([21]). *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $h \in C^n(J, E)$*, and $\Psi \in C^n(J, \mathbb{R})$. Then, the left-sided Ψ-Hilfer fractional derivative ${}^H D^{\alpha,\beta;\Psi}$ of a function h of order α and type $\beta \in [0,1]$ is defined by

$${}^H D_{a+}^{\alpha,\beta;\Psi} h(t) = I_{a+}^{\beta(n-\alpha);\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\Psi} h(t).$$

Lemma 1 ([15, 21]). *Let $\alpha, \beta, \delta > 0$. Then*

1) $I_{a+}^{\alpha;\Psi} I_{a+}^{\beta;\Psi} h(t) = I_{a+}^{\alpha+\beta;\Psi} h(t)$.
2) $I_{a+}^{\alpha;\Psi} (\Psi(t) - \Psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\Psi(t) - \Psi(a))^{\alpha+\delta-1}$.

Lemma 2 ([21]). *Let $\gamma > 0$, consider the function $f(t) = (\Psi(t) - \Psi(a))^{\gamma-1}$, where $\gamma > n$. Then for $n-1 < \alpha < n$ and $0 \le \beta \le 1$, we have*

$${}^H D^{\alpha,\beta;\Psi} (\Psi(t) - \Psi(a))^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} (\Psi(t) - \Psi(a))^{\gamma-\alpha-1}.$$

In particular, if $\alpha \in (1,2)$ and $1 < \gamma < 2$, we have

$${}^H D^{\alpha,\beta;\Psi} (\Psi(t) - \Psi(a))^{\gamma-1} = 0.$$

Lemma 3 ([21]). *If $h \in C^n(J, \mathbb{R})$, $n-1 < \alpha < n$ and $0 \le \beta \le 1$, then*

1) $I_{a+}^{\alpha;\Psi}\, {}^H D^{\alpha,\beta;\Psi} h(t) = h(t) - \sum\limits_{k=1}^{n} \frac{(\Psi(t)-\Psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} h_{\Psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\Psi} h(a)$ *where* $h_{\Psi}^{[n-k]} h(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^{n-k} h(t)$.
2) ${}^H D^{\alpha,\beta;\Psi}\, I_{a+}^{\alpha;\Psi} h(t) = h(t)$.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 4 ([5, 8]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to$

$[0,\infty)$ defined by

$$\mu(B)=\inf\{\epsilon>0: B\subseteq\cup_{i=1}^{n}B_i \text{ and } diam(B_i)\le\epsilon\}, \text{ here } B\in\Omega_E.$$

This measure of noncompactness satisfies some important properties.

(a) $\mu(B)=0\Leftrightarrow\overline{B}$ is compact (B is relatively compact),

(b) $\mu(B)=\mu(\overline{B})$,

(c) $A\subset B\Rightarrow\mu(A)\le\mu(B)$,

(d) $\mu(A+B)\le\mu(A)+\mu(B)$,

(e) $\mu(cB)=|c|\,\mu(B)$, $c\in\mathbb{R}$,

(f) $\mu(convB)=\mu(B)$.

Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in [5, 8].

Definition 5. A map $f: J\times E\to E$ is said to be Carathéodory if

(i) $t\to f(t,x)$ is measurable for each $x\in E$.

(ii) $x\to f(t,x)$ is continuous for almost all $t\in J$.

To prove the existence of solutions of (1.1), we need the following results.

Theorem 1 ([4]). *Let D be a bounded, closed and convex subset of the Banach space such that $0\in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V=\overline{conv}N(V) \text{ or } V=N(V)\cup\{0\}\Rightarrow\mu(V)=0,$$

holds for every V of D, then N has a fixed point.

Lemma 4 ([23]). *Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$. Let G be a continuous function on $J\times J$ and f a function from $J\times E\to E$, which satisfies the Carathéodory conditions, and assume there exists $p\in L^1(J,\mathbb{R}^+)$ such that, for each $t\in J$ and each bounded set $B\subset E$, we have*

$$\lim_{h\to0^+}\mu(f(J_{t,h}\times B))\le p(t)\,\mu(B), \text{ here } J_{t,h}=[t-h,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)\,f(s,y(s))\,ds : y\in V\right\}\right)\leq \int_J \|G(s,t)\|\,p(s)\,\mu(V(s))\,ds.$$

To obtain our results, we need the following lemmas.

