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A PARAMETRIZATION OF UNIPOTENT REPRESENTATIONS

GEORGE LUSZTIG

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA.
E-mail: gyuri@math.mit.edu

Abstract

We define a map from the unipotent representations of a split semisimple group over a finite field to (essentially) the set of pairs of left cell representations of the Weyl group in the same two-sided cell. We use this map to parametrize the unipotent representations.

0.1. Let G be a simple algebraic group defined and split over a finite field F_q. Let $\mathcal{U}$ be the set of isomorphism classes of irreducible unipotent representations (over $\mathbf{C}$) of the finite group $G(F_q)$. Let W be the Weyl group of G and let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations (over $\mathbf{C}$) of W. In [2] a partition of $\mathrm{Irr}(W)$ into families is described and in [4] a partition $\mathcal{U} = \sqcup_c \mathcal{U}_c$ of $\mathcal{U}$ (with c running over the families of $\mathrm{Irr}(W)$) is introduced. Moreover, in [4, §4] to any family c we have associated a finite group $\mathcal{G}_c$ and a bijection

(a) $$\mathcal{U}_c \leftrightarrow M(\mathcal{G}_c).$$

Here, for any finite group Γ, $M(\Gamma)$ is the set of Γ-conjugacy classes of pairs (x, ρ) were $x \in \Gamma$ and ρ is an irreducible representation (over $\mathbf{C}$) of the centralizer $Z_\Gamma(x)$ of x in Γ; let $\mathbf{C}[M(\Gamma)]$ (resp. $\mathbf{N}[M(\Gamma)]$) be the vector space of formal $\mathbf{C}$-linear combinations of elements in $M(\Gamma)$ and let $A_\Gamma : \mathbf{C}[M(\Gamma)] \to \mathbf{C}[M(\Gamma)]$ be the non-abelian Fourier transform of [2] (a linear isomorphism with square 1). Let $\mathbf{N}[M(\Gamma)]$ (resp. $\mathbf{R}_{\geq 0}[M(\Gamma)]$) be the set of vectors of $\mathbf{C}[M(\Gamma)]$ which are linear combinations with coefficients in $\mathbf{N}$

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(resp. $\mathbf{R}_{\geq 0}$) of elements in the basis $M(\Gamma)$ of $\mathbf{C}[M(\Gamma)]$. For any family c we write A_c instead of $A_{\mathcal{G}_c} : \mathbf{C}[M(\mathcal{G}_c)] \to \mathbf{C}[M(\mathcal{G}_c)]$.

In this paper we are interested in defining for any family c a basis β_c of the $\mathbf{C}$-vector space $\mathbf{C}[M(\mathcal{G}_c)]$ which has the properties (b)-(e) below.

(b) There is a unique bijection $\iota : M(\mathcal{G}_c) \xrightarrow{\sim} \beta_c$, $(x, \rho) \mapsto \theta(\xi, \rho)$ such that any $(x, \rho) \in M(\mathcal{G}_c)$ appears with nonzero coefficient in $\iota(\xi, \rho)$; this coefficient is actually 1.

(c) There exists a partial order $\leq$ on $M(\mathcal{G}_c)$ such that for any $(x, \rho) \in M(\mathcal{G}_c)$ we have $\iota(x, \rho) = (x, \rho$ plus an $\mathbf{N}$-linear combination of elements $(x', \rho') \in M(\mathcal{G}_c)$ which are $< (x, \rho)$. In particular we have $\beta_c \subset \mathbf{N}[M(\mathcal{G}_c)]$.

(d) We have $A_c(\beta_c) \subset \mathbf{R}_{\geq 0}[M(\mathcal{G}_c)]$.

(e) The matrix of A_c with respect to the basis β_c is upper triangular (for some partial order on β_c).

Such a basis has been constructed in [8] (where it is denoted by $\tilde{\mathbf{B}}_c$ and is called the *new basis*) extending the results of [7]; property (e) for $\tilde{\mathbf{B}}_c$ is verified in [9]. The basis β_c considered in this paper is a somewhat modified form of $\tilde{\mathbf{B}}_c$. If G is of type A, B, C we have $\beta_c = \tilde{\mathbf{B}}_c$. If G is of type D, [8] contains in addition to the definition of $\tilde{\mathbf{B}}_c$ in [8, §1], a variant of that definition given in [8, §2]; in this paper, β_c will be the same as the variant in [8, §2] (this is better suited for an extension of our results to nonsplit even orthogonal groups, as will be shown elsewhere). If G is of exceptional type and $|c| \notin \{4, 17\}$ we have $\beta_c = \tilde{\mathbf{B}}_c$. If G is of exceptional type and $|c| = 4$ (resp. $|c| = 17$), two (resp. three) elements in β_c differ from the corresponding elements in $\tilde{\mathbf{B}}_c$ (see the definition of $Prim(S_3), Prim(S_5)$ in 5.7); the reason for the change (at least in the case where $|c| = 17$) is to make formulas more symmetric.

While the definition of $\tilde{\mathbf{B}}_c$ in [8] for classical types was quite different from that for exceptional types, in this paper the definition of β_c for classical types is similar to that for exceptional types (the fact that such an approach is possible was stated without proof in [8]). The basis β_c will be called the *second basis* of $\mathbf{C}[M(\chi\ g_c)]$.

0.2. Let $\mathbf{P}$ be the set of pairs (P, R) where P is a parabolic subgroup of G defined over F_q with reductive quotient $\bar{P}$ and R is an irreducible unipotent cuspidal representation over $\mathbf{C}$ (up to isomorphism) of $\bar{P}(F_q)$. Let $\underline{\mathbf{P}}$ be the set of orbits of the conjugation action of $G(F_q)$ on $\mathbf{P}$. If $(P,R) \in \mathbf{P}$, we can view R as a representation of $P(F_q)$ and induce this from $P(F_q)$ to $G(F_q)$. The irreducible representations of $G(F_q)$ appearing in this induced representation form a subset $\mathcal{U}^{P,R}$ of $\mathcal{U}$ which depends only on the image of (P,R) in $\underline{\mathbf{P}}$. Hence for any $\mathbf{g} \in \underline{\mathbf{P}}$ the subset $\mathcal{U}^{\mathbf{g}}$ of $\mathcal{U}$ is well defined. We have $\mathcal{U} = \sqcup_{\mathbf{g}\in\underline{\mathbf{P}}}\mathcal{U}^{\mathbf{g}}$ (see [1, 3.25]).

In this paper we fix a family c of $\mathrm{Irr}(W)$.

We have a partition

(a) $$\mathcal{U}_c = \sqcup_{\mathbf{g}\in\underline{\mathbf{P}}}\mathcal{U}_c^{\mathbf{g}}$$

where $\mathcal{U}_c^{\mathbf{g}} = \mathcal{U}_c \cap \mathcal{U}^{\mathbf{g}}$. Under the bijection 0.1(a), this corresponds to a partition

(b) $$M(\mathcal{G}_c) = \sqcup_{\mathbf{g}\in\underline{\mathbf{P}}}M(\mathcal{G}_c)^{\mathbf{g}}.$$

0.3. In this paper we define a second partition of $\mathcal{U}_c$ which in some sense is transversal to the partition 0.2(a) of $\mathcal{U}_c$ (see 0.6(b)). More precisely, we will describe a collection $\mathbf{H}_c$ of subgroups of $\mathcal{G}_c$ and a certain subset $\breve{\mathbf{H}}_c$ of $\mathbf{H}_c \times \mathbf{H}_c$ such that for any $(H,H') \in \breve{\mathbf{H}}_c$, H is a normal subgroup of H'; we will also describe for any $(H,H') \in \breve{\mathbf{H}}_c$ a nonempty subset $Prim(H,H')$ of $M(H'/H)$ such that

(a) $$\Xi(c) := \{(H,H',e); (H,H') \in \breve{\mathbf{H}}_c, e \in Prim(H,H')\}$$

is in canonical bijection $\Theta : \Xi(c) \xrightarrow{\sim} M(\mathcal{G}_c)$ (see 0.6(a)) with $M(\mathcal{G}_c)$ (hence also with $\mathcal{U}_c$). If we identify $M(\mathcal{G}_c)$ with $\Xi(c)$ using this bijection, we obtain a surjective map $\alpha_c : M(\mathcal{G}_c) \to \breve{\mathbf{H}}_c$ given by $(H,H',e) \mapsto (H,H')$. For $\mathbf{h} \in \breve{\mathbf{H}}_c$ we set ${}^{\mathbf{h}}M(\mathcal{G}_c) = \alpha_c^{-1}(\mathbf{h})$. Under our identification $M(\mathcal{G}_c) = \mathcal{U}_c$, ${}^{\mathbf{h}}M(\mathcal{G}_c)$ becomes a subset ${}^{\mathbf{h}}\mathcal{U}_c$ of $\mathcal{U}_c$; our second partition of $\mathcal{U}_c$ is

(c) $$\mathcal{U}_c = \sqcup_{\mathbf{h}\in\breve{\mathbf{H}}_c}{}^{\mathbf{h}}\mathcal{U}_c.$$

0.4. The set $\mathbf{H}_c$ is in bijection with a subset Con_c^+ of $\mathbf{N}[M(\mathcal{G}_c)]$ defined as follows.

Let $\mathbf{N}[c]$ be the set of formal $\mathbf{N}$-linear combinations of elements in c. In [4, §4] an imbedding $c \subset M(\mathcal{G}_c)$ is described; this induces an imbedding $\mathbf{N}[c] \subset \mathbf{N}[M(\mathcal{G}_c)]$. Let Con_c be the set of constructible representations of W associated to c in [3]; we view the elements of Con_c as elements of $\mathbf{N}[c]$ hence, using the imbedding above, as elements of $\mathbf{N}[M(\mathcal{G}_c)]$. As shown in [5],

(a) *the representations in Con_c are precisely the representations of W carried by the left cells of W contained in the two-sided cell attached to c.*

We set

$$r_! = \sum_{(x,\rho)\in M(\mathcal{G}_c); x=1} \dim(\rho)(x,\rho) \in \mathbf{N}[M(\mathcal{G}_c)],$$

(see also 0.5(a)),

$$Con_c^+ = Con_c \cup \{r_!\} \subset \mathbf{N}[M(\mathcal{G}_c)].$$

When G is of classical type (but not in general) we have $r_! \in Con_c$ hence $Con_c^+ = Con_c$.