Lemma 5. *Let*

$$\Lambda=\frac{(\Psi(b)-\Psi(a))}{\Gamma(2)}-\sum_{i=1}^{m}\frac{\theta_i}{\Gamma(\gamma+\eta_i)}(\Psi(\delta_i)-\Psi(a))^{\gamma+\eta_i-1}\neq 0, \tag{2.1}$$

and for any $q\in C(J,E)$, then the nonlocal boundary value problem

$$\begin{cases} {}^{H}D^{\alpha,\beta;\Psi}x(t)=q(t),\ t\in(a,b),\\ x(a)=0,\ I_{a+}^{2-\gamma;\Psi}x(b)=\sum\limits_{i=1}^{m}\theta_i I_{a+}^{\eta_i;\Psi}x(\delta_i), \end{cases} \tag{2.2}$$

is equivalent to the integral equation

$$x(t)=\frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left(\sum_{i=1}^{m}\theta_i I_{a+}^{\alpha+\eta_i;\Psi}q(\delta_i)-I_{a+}^{2+\alpha-\gamma;\Psi}q(b)\right)+I_{a+}^{\alpha;\Psi}q(t). \tag{2.3}$$

Proof. Taking Ψ-fractional integral $I_{a+}^{\alpha;\Psi}$ to the first equation of (2.2), and from Lemma 3, we get

$$x(t)-\sum_{k=1}^{2}\frac{(\Psi(t)-\Psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}h_{\Psi}^{[2-k]}I_{a+}^{(1-\beta)(2-\alpha);\Psi}x(a)=I_{a+}^{\alpha;\Psi}q(t),\ t\in J. \tag{2.4}$$

We have $(1-\beta)(2-\alpha)=2-\gamma$. Therefore

$$\begin{aligned} x(t)&=\frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{\Gamma(\gamma)}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)I_{a+}^{2-\gamma;\Psi}x(t)\Big|_{t=a}\\ &\quad+\frac{(\Psi(t)-\Psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}\,I_{a+}^{2-\gamma;\Psi}x(t)\Big|_{t=a}+I_{a+}^{\alpha;\Psi}q(t)\\ &=\frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{\Gamma(\gamma)}\,D^{\gamma-1;\Psi}x(t)\big|_{t=a}\\ &\quad+\frac{(\Psi(t)-\Psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}\,I_{a+}^{2-\gamma;\Psi}x(t)\Big|_{t=a}+I_{a+}^{\alpha;\Psi}q(t). \end{aligned}$$

Put

$$c_1 = D^{\gamma-1;\Psi} x(t)\big|_{t=a} \text{ and } c_2 = I_{a+}^{2-\gamma;\Psi} x(t)\Big|_{t=a},\ t \in J.$$

Then

$$x(t) = \frac{(\Psi(t) - \Psi(a))^{\gamma-1}}{\Gamma(\gamma)} c_1 + \frac{(\Psi(t) - \Psi(a))^{\gamma-2}}{\Gamma(\gamma-1)} c_2 + I_{a+}^{\alpha;\Psi} q(t).$$

Because $\lim_{t\to a} (\Psi(t) - \Psi(a))^{\gamma-2} = \infty$, in the view of boundary conditions $x(a) = 0$, we must have

$$c_2 = 0.$$

Replacing c_2 by their value in (2.4), we get

$$x(t) = \frac{(\Psi(t) - \Psi(a))^{\gamma-1}}{\Gamma(\gamma)} c_1 + I_{a+}^{\alpha;\Psi} q(t). \tag{2.5}$$

Next, we use the second boundary condition to determine the constant c_1. Applying $I_{a+}^{\eta_i;\Psi}$ on both sides of equation (2.5), we get

$$I_{a+}^{\eta_i;\Psi} x(t) = \frac{c_1}{\Gamma(\gamma+\eta_i)} (\Psi(t) - \Psi(a))^{\gamma+\eta_i-1} + I_{a+}^{\alpha+\eta_i;\Psi} q(t). \tag{2.6}$$

From the condition $x(b) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\Psi} x(\delta_i)$ and (2.6), we have

$$\begin{aligned} x(b) &= \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\Psi} x(\delta_i) \\ &= c_1 \sum_{i=1}^{m} \frac{\theta_i}{\Gamma(\gamma+\eta_i)} (\Psi(\delta_i) - \Psi(a))^{\gamma+\eta_i-1} + \sum_{i=1}^{m} \theta_i I_{a+}^{\alpha+\eta_i;\Psi} q(\delta_i). \end{aligned} \tag{2.7}$$

From equations (2.5) and (2.7), we have

$$\begin{aligned} I_{a+}^{2-\gamma;\Psi} x(b) &= \frac{(\Psi(b) - \Psi(a))}{\Gamma(2)} c_1 + I_{a+}^{2+\alpha-\gamma;\Psi} q(b) \\ &= c_1 \sum_{i=1}^{m} \frac{\theta_i}{\Gamma(\gamma+\eta_i)} (\Psi(\delta_i) - \Psi(a))^{\gamma+\eta_i-1} + \sum_{i=1}^{m} \theta_i I_{a+}^{\alpha+\eta_i;\Psi} q(\delta_i). \end{aligned}$$

Thus, we find

$$c_1 = \frac{1}{\Lambda}\left(\sum_{i=1}^{m}\theta_i I_{a+}^{\alpha+\eta_i;\ \Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\ \Psi} q\left(b\right)\right).$$

Substituting the value of c_1 into (2.5), we obtain the equivalent fractional integral equation (2.3) to the problem (2.2).

Conversely, suppose that x is the solution of the fractional integral equation (2.3). Applying fractional derivative ${}^{H}D^{\alpha,\beta;\Psi}$ on both sides of equation (2.3) and using the Lemma 2 and Lemma 3, we get

$$\begin{aligned}
&{}^{H}D^{\alpha,\beta;\Psi}x\left(t\right)\\
&=\frac{1}{\Lambda\Gamma\left(\gamma\right)}\left(\sum_{i=1}^{m}\theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right)\ {}^{H}D^{\alpha,\beta;\Psi}\left(\Psi\left(t\right)-\Psi\left(a\right)\right)^{\gamma-1}\\
&\quad+\ {}^{H}D^{\alpha,\beta;\Psi} I_{a+}^{\alpha;\Psi} q\left(t\right) = q\left(t\right),\ t\in J. \qquad (2.8)
\end{aligned}$$

This proves x satisfies the first equation of (2.2). Next, we prove that x given by equation (2.3) verifies the boundary conditions. From equation (2.3), clearly

$$x\left(a\right) = 0. \qquad (2.9)$$

Now we prove that x satisfies the nonlocal integral boundary conditions. From equation (2.3), we have

$$\begin{aligned}
I_{a+}^{\eta_i;\Psi}x\left(t\right) &= \frac{\left(\Psi\left(t\right)-\Psi\left(a\right)\right)^{\eta_i+\gamma-1}}{\Lambda\Gamma\left(\gamma+\eta_i\right)}\left(\sum_{i=1}^{m}\theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right)\\
&\quad + I_{a+}^{\eta_i+\alpha;\Psi} q\left(t\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{i=1}^{m}\theta_i I_{a+}^{\eta_i;\Psi}x\left(\delta_i\right)\\
&=\frac{1}{\Lambda}\sum_{i=1}^{m}\frac{\theta_i\left(\Psi\left(\delta_i\right)-\Psi\left(a\right)\right)^{\eta_i+\gamma-1}}{\Gamma\left(\gamma+\eta_i\right)}\left(\sum_{i=1}^{m}\theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right)\\
&\quad+\sum_{i=1}^{m}\theta_i I_{a+}^{\eta_i+\alpha;\Psi} q\left(\delta_i\right). \qquad (2.10)
\end{aligned}$$

From equation (2.1), we have

$$\sum_{i=1}^{m} \frac{\theta_i \left(\Psi\left(\delta_i\right)-\Psi\left(a\right)\right)^{\gamma+\eta_i-1}}{\Gamma\left(\gamma+\eta_i\right)} = \frac{\left(\Psi\left(b\right)-\Psi\left(a\right)\right)}{\Gamma\left(2\right)} - \Lambda. \tag{2.11}$$

Thus, equation (2.10) reduces to

$$\begin{aligned}
&\sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\Psi} x\left(\delta_i\right) \\
&= \frac{1}{\Lambda}\left(\frac{\left(\Psi\left(b\right)-\Psi\left(a\right)\right)}{\Gamma\left(2\right)} - \Lambda\right)\left(\sum_{i=1}^{m} \theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right) \\
&\quad + \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i+\alpha;\Psi} q\left(\delta_i\right) \\
&= \frac{1}{\Lambda}\frac{\left(\Psi\left(b\right)-\Psi\left(a\right)\right)}{\Gamma\left(2\right)}\left(\sum_{i=1}^{m} \theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right) + I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right).
\end{aligned} \tag{2.12}$$