0.5. We explain how the subgroups in $\mathbf{H}_c$ are attached to the various elements of Con_c^+.

Let Γ be a finite group and let H be a subgroup of Γ. Following [4, p.312] we define a linear map $i_{H,\Gamma} : \mathbf{C}[M(H)] \to \mathbf{C}[M(\Gamma)]$ by

$$(x,\sigma) \mapsto \sum_\rho \Big(\rho : \mathrm{Ind}_{Z_H(x)}^{Z_\Gamma(x)}(\sigma)\Big)(x,\rho).$$

where ρ runs over the irreducible representations of $Z_\Gamma(x)$ up to isomorphism and : denotes multiplicity.

Assume now that H is a normal subgroup of Γ; let $\pi : \Gamma \to \Gamma/H$ be the canonical map. Following *loc.cit.* we define a linear map $\pi_{H,\Gamma}$: $\mathbf{C}[M(\Gamma/H)] \to \mathbf{C}[M(\Gamma)]$ by

$$(x,\sigma) \mapsto \sum_{y\in\pi^{-1}(x)} \sum_{\tau\in \mathrm{Irr}(Z_\Gamma(y))} |Z_\Gamma(y)||Z_{\Gamma/H}(x)|^{-1}|H|^{-1}(\tau:\sigma)(y,\tau)$$

where τ runs over the irreducible representations of $Z_\Gamma(y)$ up to isomorphism and $(\tau : \sigma)$ denotes the multiplicity of τ in σ viewed as a representation of $Z_\Gamma(y)$ via the obvious homomorphism $Z_\Gamma(y) \to Z_{\Gamma/H}(x)$. Now let $H \subset H'$ be two subgroups of Γ such that H is normal in H'. We define a linear map $\mathbf{s}_{H,H';\Gamma} : \mathbf{C}[M(H'/H)] \to \mathbf{C}[M(\Gamma)]$ by $f \mapsto i_{H',\Gamma}(\pi_{H,H'}(f))$. Note that

(a) $r_! = i_{\{1\},\Gamma}(1,1) = \mathbf{s}_{\{1\},\{1\};\Gamma}(1,1)$.

In [6] to each $r \in Con_c$ we have attached a conjugacy class $[r]$ of subgroups of $\mathcal{G}_c$. Its main property is that for any $H \in [r]$ we have

(b) $r = \mathbf{s}_{H,H;\mathcal{G}_c}(1,1)$.

When $r = r_!$ we define $[r]$ to consist of $\{1\}$; then (b) continues to hold (see (a)). Thus $[r]$ is defined for any $r \in Con_c^+$. For each r we choose a specific $H(r) \in [r]$ as follows. When G is of classical type, $\mathcal{G}_c$ is abelian and $H(r)$ is the unique subgroup in $[r]$; when G is of exceptional type, the subgroups $H(r)$ are described in §5.

By definition, we have

$$\mathbf{H}_c = \{H(r); r \in Con_c^+\}.$$

Note that $r \mapsto H(r)$ is a bijection $Con_c^+ \xrightarrow{\sim} \mathbf{H}_c$.

0.6. The subset $\breve{\mathbf{H}}_c$ of $\mathbf{H}_c \times \mathbf{H}_c$ and the subsets $Prim(H, H')$ of $M(H'/H)$ (for $(H, H') \in \breve{\mathbf{H}}_c$) in 0.3 are described:

- (i) in 3.7 when G is isogenous to a symplectic or odd special orthogonal group;
- (ii) in 3.8 when G is isogenous to an even special orthogonal group;
- (iii) in §5 when G is of exceptional type.

(When G is of type A, $\mathbf{H}_c$ consists of $\{1\}$, $\breve{\mathbf{H}}_c$ consists of $(\{1\}, \{1\})$ and Ξ_c consists of one element.)

We now state one of our main results.

(a) *There is a unique bijection* $\Xi_c \xrightarrow{\Theta} M(\mathcal{G}_c)$ *(notation of* 0.3(a)) *such that for any* $(H, H', e) \in \Xi_c$*, the element* $\Theta(H, H', e) \in M(\mathcal{G}_c)$ *appears with nonzero coefficient in* $\boldsymbol{s}_{H,H';\mathcal{G}_c}(e) \in \boldsymbol{C}[M(\mathcal{G}_c)]$.

In view of 0.1(a), this can be viewed as a parametrization of $\mathcal{U}_c$.

The proof of (a) when G is as in (i) is given in 3.7; it is based on the results on the new basis in [8]. The proof of (a) when G is as in (ii) is given in 3.8; it is based on an extension of the results on the new basis in [8]. The proof of (a) when G is as in (iii) is given in §5; a result close to (a) was already known in this case from [8]. When G is of type A, the statement (a) is obvious.

Here is one of the main properties of the bijection Θ.

(b) *The partition 0.3(b) of $\mathcal{U}_c$ is transversal to the partition 0.2(a) of $\mathcal{U}_c$, in the sense that for any $\boldsymbol{g} \in \underline{\boldsymbol{P}}$ and any $\boldsymbol{h} \in \breve{\boldsymbol{H}}_c$ we have $|{}^{\boldsymbol{h}}\mathcal{U}_c \cap \mathcal{U}_c^{\boldsymbol{g}}| \le 1$.*

From the proof of (a) and the results of [8] we see that
$\beta_c := \{\mathbf{s}_{H,H';\mathcal{G}_c}(e); (H, H', e) \in \Xi_c\}$ (see 0.6(a)) is a basis of $\mathbf{C}[M(\mathcal{G}_c)]$ (which could be called the *second basis* of $\mathbf{C}[M(\mathcal{G}_c)]$) in bijection with Ξ_c and which satisfies 0.1(b)-(d); it is also in canonical bijection

$\mathbf{s}_{H,H';\mathcal{G}_c}(e) \mapsto \Theta(H, H', e)$ with the obvious basis of $\mathbf{C}[M(\mathcal{G}_c)]$) consisting of the various elements of $M(\mathcal{G}_c)$. From [9] or some variant of it (in type D) we see that β_c satisfies 0.1(e).

For any $(x, \rho) \in M(\mathcal{G}_c)^{\mathbf{g}}$ (see 0.2(b)) let $g_{x,\rho}$ be the element of the second basis corresponding as above to (x, ρ); we define an element $f_{x,\rho} \in \mathbf{C}[M(\mathcal{G}_c)]$ by the following requirement: for any $(x', \rho') \in M(\mathcal{G}_c)^{\mathbf{g}'}$ the coefficient of (x', ρ') in $f_{x,\rho}$ is equal to the coefficient of (x', ρ') in $g_{x,\rho}$ if $\mathbf{g} = \mathbf{g}'$ and is equal to 0 if $\mathbf{g} \ne \mathbf{g}'$.

We have the following result.

(c) *$\{f_{x,\rho}; (x, \rho) \in M(\mathcal{G}_c)\}$ is a basis of $\boldsymbol{C}[M(\mathcal{G}_c)]$.*

(This could be called the *third* basis of $\mathbf{C}[M(\mathcal{G}_c)]$.)

The proof in the case 0.6(i) is given in §7. The proof in the case 0.6(ii) is similar. The proof in the case 0.6(iii) is obtained by examining the tables.

0.7. Let $\mathbf{g} \in \underline{\mathbf{P}}$. Let

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{\mathbf{h} \in \breve{\mathbf{H}}_c; |{}^{\mathbf{h}}\mathcal{U}_c \cap \mathcal{U}_c^{\mathbf{g}}| = 1\}$$

.

From 0.6(a) we see that we have a bijection

(a) $\breve{\mathbf{H}}_c^{\mathbf{g}} \to \mathcal{U}_c^{\mathbf{g}}$

which to any $\mathbf{h} \in \breve{\mathbf{H}}_c^{\mathbf{g}}$ associates the unique element in ${}^{\mathbf{h}}\mathcal{U}_c \cap \mathcal{U}_c^{\mathbf{g}}$. We thus obtain the following statement which is one of the main results of this paper.

(b) *$\mathcal{U}_c^{\mathbf{g}}$ is in natural bijection with a subset of $\breve{\boldsymbol{H}}_c$, hence with a subset of $\boldsymbol{H}_c \times \boldsymbol{H}_c$.*

0.8. A key role in this paper is played by the subset $\breve{\mathbf{H}}_c$ of $\mathbf{H}_c \times \mathbf{H}_c$. As mentioned above, $\mathbf{H}_c$ can be identified with Con_c^+. Therefore $\breve{\mathbf{H}}_c$ can be identified with a subset of $Con_c^+ \times Con_c^+$ which we can denote by $\breve{Con}_c^+$ and also with the image of this set in $(Con_c^+ \times Con_c^+)_{unord}$ (unordered pairs in $Con_c^+ \times Con_c^+$).

We will show elsewhere that $\breve{Con}_c^+$ admits a direct (inductive) definition.

We can restate 0.7(b) to say that, if $\mathbf{g} \in \underline{\mathbf{P}}$, then

(a) *$\mathcal{U}_c^{\mathbf{g}}$ is in natural bijection with a subset of $\breve{Con}_c^+$, hence with a subset of $(Con_c^+ \times Con_c^+)_{unord}$.*

0.9. *Erratum to* [8]. In the first displayed equality of 3.2 replace $Z_\Gamma(z)$ by $Z_\Gamma(x)$. In line 8 of 3.7 replace H_{221} by H_{211}.

0.10. *Notation.* $\mathbf{F}$ denotes the field with two elements. For $a, b \in \mathbf{Z}$ we set $[a, b] = \{z \in \mathbf{Z}; a \le z \le b\}$. We write $a \ll b$ instead of $b - a \ge 2$.

Contents

1. The set $\boldsymbol{\mathcal{F}}(V_D)$

1.1. We fix $D \in \mathbf{N}$. Let V_D be the $\mathbf{F}$-vector space with basis $\{e_i; i \in [1, D]\}$. When $D' \in [0, D]$ we identify $V_{D'}$ with the subspace of V_D with basis $\{e_i, i \in [1, D']\}$. When $D \geq 2$, for any $i \in [1, D]$ there is a unique linear map $T_i : V_{D-2} \to V_D$ such that

$T_i(e_k) = e_k$ if $k \in [1, i-2]$,

$T_i(e_k) = e_{k+2}$ if $k \in [i, D-2]$,

$T_i(e_{i-1}) = e_{i-1} + e_i + e_{i+1}$ if $1 < i < D$.

This map is injective.

If $v_1, v_2, \dots, v_k$ are vectors in V_D we denote by $\langle v_1, v_2, \dots, v_k \rangle$ the subspace of V_D generated by $v_1, v_2, \dots, v_k$.

1.2. A subset $I \subset \mathbf{Z}$ is said to be an *interval* if it is of the form $[a, b]$ for some $a \leq b$ in $\mathbf{Z}$. Let $\mathcal{I}_D$ be the set of intervals contained in $[1, D]$. We say that $I = [a, b], I' = [a', b']$ in $\mathcal{I}_D$ are non-touching if $b \ll a'$ or if $b' \ll a$. For $I = [a, b], I' = [a', b']$ in $\mathcal{I}_D$ we write $I \prec I'$ if $a' < a \leq b < b'$.