Now from equation (2.3), we have

$$\begin{aligned}
I_{a+}^{2-\gamma;\Psi} x\left(b\right) = &\frac{1}{\Lambda}\frac{\left(\Psi\left(b\right)-\Psi\left(a\right)\right)}{\Gamma\left(2\right)}\left(\sum_{i=1}^{m} \theta_i I_{a+}^{\alpha+\eta_i;\Psi} q\left(\delta_i\right) - I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right)\right) \\
&+ I_{a+}^{2+\alpha-\gamma;\Psi} q\left(b\right).
\end{aligned} \tag{2.13}$$

From equation (2.12) and (2.13), we obtain

$$I_{a+}^{2-\gamma;\Psi} x\left(b\right) = \sum_{i=1}^{m} \theta_i I_{a+}^{\eta_i;\Psi} x\left(\delta_i\right). \tag{2.14}$$

From (2.8), (2.9) and (2.14), it follows that the x defined by equation (2.3) satisfies the problem (2.2). $\square$

3. Main Results

In the following, we prove existence results for the boundary value problem (1.1) by using the Mönch's fixed point theorem. The following assumptions will be used in our main results.

(H_1) The function $f : J \times E \to E$ satisfies the Carathéodory condition.

(H_2) There exists a $p_f \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\|f(t,x)\| \leq p_f(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E.$$

(H_3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) \leq p_f(t)\,\mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J.$$

Theorem 2. *Assume that the assumptions* (2.1) *and* $(H_1) - (H_3)$ *hold. If*

$$k_1 = \sum_{i=1}^{m} |\theta_i| \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{\alpha+\eta_i+\gamma-1}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(\alpha+\eta_i+1)} + \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{1+\alpha}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(3+\alpha-\gamma)} + \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{3.1}$$

then the boundary value problem (1.1) *has at least one solution.*

Proof. We transform the problem (1.1) into a fixed point problem by defining an operator

$$\Phi : C(J, E) \to C(J, E),$$

as

$$(\Phi x)(t) = \frac{(\Psi(t) - \Psi(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \left(\sum_{i=1}^{m} \theta_i \, I_{a+}^{\alpha+\eta_i;\Psi} f(t, x(t))\Big|_{t=\delta_i} + I_{a+}^{2+\alpha-\gamma;\Psi} f(t, x(t))\Big|_{t=b} \right) + I_{a+}^{\alpha;\Psi} f(t, x(t)).$$

Clearly, the fixed points of operator Φ are solutions of the problem (1.1). Let $M > 0$ and consider the set

$$\Omega = \{x \in C(J, E) : \|x\|_\infty \leq M\}.$$

Clearly, the subset Ω is closed, bounded, and convex. We will show that Φ satisfies the assumptions of Theorem 1. The proof will be given in three steps.

Step 1. Φ maps Ω into itself.

For each $x \in \Omega$, by (H2) and (3.1) we have for each $t \in J$

$$
\begin{aligned}
&\left\| (\Phi x)(t) \right\| \\
&\leq \frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \Bigg(\sum_{i=1}^{m} |\theta_i| \, I_{a+}^{\alpha+\eta_i;\Psi} \left\| f(t,x(t)) \right\| \Big|_{t=\delta_i} \\
&\quad + I_{a+}^{2+\alpha-\gamma;\Psi} \left\| f(t,x(t)) \right\| \Big|_{t=b} \Bigg) + I_{a+}^{\alpha;\Psi} \left\| f(t,x(t)) \right\| \\
&\leq \sum_{i=1}^{m} |\theta_i| \frac{M \|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha+\eta_i+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+\eta_i+1)} + \frac{M \|p_f\|_\infty (\Psi(b)-\Psi(a))^{1+\alpha}}{|\Lambda|\Gamma(\gamma)\Gamma(3+\alpha-\gamma)} \\
&\quad + \frac{M \|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq M \Bigg(\sum_{i=1}^{m} |\theta_i| \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha+\eta_i+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+\eta_i+1)} + \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{1+\alpha}}{|\Lambda|\Gamma(\gamma)\Gamma(3+\alpha-\gamma)} \\
&\quad + \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha}}{\Gamma(\alpha+1)} \Bigg) \\
&\leq M k_1 < M,
\end{aligned}
$$

where k_1 is given by (3.1).