Let R_D be the set whose elements are the subsets of $\mathcal{I}_D$. We define a subset $\mathbf{S}_D^{prim}$ of R_D as follows. When D is even, $\mathbf{S}_D^{prim}$ consists of $\emptyset$ and of

(a) $$\{[1, D], [2, D-1], \dots, [k, D+1-k]\}$$

for various $k \in [1, D/2]$. When D is odd, $\mathbf{S}_D^{prim}$ consists of $\emptyset$, of

(b) $$\{[1, D-1], [2, D-2], \dots, [k, D-k]\}$$

and of

(c) $$\{[2, D], [3, D-1], \dots, [k+1, D+1-k]\}$$

for various odd $k \in [1, (D-1)/2]$.

For example, if $D = 2$, $\mathbf{S}_D^{prim}$ consists of $\emptyset, \{[1, 2]\}$.

If $D = 4$, $\mathbf{S}_D^{prim}$ consists of $\emptyset, \{[1, 4]\}, \{[1, 4], [2, 3]\}$.

If $D = 6$, $\mathbf{S}_D^{prim}$ consists of $\emptyset, \{[1, 6]\}, \{[1, 6], [2, 5]\}, \{[1, 6], [2, 5], [3, 4]\}$.

If $D = 3$, $\mathbf{S}_D^{prim}$ consists of $\emptyset, \{[1, 2]\}, \{[2, 3]\}$.

If $D=5$, $\mathbf{S}_D^{prim}$ consists of $\emptyset, \{[1,4]\}, \{[2,5]\}$.

If $D=7$, $\mathbf{S}_D^{prim}$ consists of

$\emptyset, \{[1,6]\}, \{[2,7]\},\ \{[1,6],[2,5],[3,4]\}, \{[2,7],[3,6],[4,5]\}$.

The elements of $\mathbf{S}_D^{prim}$ are said to be *primitive.*

When $D \geq 2$ and $i \in [1,D]$ we define an (injective) map $\xi_i : \mathcal{I}_{D-2} \to \mathcal{I}_D$ by

$\xi_i([a',b']) = [a'+2, b'+2]$ if $i \leq a'$,

$\xi_i([a',b']) = [a',b']$ if $i \geq b'+2$,

$\xi_i([a',b']) = [a',b'+2]$ if $a' < i < b'+2$.

We define $t_i : R_{D-2} \to R_D$ by $B' \mapsto \{\xi_i(I'); I' \in B'\} \sqcup \{i\}$ (see [8, 1.1]).

We define a subset $\mathbf{S}_D$ of R_D by induction on D as follows.

If $D=0$, $\mathbf{S}_D$ consists of $\emptyset \in R_D$. If $D=1$, $\mathbf{S}_D$ consists of $\emptyset$ and of $\{1\} \in R_D$. If $D \geq 2$, a subset B of $\mathcal{I}_D$ is in $\mathbf{S}_D$ if either $B \in \mathbf{S}_D^{prim}$ or if there exists $i \in [1,D]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$. (When D is even this definition appears in [8].)

We note the following result (when D is even this appears in [8]).

(d) *If $B \in \boldsymbol{S}_D$ and $I \in B, \tilde{I} \in B$, then either $I = \tilde{I}$ or I, I' are non-touching or $I \prec \tilde{I}$ or $\tilde{I} \prec I$.*

We use induction on D. If B is primitive then (d) is obvious. We now assume that B is not primitive. Then $D \geq 2$ and there exists $i \in [1,D]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$. By the induction hypothesis, (d) holds when B is replaced by B'. It follows immediately that (d) holds for B.

For any $I \subset [1,D]$ we set $e_I = \sum_{i \in I} e_i \in V_D$. For $B \in \mathbf{S}_D$ let $\mathbf{E}_B$ be the subspace of V_D spanned by $\{e_I; I \in B\}$. Let $\boldsymbol{\mathcal{F}}(V_D)$ be the set of subspaces of V_D of the form $\mathbf{E}_B$ for some $B \in \mathbf{S}_D$. We have the following result (when D is even this appears in [8]).

(e) *If $B \in \boldsymbol{S}_D$ then $\{e_I; I \in B\}$ is a basis of $\boldsymbol{E}_B$.*

We can assume that $D \geq 2$. Assume that there exists a nonempty subset $\mathcal{X} \subset \mathcal{B}$ such that $\sum_{I \in \mathcal{X}} e_I = 0$. Let $I_1 = [a,b] \in \mathcal{X}$ be such that $|I_1|$ is maximum. If $I_2 \in \mathcal{X}$, $I_2 \neq I_1$ and $a \in I_2$, then $a \in I_1 \cap I_2$ so that by (d) we have $I_2 \prec I_1$ or $I_1 \prec I_2$; now $I_2 \prec I_1$ contradicts $a \in I_1, a \in I_2$ and $I_1 \prec I_2$

contradicts the maximality of $|I_1|$. We see that if $I_2 \in \mathcal{X}$, $I_2 \neq I_1$, then $a \notin I_2$ (and similarlly $b \notin I_2$). It follows that the coefficient of e_a (and that of e_b) in $\sum_{I\in\mathcal{X}} e_I$ is 1. But the last sum is zero. This contradiction proves (e).

We have the following result.

(f) *If $B \in \boldsymbol{S}_D$ and $J \in \mathcal{I}_D$ is such that $e_J \in \boldsymbol{E}_B$ then $J \in B$.*

Assume that $J \notin B$. We can find a nonempty subset $\mathcal{X} \subset \mathcal{B}$ such that $\sum_{I\in\mathcal{X}} e_I = e_J$. Let $I_1 = [a,b] \in \mathcal{X}$ be such that $|I_1|$ is maximum. As in the proof of (e) we see that the coefficient of e_a (and that of e_b) in $\sum_{I\in\mathcal{X}} e_I$ is 1. Since the last sum is equal to e_J it follows that $a \in J, b \in J$. Since J is an interval we have $[a,b] \subset J$. Since J is an interval different from $[a,b]$ we must have $a-1 \in J$ or $b+1 \in J$. Assume first that $a-1 \in J$. Since $\sum_{I\in\mathcal{X}} e_I = e_J$ we can find $I_2 \in \mathcal{X}$ such that $a-1 \in I_2$. We have $I_2 \not\subset I_1$ (since $a-1 \in I_2, a \notin I_1$); we cannot have $I_1 \prec I_2$ (this would contradict the maximality of $|I_1|$). Using (d) we deduce that I_1, I_2 are non-touching, but this contradicts $a-1 \in I_2, I_1 = [a,b]$. We see that $a-1 \in J$ leads to a contradiction. Similarly $b+1 \in J$ leads to a contradiction. This contradiction proves (f).

(g) *The map $\boldsymbol{S}_D \to \boldsymbol{\mathcal{F}}(V_D)$, $B \mapsto \boldsymbol{E}_B$ is a bijection.*

It is enough to show that this map is injective. Let $B \in \mathbf{S}_D, B' \in \mathbf{S}_D$ be such that $\mathbf{E}_B = \mathbf{E}_{B'}$. By (f) we have

$B = \{J \in \mathcal{I}_D; e_J \in \mathbf{E}_B\} = \{J \in \mathcal{I}_D; e_J \in \mathbf{E}_{B'}\} = B'$. We see that $B = B'$. This proves (g).

In the case where D is even, (g) appears also in [8] but the present proof is simpler.

1.3. The set $\boldsymbol{\mathcal{F}}(V_D)$ has an alternative definition (by induction on D). When $D \geq 2$ we define a set $P(V_D)$ of subspaces of V_D (said to be *primitive* subspaces) as follows. If D is even, $P(V_D)$ consists of $\{0\}$ and of the subspaces $\mathbf{E}_B$ with B as in 1.2(a) and $k \in [1, D/2]$. If D is odd, $P(V_D)$ consists of $\{0\}$ and of the subspaces $\mathbf{E}_B$ with B as in 1.2(b) or 1.2(c) and odd $k \in [1,(D-1)/2]$. If $D = 0$, $\boldsymbol{\mathcal{F}}(V_D)$ consists of the subspace $\{0\}$. If $D = 1$, $\boldsymbol{\mathcal{F}}(V_D)$ consists of the subspace $\{0\}$ and of V. If $D \geq 2$, a subspace $\mathbf{E}$

of V_D is in $\boldsymbol{\mathcal{F}}(V_D)$ if it is either primitive or if there exists $i \in [1, D]$ and $\mathbf{E}' \in \boldsymbol{\mathcal{F}}(V_{D-2})$ such that $\mathbf{E} = T_i(\mathbf{E}') \oplus \mathbf{F}e_i$.

1.4. For $\delta \in \{0, 1\}$ we set $\mathcal{I}_D^\delta = \{I \in \mathcal{I}_D; |I| = \delta \bmod 2\}$. For $B \in \mathbf{S}_D$ we set $B^\delta = B \cap \mathcal{I}_D^\delta$. Let

$$S_D = \{B \in \mathbf{S}_D; B = B^1\}.$$

We can also define S_D by induction on D as follows. We have $S_0 = \{\emptyset\}$; S_1 consists of $\{\emptyset\}$ and $\{1\}$. If $D \geq 2$ then a subset B of $\mathcal{I}_D$ is in S_D if either $B = \emptyset$ or if there exists $i \in [1, D]$ and $B' \in S_{D-2}$ such that $B = t_i(B')$.

We define a collection $\mathcal{F}(V_D)$ of subspaces of V_D by induction on D as follows. If $D = 0$, $\mathcal{F}(V_D)$ consists of $\{0\}$. If $D = 1$, $\mathcal{F}(V_D)$ consists of $\{0\}$ and V_D. If $D \geq 2$, a subspace E of V_D is said to be in $\mathcal{F}(V_D)$ if either $E = 0$ or if there exists $i \in [1, D]$ and $E' \in \mathcal{F}(V_{D-2})$ such that $E = T_i(E') \oplus \mathbf{F}e_i$. We have $\mathcal{F}(V_D) \subset \boldsymbol{\mathcal{F}}(V_D)$. Note that, for $\mathbf{E} \in \boldsymbol{\mathcal{F}}(V_D)$ with $\mathbf{E} = \mathbf{E}_B$, $B \in \mathbf{S}_D$ we have $\mathbf{E} \in \mathcal{F}(V_D)$ if and only $B \in S_D$.