Step 2. $\Phi(\Omega)$ is bounded and equicontinuous.

By Step 1, we have $\Phi(\Omega) = \{\Phi x : x \in \Omega\} \subset \Omega$. Thus, for each $x \in \Omega$, we have $\|\Phi x\|_\infty \leq M$, which means that $\Phi(\Omega)$ is bounded. For the equicontinuity of $\Phi(\Omega)$. Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and for $x \in \Omega$, we get

$$
\begin{aligned}
&\left\| (\Phi x)(t_2) - (\Phi x)(t_1) \right\| \\
&\leq \frac{(\Psi(t_2)-\Psi(a))^{\gamma-1} - (\Psi(t_1)-\Psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \\
&\quad \times \Bigg(\sum_{i=1}^{m} \frac{|\theta_i|}{\Gamma(\alpha+\eta_i)} \int_a^{\delta_i} \Psi'(s) (\Psi(\delta_i)-\Psi(s))^{\alpha+\eta_i-1} \left\| f(s,x(s)) \right\| ds \\
&\quad + \frac{1}{\Gamma(2+\alpha-\gamma)} \int_a^b \Psi'(s) (\Psi(b)-\Psi(s))^{2+\alpha-\gamma} \left\| f(s,x(s)) \right\| ds \Bigg) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \Psi'(s) \Big((\Psi(t_2)-\Psi(s))^{\alpha-1} - (\Psi(t_1)-\Psi(s))^{\alpha-1} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \|f(s,x(s))\|\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha-1}\|f(s,x(s))\|\,ds\\
\leq&\frac{\left((\Psi(t_2)-\Psi(a))^{\gamma-1}-(\Psi(t_1)-\Psi(a))^{\gamma-1}\right)M\|p_f\|_\infty}{|\Lambda|\Gamma(\gamma)}\\
&\times\left(\sum_{i=1}^{m}\frac{|\theta_i|}{\Gamma(\alpha+\eta_i)}\int_a^{\delta_i}\Psi'(s)(\Psi(\delta_i)-\Psi(s))^{\alpha+\eta_i-1}\,ds\right.\\
&\left.+\frac{1}{\Gamma(2+\alpha-\gamma)}\int_a^b\Psi'(s)(\Psi(b)-\Psi(s))^{2+\alpha-\gamma}\,ds\right)\\
&+\frac{M\|p_f\|_\infty}{\Gamma(\alpha+1)}\left(((\Psi(t_2)-\Psi(a))^{\alpha}-(\Psi(t_1)-\Psi(a))^{\alpha})\right).
\end{aligned}
$$

As $t_1\to t_2$, we see that the right-hand side of the above equation tends to zero and the convergence is independent of x in Ω, which means $\Phi(\Omega)$ is equicontinuous.

Step 3. Φ is continuous.

Let $\{x_n\}$ be sequence such that $x_n\to x$ in $C(J,E)$. Then, for each $t\in J$, we have

$$
\begin{aligned}
&\|(\Phi x_n)(t)-(\Phi x)(t)\|\\
\leq&\frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\left(\sum_{i=1}^{m}|\theta_i|\,I_{a+}^{\alpha+\eta_i;\Psi}\|f(t,x_n(t))-f(t,x(t))\|\Big|_{t=\delta_i}\right.\\
&\left.+I_{a+}^{2+\alpha-\gamma;\Psi}\|f(t,x_n(t))-f(t,x(t))\|\Big|_{t=b}\right)+I_{a+}^{\alpha;\Psi}\|f(t,x_n(t))-f(t,x(t))\|.
\end{aligned}
$$

Since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies that

$$\|(\Phi x_n)(t)-(\Phi x)(t)\|\to 0 \text{ as } n\to\infty.$$

This shows that (Φx_n) converges pointwise to Φx on J. Moreover, the sequence (Φx_n) is equicontinuous by a similar proof of Step 2. Therefore (Φx_n) converges uniformly to Φx and hence Φ is continuous.