1.5. Let $\delta \in \{0, 1\}$. Let $\mathbf{Z}^\delta = \delta + 2\mathbf{Z}$. For any $I \in \mathcal{I}_D$ we set $I^\delta = I \cap \mathbf{Z}^\delta$; we have $I = I^0 \sqcup I^1$. Let $V_D^\delta = \langle e_i; i \in [1, D]^\delta \rangle$. We have $V_D = V_D^0 \oplus V_D^1$.

Assume that $D \geq 2$, $i \in [1, D]$. There is a unique linear map $T_i^\delta : V_{D-2}^\delta \to V_D^\delta$ such that

$T_i^\delta(e_k) = e_k$ if $k \in [1, i-2]^\delta$;

$T_i^\delta(e_k) = e_{k+2}$ if $k \in [i, D-2]^\delta$;

$T_i^\delta(e_{i-1}) = e_{i-1} + e_{i+1}$ if $i \in [2, D-1]^{1-\delta}$.

Note that for $x \in V_{D-2}^0, y \in V_{D-2}^1$ we have

(a) $T_i(x + y) = T_i^0(x) + T_i^1(y) \bmod \mathbf{F}e_i$.

1.6. We define a collection $\mathcal{C}(V_D^\delta)$ of subspaces of V_D^δ by induction on D as follows. If $D = 0$, $\mathcal{C}(V_D^\delta)$ consists of $\{0\}$. If $D = 1$, $\delta = 0$, $\mathcal{C}(V_D^\delta)$ consists of $\{0\}$. If $D = 1$, $\delta = 1$, $\mathcal{C}(V_D^\delta)$ consists of $\{0\}$ and V_D^δ. If $D \geq 2$, a subspace $\mathcal{L}$ of V_D^δ is said to be in $\mathcal{C}(V_D^\delta)$ if either there exists $i \in [1, D]^\delta$ and $\mathcal{L}' \in \mathcal{C}(V_{D-2}^\delta)$ such that $\mathcal{L} = T_i^\delta(\mathcal{L}') \oplus \mathbf{F}e_i$, or there exists $i \in [1, D]^{1-\delta}$ and $\mathcal{L}' \in \mathcal{C}(V_{D-2}^\delta)$ such that $\mathcal{L} = T_i^\delta(\mathcal{L}')$.

For $E \in \mathcal{F}(V_D)$ we set $E^\delta = E \cap V_D^\delta$. Then

(a) $E^\delta \in \mathcal{C}(V_D^\delta)$, $E = E^0 \oplus E^1$.

When D is even this is shown in [7, 2.2(c), 2.3(b)]. The case where D is odd is similar.

1.7. We define by an induction on D a collection $\tilde{\mathcal{F}}(V_D)$ of pairs $(\mathcal{M}^0, \mathcal{M}^1)$ where $\mathcal{M}^0, \mathcal{M}^1$ are subspaces of V_D^0, V_D^1 respectively. If $D = 0$, $\tilde{\mathcal{F}}(V_D)$ consists of $(\{0\}, \{0\})$. If $D = 1$, $\tilde{\mathcal{F}}(V_D)$ consists of $(\{0\}, \{0\})$ and $(\{0\}, V_D^1)$. If $D \geq 2$ a pair $(\mathcal{M}^0, \mathcal{M}^1)$ of subspaces of V_D^0, V_D^1 is said to be in $\tilde{\mathcal{F}}(V_D)$ if either $(\mathcal{M}^0, \mathcal{M}^1) = (\{0\}, \{0\})$ or there exists $i \in [1, D]$ and $(\mathcal{M}'^0, \mathcal{M}'^1) \in \tilde{\mathcal{F}}(V_{D-2})$ such that

if $\delta = i \bmod 2$ then $\mathcal{M}^\delta = T_i^\delta(\mathcal{M}'^\delta) \oplus \mathbf{F}e_i$;

if $\delta = i + 1 \bmod 2$, then $\mathcal{M}^\delta = T_i^\delta(\mathcal{M}'^\delta)$.

Using 1.5(a) and the definitions we see that

(a) $E \mapsto (E^0, E^1)$ is a bijection $\mathcal{F}(V_D) \xrightarrow{\sim} \tilde{\mathcal{F}}(V_D)$. Moreover, if $(\mathcal{M}^0, \mathcal{M}^1) \in \tilde{\mathcal{F}}(V_D)$, then $\mathcal{M}^\delta \in \mathcal{C}(V_D^\delta)$ for $\delta = 0, 1$.

1.8. There is a unique symplectic form $(,) : V_D \times V_D \to \mathbf{F}$ such that for i, j in $[1, D]$ we have $(e_i, e_j) = 1$ if $i - j = \pm 1$, $(e_i, e_j) = 0$ if $i - j \neq \pm 1$. This form is nondegenerate if D is even and has a one dimensional radical spanned by

$$\eta_D := e_1 + e_3 + e_5 + \cdots + e_D$$

if D is odd. The next result follows from 1.2(d).

(a) *If $B \in \boldsymbol{S}_D$, then $(,)$ is identically zero on $\boldsymbol{E}_B \times \boldsymbol{E}_B$,* If $D \geq 2$ then for $i \in [1, D]$ we have

(b) $(x, y) = (T_i(x), T_i(y))$ for any x, y in V_{D-2},

(c) $(T_i^0(x), T_i^1(y)) = (x, y)$ for any $x \in V_{D-2}^0, y \in V_{D-2}^1$.

For any subspace Z of V_D we set $Z^\perp = \{x \in V_D; (x, y) = 0 \quad \forall y \in Z\}$. When $Z \subset V_D^{1-\delta}$, we set $Z^! = \{x \in V_D^\delta; (x, y) = 0 \quad \forall y \in Z\} = Z^\perp \cap V_D^\delta$. Let

$\mathcal{F}_*(V_D) = \{E \in \mathcal{F}(V_D); \dim(E) = D/2\}$ (if D is even),

$\mathcal{F}_*(V_D) = \{E \in \mathcal{F}(V_D); \dim(E) = (D+1)/2\}$ (if D is odd).

We have the following result.

(d) *Assume that D is even or that D is odd and $\delta = 0$. If $\mathcal{L} \in \mathcal{C}(V_D^\delta)$, then $\mathcal{L}^! \in \mathcal{C}(V_D^{1-\delta})$ and $\mathcal{L} \oplus \mathcal{L}^! \in \mathcal{F}(V_D)$. Moreover, $\mathcal{L} \mapsto \mathcal{L} \oplus \mathcal{L}^!$ is a bijection $\mathcal{C}(V_D^\delta) \to \mathcal{F}_*(V_D)$.*

When D is even this is proved in [7]. The proof for D odd is similar.

We show:

(e) *If D is odd, then $|\mathcal{F}_*(V_D)| = |\mathcal{F}_*(V_{D-1})|$.*

Using (d) we see that it is enough to show that $|\mathcal{C}(V_D^0)| = |\mathcal{C}(V_{D-1}^0)|$. But from the definitions we actually have $V_D^0 = V_{D-1}^0$ and $\mathcal{C}(V_D^0) = \mathcal{C}(V_{D-1}^0)$.

1.9. Let $\delta \in \{0, 1\}$. Assume that D is even or that D is odd and $\delta = 1$. We define a collection $\breve{\mathcal{C}}(V_D^\delta)$ of pairs $(\mathcal{L} \subset \tilde{\mathcal{L}})$ of subspaces of V_D^δ. Namely, $\breve{\mathcal{C}}(V_D^\delta)$ consists of all pairs $(\mathcal{M}^\delta \subset (\mathcal{M}^{1-\delta})^!)$ with $(\mathcal{M}^0, \mathcal{M}^1) \in \tilde{\mathcal{F}}(V_D)$. (We use that $\mathcal{M}^0 \oplus \mathcal{M}^1$ is an isotropic subspace of V_D, see 1.7(a), 1.8(a).)

From 1.7(a), 1.8(d) we see that

(a) $\breve{\mathcal{C}}(V_D^\delta) \subset \mathcal{C}(V_D^\delta) \times \mathcal{C}(V_D^\delta)$.

From 1.7(a) we see that we have a bijection

(b) $\mathcal{F}(V_D) \xrightarrow{\sim} \breve{\mathcal{C}}(V_D^\delta)$, $E \mapsto (E^\delta \subset (E^{1-\delta})^!)$.

The inverse of the map (b) is $(\mathcal{L} \subset \tilde{\mathcal{L}}) \mapsto \mathcal{L} \oplus \tilde{\mathcal{L}}^!$.

1.10. In this subsection we assume that $D \geq 1$. We show:

(a) *If $B \in S_{D-1}$ then $B \in S_D$.*

This makes sense since $\mathcal{I}_{D-1} \subset \mathcal{I}_D$. We argue by induction on D. If $D \leq 2$, the result is obvious. Assume now that $D \geq 3$. The maps $R_{D-3} \to R_{D-1}$ analogous to $t_i : R_{D-2} \to R_D$ (see 1.2) are denoted by $\tilde{t}_i$; they are defined for $i \in [1, D-1]$. If $B = \emptyset$ the result is obvious. Thus we can assume that $B \neq \emptyset$. We can find $B' \in S_{D-3}$ and $i \in [1, D-1]$ such that $B = \tilde{t}_i(B')$. From the definitions we have $\tilde{t}_i(B') = t_i(B')$ so that $B = t_i(B')$. By the induction hypothesis we have $B' \in S_{D-2}$ so that $B \in S_D$. This proves (a).

Note that $\mathbf{S}_{D-1}$ is not in general contained in $\mathbf{S}_D$.

1.11. We now give an alternative (non-inductive) definition fo S_D.

(a) *Let $B \in R_D$ be such that $|I|$ is odd for any $I \in B$. Then $B \in S_D$ if and only if B satisfies properties $(P_0), (P_1)$ below.*

(P_0). *If $I \in B, \tilde{I} \in B$, then either $I = \tilde{I}$, or I, I' are non-touching, or $I \prec \tilde{I}$, or $\tilde{I} \prec I$.*

(P_1). *If $[a, b] \in B$ and $c \in \boldsymbol{N}$ satisfy $a < c < b$, $c \neq a \bmod 2$, then there exists $[a_1, b_1] \in B$ such that $a < a_1 \leq c \leq b_1 < b$.*

When D is even this is proved in [7, 1.3(c)]. The proof in the case where D is odd is similar. (The fact that any $B \in S_D$ satisfies (P_0) is also contained in 1.2(d).)

Note that (a) provides an alternative proof of the inclusion $S_{D-1} \subset S_D$ in 1.10.