Now let V be a subset of Ω such that $V\subset\overline{conv}((\Phi V)\cup\{0\})$. V is bounded and equicontinuous, and therefore the function $v\to v(t)=\mu(v(t))$ is continuous on J. By assumption (H3), Lemma 4 and the properties of the measure μ we have for each $t \in J$

$$\begin{aligned}
v(t) &\le \mu\left((\Phi V)(t) \cup \{0\}\right) \le \mu\left((\Phi V)(t)\right) \\
&\le \frac{(\Psi(t)-\Psi(a))^{\gamma-1}}{|\Lambda|\,\Gamma(\gamma)} \left(\sum_{i=1}^{m} \|\theta_i\|\; I_{a+}^{\alpha+\eta_i;\Psi} p_f(t)\,\mu(V(t))\Big|_{t=\delta_i} \right. \\
&\quad \left. + I_{a+}^{2+\alpha-\gamma;\Psi} p_f(t)\,\mu(V(t))\Big|_{t=b} \right) + I_{a+}^{\alpha;\Psi} p_f(t)\,\mu(V(t)) \\
&\le \sum_{i=1}^{m} |\theta_i| \frac{\|v\|_\infty \|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha+\eta_i+\gamma-1}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(\alpha+\eta_i+1)} \\
&\quad + \frac{\|v\|_\infty \|p_f\|_\infty (\Psi(b)-\Psi(a))^{1+\alpha}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(3+\alpha-\gamma)} + \frac{\|v\|_\infty \|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \|v\|_\infty \left(\sum_{i=1}^{m} |\theta_i| \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha+\eta_i+\gamma-1}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(\alpha+\eta_i+1)} \right. \\
&\qquad \left. + \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{1+\alpha}}{|\Lambda|\,\Gamma(\gamma)\,\Gamma(3+\alpha-\gamma)} + \frac{\|p_f\|_\infty (\Psi(b)-\Psi(a))^{\alpha}}{\Gamma(\alpha+1)} \right) \\
&\le \|v\|_\infty k_1,
\end{aligned}$$

where k_1 is given by (3.1). This means that

$$\|v\|_\infty (1-k_1) \le 0.$$

By (3.1), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in Ω. Applying now Theorem 1, we conclude that Φ has a fixed point, which is a solution of the problem (1.1). $\square$

4. Examples

In this section, we consider some particular cases of the nonlinear fractional differential equation to apply our results in the study of existence. Consider the nonlinear fractional differential equation (FDEs) of the form

$$\begin{cases} {}^{H}D_{a+}^{\alpha,\beta,\Psi} x(t) = f(t, x(t)), \; t \in (a,b), \\ x(a) = 0, \; I_{a+}^{2-\gamma,\Psi} x(b) = \sum\limits_{i=1}^{m} \theta_i I_{a+}^{\eta_i,\Psi} x(\delta_i). \end{cases} \tag{4.1}$$

The following examples are particular cases of the (FDEs) given by (4.1).

Example 1. Consider the (FDEs) given by (4.1). Taking $\Psi(t) = t$, $\beta \to 0$, $a = 0$, $b = 1$, $\alpha = \frac{3}{2}$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{10}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$, and f is a continuous function defined by

$$f(t, x) = \frac{\exp\left(-t^2\right)}{2} x, \text{ for } x \in \mathbb{R}, \ t \in [0, 1].$$

Then, the problem (4.1) reduces to the following problem

$$\begin{cases} {}^{RL}D_{0+}^{\frac{3}{2},0,t} x(t) = \frac{1}{2\exp(t^2)} x(t), \ t \in (0, 1), \\ x(0) = 0, \ I_{0+}^{\frac{1}{2};t} x(1) = \frac{1}{2} I_{0+}^{\frac{1}{4};t} x\left(\frac{1}{4}\right) + \frac{1}{10} I_{0+}^{\frac{5}{2};t} x\left(\frac{1}{2}\right), \end{cases} \tag{4.2}$$

which is a nonlinear fractional differential equation involving Riemann-Liouville fractional derivative. In this case $\gamma = \frac{3}{2}$. Let

$$E = l^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

equipped with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, \ldots, x_n, \ldots) \text{ and } f = (f_1, f_2, \ldots, f_n, \ldots),$$

and

$$f_n(t, x_n) = \frac{1}{2\exp(t^2)} x_n, \ t \in J.$$

For each x_n and $t \in J$, we have

$$|f_n(t, x_n)| \leq \frac{1}{2\exp(t^2)} |x_n|. \tag{4.3}$$

Hence conditions (H1) and (H2) are satisfied with $p_f(t) = \frac{\exp\left(-t^2\right)}{2}$. By (4.3), for any bounded set $B \subset l^1$, we have

$$\mu(f(t, B)) \leq \frac{1}{2\exp(t^2)} \mu(B) \text{ for each } t \in [0, 1].$$

Hence (H3) is satisfied. The condition

$$\begin{aligned} k_1 &= \sum_{i=1}^{m} \|\theta_i\| \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{\alpha+\eta_i+\gamma-1}}{\|\Lambda\| \Gamma(\gamma) \Gamma(\alpha+\eta_i+1)} + \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{1+\alpha}}{\|\Lambda\| \Gamma(\gamma) \Gamma(3+\alpha-\gamma)} \\ &\quad + \frac{\|p_f\|_\infty (\Psi(b) - \Psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\ &\simeq 0.95 < 1, \end{aligned}$$

is satisfied with $\|p_f\|_\infty = \frac{1}{2}$. Consequently, Theorem 2 implies that problem (4.2) has a solution defined on $[0,1]$.

Example 2. Consider the (FDEs) given by (4.1). Taking $\Psi(t) = \log t$, $\beta \to 0$, $a = 1$, $b = e$, $\alpha = \frac{7}{4}$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{10}$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{2}$, $\delta_1 = \frac{3}{2}$, $\delta_2 = 2$ and f is continuous function defined by

$$f(t,x) = \frac{\cos(\log(t))\, x}{4t}, \text{ for } x \in \mathbb{R},\ t \in [1,e].$$

Then, the problem (4.1) reduces to the following problem

$$\begin{cases} {}^{Ha}D_{1+}^{\frac{7}{4},0,t} x(t) = \frac{\cos(\log(t))}{4t} x(t),\ t \in (1,e), \\ x(1) = 0,\ I_{0+}^{\frac{1}{4};\log t} x(e) = \frac{1}{2} I_{0+}^{\frac{1}{4};\log t} x\left(\frac{5}{2}\right) + \frac{1}{10} I_{0+}^{\frac{5}{2};\log t} x(2), \end{cases} \tag{4.4}$$

which is a nonlinear fractional differential equation involving Hadamard fractional derivative. In this case $\gamma = \frac{3}{2}$. Let

$$E = l^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

equipped with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, \ldots, x_n, \ldots) \text{ and } f = (f_1, f_2, \ldots, f_n, \ldots),$$

and

$$f_n(t, x_n) = \frac{\cos(\log(t))}{4t} x_n,\ t \in [1,e].$$

For each x_n and $t \in J$, we have

$$\left|f_n\left(t, x_n\right)\right| \leq \frac{\cos\left(\log\left(t\right)\right)}{4t}\left|x_n\right|. \tag{4.5}$$

Hence conditions (H1) and (H2) are satisfied with $p_f\left(t\right) = \frac{1}{4t}$. By (4.5) , for any bounded set $B \subset l^1$, we have

$$\mu\left(f\left(t, B\right)\right) \leq \frac{1}{4t}\mu\left(B\right) \text{ for each } t \in \left[0, 1\right].$$

Hence (H3) is satisfied. The condition

$$\begin{aligned} k_1 = &\sum_{i=1}^{m}\left\|\theta_i\right\|\frac{\left\|p_f\right\|_\infty\left(\Psi\left(b\right)-\Psi\left(a\right)\right)^{\alpha+\eta_i+\gamma-1}}{\left\|\Lambda\right\|\Gamma\left(\gamma\right)\Gamma\left(\alpha+\eta_i+1\right)} + \frac{\left\|p_f\right\|_\infty\left(\Psi\left(b\right)-\Psi\left(a\right)\right)^{1+\alpha}}{\left\|\Lambda\right\|\Gamma\left(\gamma\right)\Gamma\left(3+\alpha-\gamma\right)} \\ &+ \frac{\left\|p_f\right\|_\infty\left(\Psi\left(b\right)-\Psi\left(a\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)} \\ \simeq\, & 0.41 < 1 \end{aligned}$$

is satisfied with $\left\|p_f\right\|_\infty = \frac{1}{4}$. Consequently, Theorem 2 implies that problem (4.4) has a solution defined on $[1, e]$.

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