2. The bijection $\mu : \underline{\boldsymbol{\mathcal{F}}}(V_D) \xrightarrow{\sim} \boldsymbol{\mathcal{F}}(V_D)$

2.1. Until the end of 2.6 we fix $B \in S_D$. Let $E = \mathbf{E}_B \in \mathcal{F}(V_D)$. We have $|B| \leq D/2$ if D is even, $|B| \leq (D+1)/2$ if D is odd. We set $\sigma \in \cup_{I \in B}$. We can write uniquely

$$\sigma = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_s, b_s]$$

where $1 \leq a_1 \leq b_1 \ll a_2 \leq b_2 \ll \cdots \ll a_s \leq b_s \leq D$, $[a_1, b_1] \in B, [a_2, b_2] \in B, \ldots, [a_s, b_s] \in B$. Note that $c_i := b_i - a_i \in 2\mathbf{N}$ for $i = 1, \ldots, s$. We set $b_0 = -1, a_{s+1} = D + 2$. Note that

$$a_i - b_{i-1} \geq 2 \text{ for } i \in [1, s+1].$$

We have

(a) $$|B| = \sum_{1 \in [1,s]} (c_i + 2)/2.$$

This can be proved by induction on D in the same way as [8, 1.3(g)].

In the remainder of this subsection we assume that D is odd. We show:

(b) *If $\eta_D \in E$ (notation of 1.8) then $|B| = (D+1)/2$.*

By our assumption, $\cup_{I\in B}$ contains $\{1,3,\dots,D\}$. It follows that

$a_1, b_1, a_2, b_2, \dots, a_s, b_s$

are all odd and

$$a_1 = 1, a_2 = b_1 + 2, a_3 = b_2 + 2, \dots, a_s = b_{s-1} + 2, b_s = D.$$

Using (a) we deduce

$$\begin{aligned} 2|B| &= \sum_{1\in[1,s]} (b_i - a_i + 2) = -1 - 2 - 2 - \cdots - 2 + 2s + D \\ &= -1 - 2(s-1) + 2s + D = D + 1 \end{aligned}$$

and (b) is proved.

We show

(c) *If* $|B| \le (D-1)/2$ *then* $\dim(E^\perp) = D - \dim(E)$.

Since the radical of $(,)$ is spanned by η_D, it is enough to show that $\eta_D \notin E$. This follows from (b).

We show:

(d) *If* $|B| = (D+1)/2$*, then* $\eta_D \in E$.

We argue by induction on D. If $D = 1$ we have $E = V_1$ so that there is nothing to prove. Assume now that $D \ge 3$. We can find $i \in [1, D]$ and $E' \in \mathcal{F}(V_{D-2})$ such that $E = T_i(E') \oplus \mathbf{F}e_i$. We have $\dim(E') = (D-1)/2$ hence by the induction hypothesis we have $\eta_{D-2} \in E'$. From the definitions we have $T_i(\eta_{D-2}) = \eta_D \bmod \mathbf{F}e_i$ so that $\eta_D \in E$. This proves (d).

We show

(e) *If* $|B| = (D-1)/2$*, then* $\eta_D \notin E$ *and* $\boldsymbol{F}\eta_D \oplus E \in \mathcal{F}(V_D)$.

The first assertion follows from (b). For the second assertion we argue by induction on D. If $D = 1$, we have $E = 0$ and there is nothing to prove. Assume now that $D \ge 3$. We can find $i \in [1, D]$ and $E' \in \mathcal{F}(V_{D-2})$ such that $E = T_i(E') \oplus \mathbf{F}e_i$. We have $\dim(E') = (D-3)/2$ hence by the induction hypothesis we have $\mathbf{F}\eta_{D-2} + E' \in \mathcal{F}(V_{D-2})$. It follows that $\mathbf{F}T_i(\eta_{D-2}) + T_i(E') + \mathbf{F}e_i \in \mathcal{F}(V_D)$ hence (as in the proof of (d)) we have $\mathbf{F}\eta_D + T_i(E') + \mathbf{F}e_i \in \mathcal{F}(V_D)$, that is $\mathbf{F}\eta_D + E \in \mathcal{F}(V_D)$. This proves (e).

We show:

(f) *If* $\dim(E) = (D+1)/2$, *then there is a unique* $E' \in \mathcal{F}(V_{D-1})$ *such that* $\dim(E') = (D-1)/2$ *and* $E = \boldsymbol{F}\eta_D + E'$.

We define B' to be B from which the unique interval of B containing D is removed. We have $B' \in S_{D-1}$ (viewed as a subset of S_D, see 1.10). Let $E' \in \mathcal{F}(V_{D-1})$ be the subspace of V_{D-1} with basis $\{e_I; I \in B'\}$. We have $\dim(E') = (D-1)/2$ and $E' \subset E$. By (d) we have $\eta_D \in E$. It follows that $\mathbf{F}\eta_D \oplus E' \subset E$ (the sum is direct by (e)). Since $\dim(\mathbf{F}\eta_D \oplus E') = \dim(E) = (D+1)/2$, it follows that $\mathbf{F}\eta_D \oplus E' = E$. This proves the existence in (f). We now define a map

$$\{E'_1 \in \mathcal{F}(V_{D-1}); \dim(E'_1) = (D-1)/2\} \to \{E_1 \in \mathcal{F}(V_D); \dim(E_1) = (D+1)/2\}$$

by $E'_1 \mapsto \mathbf{F}\eta_D + E'_1$. This map is well defined by (e) and is surjective by the first part of the proof. Our map is between two finite sets of the same cardinal (see 1.8(e)) hence it is a bijection. This proves (f).

From (e), (f), we see that we have a bijection $\mathcal{F}_*(V_{D-1}) \xrightarrow{\sim} \mathcal{F}_*(V_D)$ given by $E' \mapsto \mathbf{F}\eta_D + E'$ (a refinement of 1.8(e)).

We show:

(g) *If* $|B| < (D-1)/2$, *then there exist* $x < y < z$ *in* $[1, D] - \sigma$ *such that* x, z *are odd and* y *is even.*

Assume first that any number in $[1, D] - \sigma$ is even. Then

$$1 = a_1, b_1, a_2, b_2, \dots, a_s, b_s = D$$

are all odd and

$$a_2 = b_1 + 2, a_3 = b_1 + 2, \dots, a_s = b_{s-1} + 2.$$

We have

$$b_1 = 1 + c_1, a_2 = 3 + c_1, b_2 = 3 + c_1 + c_2, a_3 = 5 + c_1 + c_2, b_3 = 5 + c_1 + c_2 + c_3, \dots,$$
$$a_s = 2s - 1 + c_1 + c_2 + \cdots + c_{s-1}, b_s = 2s - 1 + c_1 + c_2 + \cdots + c_s.$$

The last equality implies $D = 2s - 1 + 2|B| - 2s$ that is $|B| = (D+1)/2$. This contradicts $|B| < (D-1)/2$. We see that the set of odd numbers in $[1, D] - \sigma$ is nonempty. Let x (resp. z) be the smallest (resp. largest) odd number in $[1, D] - \sigma$. We have $b_t < x < a_{t+1}$, $b_u < z < a_{u+1}$ for some $t \in [0, s], u \in [0, s], t \le u$.

Assume now that there is no even number $y \in [1, D] - \sigma$ such that $x < y < z$. Then

$a_1, b_1, a_2, b_2, \dots, a_t, b_t$ are all odd; $a_{t+1}, b_{t+1}, a_{t+2}, b_{t+2}, \dots, a_u, b_u$ are all even; $a_{u+1}, b_{u+1}, \dots, a_s, b_s$ are all odd (also $a_1 = 1$ if $t > 0$, $a_1 = 2$ if $t = 0$, $b_s = D$ if $u < s$, $b_s = D - 1$ if $u = s$);

$a_2 = b_1 + 2, a_3 = b_2 + 2, \dots, a_t = b_{t-1} + 2$; $a_{t+1} = b_t + 3$, $a_{t+2} = b_{t+1} + 2$, $a_{t+3} = b_{t+2} + 2, \dots, a_u = b_{u-1} + 2$; $a_{u+1} = b_u + 3, a_{u+2} = b_{u+1} + 2, \dots, a_s = b_{s-1} + 2$.

From this we deduce as above that $b_s = 2s - 1 + c_1 + c_2 + \cdots + c_s + e$ where $e = 1$ if $x = z$ and $e = 2$ if $x < z$. Hence $D = 2s - 1 + 2|B| - 2s + e$, that is $|B| = (D + 1 - e)/2$. If $e = 1$ this is a contradiction since $(D+2)/2 \notin \mathbf{Z}$. If $e = 2$ we see that $|B| = (D-1)/2$, contradicting $|B| < (D-1)/2$. This proves (g).

We show:

(h) *If $|B| < (D-1)/2$ and $J \in \mathcal{I}_D$ satisfies $e_J \in E + \boldsymbol{F}\eta_D$, then $J \in B$.*

Assume that $J \notin B$. By 1.2(f) we then have $e_J \notin E$ so that $e_J + \eta_D \in E$. For any $\xi \in V_D, i \in [1, D]$ let $(e_i : \xi)$ be the coefficient of e_i in ξ. Since any element of E is a linear combination of $e_i, i \in \sigma$, we have $(e_i : e_J) + (e_i : \eta_D) = 0$ for $i \notin \sigma$. Let $x < y < z$ be as in (g). Since x, y, z are not in σ we have

$$(e_x : e_J) + (e_x : \eta_D) = 0,\ (e_y : e_J) + (e_y : \eta_D) = 0\ (e_z : e_J) + (e_z : \eta_D) = 0.$$

Since x, z are odd and y is even we have

$$(e_x : \eta_D) = 1,\ (e_y : \eta_D) = 0,\ (e_z : \eta_D) = 1.$$

Hence $(e_x : e_J) = 1$, $(e_y : e_J) = 0$ $(e_z : e_J) = 1$. Thus e_x, e_z appear with nonzero coefficient in e_J so that $x \in J, z \in J$. Since J is an interval and $x < y < z$, it follows that $y \in J$ contradicting $(e_y : e_J) = 0$. This proves (h).

2.2. We no longer assume that D is odd. Let $\mathcal{H}$ (resp. $\mathcal{H}'$) be the set of all $i \in [1, s+1]$ such that $a_i - b_{i-1} \geq 3$ (resp. $a_i - b_{i-1} \geq 4$). We have $\mathcal{H}' \subset \mathcal{H}$.

Assume first that $\mathcal{H} = \emptyset$ so that $a_i - b_{i-1} = 2$ for $i \in [1, s+1]$. From (a) we then have

$$\begin{aligned}2|B| &= 2s - 1 + b_1 - a_2 + b_2 - \cdots + b_{s-1} - a_s + D \\ &= 2s - 1 - 2 - 2 - \cdots - 2 + D = 2s - 1 - 2(s-1) + D = D + 1\end{aligned}$$

so that D is odd and $|B| = (D+1)/2$. We now assume that either D is even or D is odd and $|B| < (D+1)/2$. Then $|\mathcal{H}| \geq 1$. We write the elements of $\mathcal{H}$ in a sequence $i_1 < i_2 < \cdots < i_t$ with $t \geq 1$. Let $z(B)$ be the set consisting of the intervals $[a_{i_u} - 1, b_{i_{u+1}-1} + 1]$ with $u \in [1, t-1]$ and of the intervals $[j, j]$ with $j \in \cup_{i \in \mathcal{H}'} [b_{i-1} + 2, a_i - 2]$. Note that the condition that $z(B) = \emptyset$ is the same as the condition that $\mathcal{H}' = \emptyset$ and $|\mathcal{H}| = 1$. An equivalent condition is that for some $j \in [1, s+1]$ we have $a_j - b_{j-1} = 3$ and $a_i - b_{i-1} = 2$ for all $i \in [1, s+1] - \{j\}$. From (a) we then have

$$\begin{aligned}2|B| &= 2s - 2 + b_1 - a_2 + b_2 - \cdots + b_{s-1} - a_s + D \\ &= 2s - 2 - 2 - \cdots - 2 + D = 2s - 2s + D = D\end{aligned}$$

so that D is even and $|B| = D/2$.

Until the end of 2.6 we assume further that either D is even and $|B| < D/2$ or D is odd and $|B| < (D+1)/2$. Then $|\mathcal{H}| + |\mathcal{H}'| \geq 2$ and $z(B) \neq \emptyset$. We have

$$\begin{aligned}|z(B)| &= \sum_{i \in \mathcal{H}'} (a_i - b_{i-1} - 3) + |\mathcal{H}| - 1 \\ &= \sum_{i \in \mathcal{H}} (a_i - b_{i-1} - 3) + |\mathcal{H}| - 1 = \sum_{i \in \mathcal{H}} (a_i - b_{i-1} - 2) - 1.\end{aligned}$$

If $i \in [1, s+1] - \mathcal{H}$, then $a_i - b_{i-1} - 2 = 0$, hence the last sum over $\mathcal{H}$ does not change if $\mathcal{H}$ is replaced by $[1, s+1]$, so that

$$\begin{aligned}|z(B)| &= \sum_{i \in [1, s+1]} (a_i - b_{i-1} - 2) - 1 \\ &= -b_0 + \sum_{i \in [1, s]} (a_i - b_i) + a_{s+1} - 2(s+1) - 1\end{aligned}$$

$$= D - \sum_{i\in[1,s]} (b_i - a_i + 2) = D - 2|B|.$$

(We have used 2.1(a).) We set $\Delta = |z(B)|$, so that $\Delta = D - 2|B| \geq 1$. From the definition we see that there is a well defined sequence $c_0, c_1, \dots, c_\Delta$ in $\mathbf{Z}_{>0}$ such that the intervals in $z(B)$ are:

(a)
$$\begin{aligned} I_1 = {} & [c_0, c_0 + c_1 - 1], I_2 = [c_0 + c_1, c_0 + c_1 + c_2 - 1], \dots, \\ I_\Delta = {} & [c_0 + c_1 + \dots + c_{\Delta-1}, c_0 + c_1 + \dots + c_\Delta]. \end{aligned}$$

We write $I_* = (I_1, I_2, \dots, I_\Delta) \in \mathcal{I}_D \times \mathcal{I}_D \times \ldots \times \mathcal{I}_D$ (Δ factors).

2.3. We now give several examples of the assignment $B \mapsto I_*$ in 2.2.

When $B = \{[3,5], \{4\}, [8,10], \{9\}\}$, then $I_* = (\{1\}, [2,6])$ (if $D = 10$), $I_* = (\{1\}, [2,6], [7,11])$ (if $D = 11$), $I_* = (\{1\}, [2,6], [7,11], \{12\})$ (if $D = 12$), $I_* = (\{1\}, [2,6], [7,11], \{12\}, \{13\})$ (if $D = 13$).

If $D = 10$ and $B = \{[2,4], \{3\}, [8,10], \{9\}\}$ then $I_* = ([1,5], \{6\})$.

If $D = 10$ and $B = \{[2,4], \{3\}, [6,8], \{7\}\}$ then $I_* = ([1,9], \{10\})$.

If $D = 20$ and $B = \{[4,6], \{5\}, [9,11], \{10\}, [15,17]\}, \{16\}\}$, then

$I_* = (\{1\}, \{2\}, [3,7], [8,12], \{13\}, [14,18], \{19\}, \{20\})$.

If $D \geq 1, B = \emptyset$ then $I_* = (\{1\}, \{2\}, \dots, \{D\})$.

2.4. Let $I_* = (I_1, I_2, \dots, I_\Delta)$ be associated to B as in 2.2. From 2.2(a) we see that $\{e_{I_c}; c \in [1,\Delta]\}$ are linearly independent in V_D and that for c, d in $[1,\Delta]$:

(a) (e_{I_c}, e_{I_d}) *is* 1 *if* $c - d = \pm 1$ *and is* 0 *if* $c - d \neq \pm 1$*; moreover, if* D *hence* Δ *is odd then* $\sum_{c\in[1,\Delta]} e_{I_c} = e_{[c_0,c_\Delta]}$ *satisfies* $(e_{[c_0,c_\Delta]}, e_{I_c}) = 0$ *for all* $c \in [1,\Delta]$.

Here c_0, c_Δ are as in 2.2(a). Let $\mathfrak{T}_E = \langle e_{I_c}; c \in [1,\Delta]\rangle \subset V_D$. From (a) we see that

(b) $\{e_{I_c}; c \in [1,\Delta]\}$ *is a basis of* $\mathfrak{T}_E$ *and* $\mathfrak{T}_E^\perp \cap \mathfrak{T}_E$ *is* 0 *if* D *is even and is the line* $\boldsymbol{F} e_{[c_0,c_\Delta]}$ *if* D *is odd.*

From the definitions we see that $(e_I, e_{I_c}) = 0$ for any $I \in B, c \in [1,\Delta]$. It follows that $\mathfrak{T}_E \subset E^\perp$. Hence if $x \in E \cap \mathfrak{T}_E$ then $x \in \mathfrak{T}_E^\perp \cap \mathfrak{T}_E$. Hence

$x = 0$ if D is even and $x \in \mathbf{F}e_{[c_0,c_\Delta]}$ if D is odd. From the definitions we see that some $c \in [c_0, c_\Delta]$ is not contained in $\cup_{I\in B} I$; it follows that $e_{[c_0,c_\Delta]}$ is not contained in $\mathbf{E}_B = E$. Since $x \in E$, it follows that $x = 0$. Thus we have $E \cap \mathfrak{T}_E = 0$ so that $E + \mathfrak{T}_E = E \oplus \mathfrak{T}_E$ has dimension $\dim(E) + \Delta = |B| + \Delta = |B| + D - 2|B| = D - \dim(E)$. We have $\dim(E^\perp) = D - \dim(E)$. (When D is even this follows from the nondegeneracy of $(,)$; when D is odd this follows from 2.1(c) since $|B| \le (D-1)/2$.) We see that $\dim(E^\perp) = \dim(E \oplus \mathfrak{T}_E)$. We show that

(c) $E^\perp = E \oplus \mathfrak{T}_E$.

In view of the dimension equality above it is enough to show that $E \oplus \mathfrak{T}_E \subset E^\perp$. The inclusion $E \subset E^\perp$ holds since E is isotropic (see 1.9). To prove the inclusion $\mathfrak{T}_E \subset E^\perp$ it is enough to prove that $(e_{I_c}, e_I) = 0$ for any $c \in [1,\Delta]$ and any $I \in B$. But from 2.2(a) and the definitions we see that $I_c \prec I$ or $I \prec I_c$ or I, I_c are non-touching. In each case we have $(e_{I_c}, e_I) = 0$, proving (c).

We see that $\mathfrak{T}_E$ is a canonical complement of E in $E^\perp$ and that $\mathfrak{T}_E$ has a canonical basis $\{e_{I_c}, c \in [1,\Delta]\}$.

2.5. We now assume that $D \ge 2$ and $i \in [1, D]$, $B' \in S_{D-2}$ are such that $B = t_i(B')$. Let $E' \in \mathcal{F}(V_{D-2})$ be the subspace defined by B' so that $E = T_i(E') \oplus \mathbf{F}e_i$. We have $|B'| = |B| - 1$ hence $|B'| < (D-2)/2$ if D is even and $|B'| < (D-1)/2$ if D is odd. Thus $\mathfrak{T}_{E'} \subset V_{D-2}$ is defined. From the definitions we see that

(a) $T_i : V_{D-2} \to V_D$ *carries* $\mathfrak{T}_{E'}$ *isomorphically onto* $\mathfrak{T}_E$ *compatibly with the canonical bases of* $\mathfrak{T}_{E'}$ *and* $\mathfrak{T}_E$.

2.6. For $c \le d$ in $[1,\Delta]$ we set

$$I_{[c,d]} = I_c \cup I_{c+1} \cup I_{c+2} \cup \cdots \cup I_d \in \mathcal{I}_D.$$

If Δ is even and $k \in [1, \Delta/2]$ we set

$$\mathfrak{T}_E^k = \langle e_{I_{[1,\Delta]}}, e_{I_{[2,\Delta-1]}}, \ldots, e_{I_{[k,\Delta+1-k]}} \rangle.$$

If Δ is odd and $k \in [1,(\Delta-1)/2]^1$ let

$$\mathfrak{T}_E^k = \langle e_{I_{[1,\Delta-1]}}, e_{I_{[2,\Delta-2]}}, \dots, e_{I_{[k,\Delta-k]}} \rangle,$$

$$\mathfrak{T}_E^{-k} = \langle e_{I_{[2,\Delta]}}, e_{I_{[3,\Delta-1]}}, \dots, e_{I_{[k+1,\Delta+1-k]}} \rangle.$$

Without restriction on Δ we set $\mathfrak{T}_E^0 = 0$. Thus $\mathfrak{T}_E^k$ is defined for any

(a) $k \in [0, \Delta/2]$ if Δ is even and for any $k \in [1,(\Delta-1)/2]^1 \cup (-[1,(\Delta-1)/2]^1) \cup \{0\}$ if Δ is odd.

For k as in (a) we define $E(k) = E + \mathfrak{T}_E^k = E \oplus \mathfrak{T}_E^k \subset V_D$. We define $B(k)$ to be

B if $k=0$,

$B \sqcup \{I_{[1,\Delta]}, I_{[2,\Delta-1]}, \dots, I_{[k,\Delta+1-k]}\}$ if Δ is even, $k \in [1, \Delta/2]$,

$B \sqcup \{I_{[1,\Delta-1]}, I_{[2,\Delta-2]}, \dots, I_{[k,\Delta-k]}\}$ if Δ is odd, $k \in [1,(\Delta-1)/2]^1$,

$B \sqcup \{I_{[2,\Delta]}, I_{[3,\Delta-1]}, \dots, I_{[-k+1,\Delta+1+k]}\}$ if Δ is odd, $-k \in [1,(\Delta-1)/2]^1$.

We show:

(b) *$B(k) \in \boldsymbol{S}_D$ and $E(k) = \boldsymbol{E}_{B(k)} \in \boldsymbol{\mathcal{F}}(V_D)$.*

We argue by induction on D. If $E=0$ then $E(k)$ is a primitive subspace of V_D so that (b) holds. Thus we can assume that $D \ge 2$ and $E \ne 0$. (If $D=1$ then $E=0$ since $|B| < (D+1)/2$.) We can find $i, B' \in S_{D-2}, E' \in \mathcal{F}(V_{D-2}]$ as in 2.5. By the induction hypothesis we have $B'(k) \in \mathbf{S}_{D-2}$ and $E'(k) \in \boldsymbol{\mathcal{F}}(V_{D-2})$ is the subspace of V_{D-2} defined in terms of $B'(k)$ in the same way as $E(k)$ is defined in terms of $B(k)$. We have $E = T_i(E') \oplus \mathbf{F}e_i$. Using 2.5 we see that $E(k) = T_i(E'(k)) \oplus \mathbf{F}e_i$ and $B(k) = t_i(B'(k))$. Hence (b) holds.

The same inductive proof shows that

(c) *if $I \in B(k) - B$ then $|I|$ is even.*

(We use that this holds when $B = \emptyset$.)

2.7. We no longer fix B. Let $\underline{\mathbf{S}}_D$ be the set of all pairs (B,k) where $B \in S_D$ and one of (i)-(iii) below holds.

(i) $|B| = D/2$ and $k=0$ (if D is even);

(ii) $|B| = (D+1)/2$ and $k=0$ (if D is odd);

(iii) we have $|B| < D/2$ (if D is even), $|B| < (D+1)/2$ (if D is odd) so that $\Delta := D - 2|B| \geq 1$, and we have $k \in [0, \Delta/2]$ if D is even and $k \in [1, (\Delta-1)/2]^1 \cup (-[1, (\Delta-1)/2]^1) \cup \{0\}$ if D is odd.

We define $\lambda : \underline{\mathbf{S}}_D \to \mathbf{S}_D$ by $\lambda(B, k) = B(k)$. (We have $B(0) = B$.)

Proposition 2.8. *λ is a bijection.*

We show that λ is injective. Indeed, assume that $(B, k) \in \underline{\mathbf{S}}_D$, $(B', k') \in \underline{\mathbf{S}}_D$ are such that $B(k) = B'(k')$. Intersecting the two sides with $\mathcal{I}^1_D$ and using 2.6(c) we obtain $B = B'$. Since $|B(k)| - |B| = \pm k$, $|B(k')| - |B| = \pm k'$, we see that $\pm k = \pm k'$. If D is even we have $k \geq 0, k' \geq 0$ hence $k = k'$. Assume now that D is odd and $k \neq k'$ so that $k' = -k$. But then one of $B(k) - B$, $B(k') - B$ contains $I_{[1,\Delta-1]}$ while the other one does not; this contradicts $B(k) - B = B(k') - B$. We see that we must have $k = k'$ proving the injectivity of λ.

We show that λ is surjective by induction on D. Let $B \in \mathbf{S}_D$. If B is primitive then $B = \{0\}(k)$ with $k \in [0, D/2]$ (if D is even) or with $k \in [1, (D-1)/2]^1 \cup (-[0, (D+1)/2]^1) \cup \{0\}$ (if D is odd); thus B is in the image of λ. If $B \in S_D$ then $B = B(0)$ so that B is in the image of λ. Now assume that B is not primitive and $B \notin S_D$. Then $D \geq 2$ and there exists $i \in [1, D]$ and $B' \in \mathbf{S}_{D-2}$ such that $B = t_i(B')$. We must have $B' \notin S_{D-2}$. By the induction hypothesis we have $B' = \tilde{B}'(k)$ where $\tilde{B}' \in S_{D-2}$ and $k \in [1, (D-2)/2 - |\tilde{B}'|]$ (if D is even) and $k \in [1, (D-2-2|\tilde{B}'|-1)/2]^1 \cup (-[1, (D-2-2|\tilde{B}'|-1)/2]^1)$ (if D is odd). Let $\tilde{B} = t_i(\tilde{B}')$. We have $\tilde{B} \in S_D$ and $|\tilde{B}| = |\tilde{B}'| + 1$ so that $k \in [1, D/2 - |\tilde{B}|]$ (if D is even) and $k \in [1, (D - 2|\tilde{B}| - 1)/2]^1 \cup (-[1, (D - 2|\tilde{B}| - 1)/2]^1)$ (if D is odd). As in the proof of 2.6(b) we have $\tilde{B}(k) = t_i(\tilde{B}'(k))$ hence $\tilde{B}(k) = B$. Thus λ is surjective. The proposition is proved.

2.9. In this subsection we assume that D is odd. Let

$$S'_D = S_{D-1} \sqcup \{B \in S_D - S_{D-1}; |B| < (D-1)/2\}.$$

This is a subset of S_D (we view S_{D-1} as a subset of S_D, see 1.10). Let

$$\begin{aligned}\mathcal{F}'(V_D) &= \{\mathbf{E}_B; B \in S'_D\} \\ &= \mathcal{F}(V_{D-1}) \sqcup \{E \in \mathcal{F}(V_D) - \mathcal{F}(V_{D-1}); \dim(E) < (D-1)/2\}.\end{aligned}$$

This is a subset of $\mathcal{F}(V_D)$ (we view $\mathcal{F}(V_{D-1})$ as a subset of $\mathcal{F}(V_D)$, see 1.10).

Let $\mathbf{S}'_D$ be the set consisting of all elements $B(k) \in \mathbf{S}_D$ (see 2.6) where either

(i) $B \in S_{D-1}$ and $k \in \{0\} \cup [1, (D-2|B|-1)/2]^1$ or

(ii) $B \in S_D - S_{D-1}$, $|B| < (D-1)/2$ and $k \in [1, (D-2|B|-1)/2]^1$.

(We use that $S_{D-1} \subset S_D$, see 1.10.) We define a map $\iota : \mathbf{S}'_D \to \mathbf{S}_{D-1}$ by $B(k) \mapsto B'(k')$ where $B'(k') \in \mathbf{S}_{D-1}$ (as in 2.6 with D replaced by $D-1$) is given by:

$B' = B, k' = 0$ if B, k are as in (i) and $k = 0$;

$B' = B, k' = (k+1)/2$ if B, k are as in (i) and $k > 0$;

$B' \in S_{D-1}$ is obtained by removing from B the unique interval containing D, $k' = (k+3)/2$ if B, k are as in (ii).

From the definitions we see using 2.8 that

(a) *ι is a bijection.*

2.10. Let $\underline{\boldsymbol{\mathcal{F}}}(V_D)$ be the set of all pairs (E, k) where $E \in \mathcal{F}(V_D)$ and one of (i)-(iii) below holds.

(i) $\dim(E) = D/2$ and $k = 0$ (if D is even);

(ii) $\dim(E) = (D+1)/2$ and $k = 0$ (if D is odd);

(iii) we have $\dim(E) < D/2$ (if D is even), $\dim(E) < (D+1)/2$ (if D is odd) so that $\Delta := D - 2\dim(E) \geq 1$, and we have $k \in [0, \Delta/2]$ if D is even and $k \in [1, (\Delta-1)/2]^1 \cup (-[1, (\Delta-1)/2]^1) \cup \{0\}$ if D is odd.

We define $\mu : \underline{\boldsymbol{\mathcal{F}}}(V_D) \to \boldsymbol{\mathcal{F}}(V_D)$ by $\mu(E, k) = E(k)$. (We have $E(0) = E$.) The following result is a reformulation of 2.8.

(a) *μ is a bijection.*

2.11. In the remainder of this section we assume that D is odd. We set

$$V'_D = V_D/\mathbf{F}\eta_D$$

where $\eta_D \in V_D$ is as in 1.8; let $\pi : V_D \to V'_D$ be the obvious map. Let $\mathcal{F}(V'_D)$ be the set of subspaces of V'_D of the form $\pi(E)$ for various $E \in \mathcal{F}(V_D)$. We show:

(a) *the map* $E \mapsto \pi(E)$ *defines a bijection* $\mathcal{F}'(V_D)\xrightarrow{\sim}\mathcal{F}(V'_D)$ *(notation of 2.9).*

We first show that this map is surjective. It is enough to show that for any $E \in \mathcal{F}(V_D)$ there exists $E' \in \mathcal{F}'(V_D)$ such that $\pi(E) = \pi(E')$. If $E \in \mathcal{F}'(V_D)$ then $E' = E$ satisfies our requirement. Thus we can assume that $E \notin \mathcal{F}'(V_D)$ so that $E \in \mathcal{F}(V_D) - \mathcal{F}(V_{D-1})$ and $\dim(E) \geq (D-1)/2$. It follows that we have either $\dim(E) = (D+1)/2$ or $\dim(E) = (D-1)/2$. If $\dim(E) = (D+1)/2$ then by 2.1(f) we can find $E' \in \mathcal{F}(V_{D-1})$ such that $\pi(E) = \pi(E')$; since $E' \in \mathcal{F}'(V_D)$ we see that E' satisfies our requirement. If $\dim(E) = (D-1)/2$ then by 2.1(e) we have $\eta_D \notin E$ and $\mathbf{F}\eta_D \oplus E \in \mathcal{F}(V_D)$; by the previous sentence applied to $\mathbf{F}\eta_D \oplus E$ (which has dimension $(D+1)/2$) instead of E we see that we can find $E' \in \mathcal{F}(V_{D-1})$ such that $\pi(\mathbf{F}\eta_D + E) = \pi(E')$. Since $\pi(\mathbf{F}\eta_D + E) = \pi(E)$ we see that E' satisfies our requirement. This proves the surjectivity of the map in (a).

We now prove injectivity. Let E, E' in $\mathcal{F}'(V_D)$ be such that $\pi(E) = \pi(E')$; we must show that $E = E'$.

Let $B \in S'_D, B' \in S'_D$ be such that $E = \mathbf{E}_B, E' = \mathbf{E}_{B'}$. Since $\pi(E) = \pi(E')$ we have $E + \mathbf{F}\eta_D = E' + \mathbf{F}\eta_D$. Assume first that $\dim(E') < (D-1)/2$. If $J \in B$, then $e_J \in E$ hence $e_J \in E + \mathbf{F}\eta_D = E' + \mathbf{F}\eta_D$. Using 2.1(h), we deduce that $J \in B'$. Thus we have $B \subset B'$ so that $E \subset E'$ and $\dim(E) < (D-1)/2$. Similarly, if $\dim(E) < (D-1)/2$, then $E' \subset E$ and $\dim(E') < (D-1)/2$. We see that if one of the conditions $\dim(E) < (D-1)/2$, $\dim(E') < (D-1)/2$ holds, then both conditions hold and $E = E'$. Thus, we can assume that $E \in \mathcal{F}(V_{D-1}), E' \in \mathcal{F}(V_{D-1})$. In this case we use that the restriction of π to V_{D-1} is an isomorphism $V_{D-1}\xrightarrow{\sim}V'_D$ to see that $E = E'$. This proves the injectivity of the map in (a) and completes the proof of (a).

We have $V'_D = {V'_D}^0 \oplus {V'_D}^1$ where ${V'_D}^0$ (resp. ${V'_D}^1$) is the image of V_D^0 (resp. V_D^1) under the map $\pi : V_D \to V'_D$.

Let $\mathcal{C}({V'_D}^1)$ be the set of subspaces of ${V'_D}^1$ which are images under π of subspaces of V_D^1 in $\mathcal{C}(V_D^1)$.

Let $\breve{\mathcal{C}}(V_D'^1)$ be the set of pairs of subspaces of $V_D'^1$ which are images under π of pairs of subspaces of V_D^1 in $\breve{\mathcal{C}}(V_D^1)$. We have $\breve{\mathcal{C}}(V_D'^1) \subset \mathcal{C}(V_D'^1) \times \mathcal{C}(V_D'^1)$.

By 1.9(b), $\breve{\mathcal{C}}(V_D'^1)$ consists of the pairs $\gamma_E = (\pi(E^1), \pi((E^0)^!))$ of subspaces of $V_D'^1$ for various $E \in \mathcal{F}(V_D)$. We show:

(b) *for E, E' in $\mathcal{F}(V_D)$ we have $\gamma_E = \gamma_{E'}$ if and only if $\pi(E) = \pi(E')$.*

Assume first that $\gamma_E = \gamma_{E'}$ that is $\pi(E^1) = \pi(E'^1)$ and $\pi((E^0)^!) = \pi((E'^0)^!)$. We have $E^1 + \mathbf{F}\eta_D = E'^1 + \mathbf{F}\eta_D$, $(E^0)^! = (E'^0)^!$ so that $E^0 = E'^0$ and $E + \mathbf{F}\eta_D = E' + \mathbf{F}\eta_D$ that is $\pi(E) = \pi(E')$. Conversely assume that $\pi(E) = \pi(E')$. Then $E + \mathbf{F}\eta_D = E' + \mathbf{F}\eta_D$ hence $E^1 + \mathbf{F}\eta_D = E'^1 + \mathbf{F}\eta_D$, $E^0 = E'^0$, so that $(E^0)^! = (E'^0)^!$ and $\gamma_E = \gamma_{E'}$. This proves (b).

From (b) we see that $E \mapsto \gamma_E$ induces a bijection $\mathcal{F}(V_D') \xrightarrow{\sim} \breve{\mathcal{C}}(V_D'^1)$. Composing this bijection with the bijection (a), we obtain a bijection

(c) $\mathcal{F}'(V_D) \xrightarrow{\sim} \breve{\mathcal{C}}(V_D'^1)$.

2.12. Let $\boldsymbol{\mathcal{F}}'(V_D) = \{\mathbf{E}_B; B \in \mathbf{S}_D'\} \subset \boldsymbol{\mathcal{F}}(V_D)$. Let $\underline{\boldsymbol{\mathcal{F}}'}(V_D)$ be the set of all pairs (E, k) where

(i) $E \in \mathcal{F}(V_{D-1})$ and $k \in \{0\} \cup [1, (D - 2\dim(E) - 1)/2]^1$ or

(ii) $E \in \mathcal{F}(V_D) - \mathcal{F}(V_{D-1})$, $\dim(E) < (D-1)/2$ and $k \in [1, (D-2\dim(E)-1)/2]^1$.

We define $\mu' : \underline{\boldsymbol{\mathcal{F}}'}(V_D) \to \boldsymbol{\mathcal{F}}'(V_D)$ by $\mu'(E, k) = E(k)$. The following result is a consequence of 2.10(a).

(a) *μ' is a bijection.*

2.13. Let $\mathcal{F}_*(V_D') = \{E' \in \mathcal{F}(V_D'); \dim(E') = (D - 1)/2\}$. We show:

(a) *The map $E \mapsto \pi(E)$ defines a bijection $\mathcal{F}_*(V_{D-1}) \to \mathcal{F}_*(V_D')$.*

Let $E \in \mathcal{F}_*(V_{D-1})$. Since the restriction of π to V_{D-1} is bijective we see that $\dim(\pi(E)) = (D - 1)/2$. Using 2.11(a) we see that the map in (a) is well defined and injective. Let $E' \in \mathcal{F}_*(V_D')$. By 2.11(a) we have $E' = \pi(E)$ where $E \in \mathcal{F}'(V_D)$. We have $\dim(E) \geq (D - 1)/2$ hence $E \in \mathcal{F}(V_{D-1})$ and

$\dim(E) = (D-1)/2$ so that $E \in \mathcal{F}_*(V_{D-1})$. We see that the map in (a) is surjective. This proves (a).

The following result can be deduced from (a) and 2.1(e),(f).

(b) *The map $E \mapsto \pi(E)$ defines a bijection $\mathcal{F}_*(V_D) \to \mathcal{F}_*(V'_D)$.*

3. From $\mathbf{S}_D$ to V_D

3.1. We fix a symbol $\mathbf{a}$. We set $\overline{[1,D]} = [1,D] \sqcup \{\mathbf{a}\}$. A subset I of $\overline{[1,D]}$ is said to be an interval if either $I \in \mathcal{I}_D$ or if $\overline{[1,D]} - I \in \mathcal{I}_D$. Such an I is necessarily nonempty and not equal to $\overline{[1,D]}$. Let $\tilde{\mathcal{I}}_D$ be the set of intervals of $\overline{[1,D]}$. We have $\mathcal{I}_D \subset \tilde{\mathcal{I}}_D$. For $I \in \mathcal{I}_D$ we define $I^* \in \tilde{\mathcal{I}}_D$ by $I^* = I$ if $|I|$ is odd and $I^* = \overline{[1,D]} - I$ if $|I|$ is even. For $B \in R_D$ we set $B^* = \{I^*; I \in B\}$. Let $\mathbf{S}^*_D = \{B^*; B \in \mathbf{S}_D\}$. If D is even and $B \in \mathbf{S}^*_D$ then any $I \in B$ satisfies $|I| = 1 \mod 2$. If D is odd and $B \in \mathbf{S}^*_D$ then any $I \in B$ satisfies $|I| = 1 \mod 2$ (if $\mathbf{a} \notin I$) or $|I| = 0 \mod 2$ (if $\mathbf{a} \in I$).

3.2. Let $B \in \mathbf{S}^*_D$. For any $j \in \overline{[1,D]}$ we set $n_j(B) = \sharp(I \in B; j \in I)$. We now define $\epsilon^j(B) \in \mathbf{F}$ for any $j \in \overline{[1,D]}$ as follows. If D is even we set $\epsilon^j(B) = n_j(B)(n_j(B)+1)/2 \in \mathbf{F}$. Assume now that D is odd. We denote by $[B]$ the union of all $I \in B$ such that $\mathbf{a} \in I$; then either $[B] = \emptyset$ or $\mathbf{a} \in [B]$ and $[B] \in B$. If $j \notin [B]$ or if $j = \mathbf{a}$ we set $\epsilon^j(B) = n_j(B)(n_j(B)+1)/2 \in \mathbf{F}$. Assume now that $j \in [B], j \neq \mathbf{a}$. We have $[B] - \{\mathbf{a}\} = [1,k] \sqcup [k',D]$ with $0 \le k < D$, $1 < k' \le D+1$, $k' \ge k+2$; we have either $j \in [k',D]$ (we then set $u = k'$) or $j \in [1,k]$ (we then set $u = k$). If $u = j+1 \mod 2$ we set $\epsilon^j(B) = (n_j(B))(n_j(B)+1)/2 \in \mathbf{F}$. If $u = j \mod 2$ we set $\epsilon^j(B) = (n_j(B)+1)(n_j(B)+2)/2 \in \mathbf{F}$.

We set $e_{\mathbf{a}} = e_{[1,D]} \in V_D$; we define $\epsilon_D : \mathbf{S}^*_D \to V_D$ by $\epsilon_D(B) = \sum_{j \in \overline{[1,D]}} \epsilon^j(B) e_j$.

In the case where D is odd, we define $\epsilon'_D : \mathbf{S}'^*_D \to V'_D$ by $B \mapsto \pi(\epsilon_D(B))$. We state the following result (see also the tables in §4).

Theorem 3.3.

(a) *For any D, $\epsilon_D : \mathbf{S}^*_D \to V_D$ is a bijection.*

(b) *For D odd, $\epsilon'_D : \mathbf{S}'^*_D \to V'_D$ is a bijection.*

When D is even, (a) can be deduced from [8, 1.17(b)], see 3.4; the proof of (a) for odd D is similar and will be omitted.

We prove (b). Consider the diagram

$$\begin{array}{ccc} \mathbf{S}'_D & \xrightarrow{\iota} & \mathbf{S}_{D-1} \\ \downarrow & & \downarrow \\ V_D & \longleftarrow & V_{D-1} \end{array}$$

with ι being the bijection in 2.9, the left vertical map being $B \mapsto \epsilon_D(B^*)$, the right vertical map being $B \mapsto \epsilon_{D-1}(B^*)$, and the lower horizontal map being the obvious inclusion. From the definitions this diagram is commutative. Combining this with (a) (with D replaced by $D-1$) and with the bijection $\mathbf{S}_D \to \mathbf{S}^*_D$, $B \mapsto B^*$, we see that the restriction of ϵ_D to $\mathbf{S}'^*_D$ is a bijection $\mathbf{S}'^*_D \to V_{D-1}$. It remains to use that π restricts to a bijection $V_{D-1} \to V'_D$.

3.4. In this subsection we assume that D is even. Let $B \in \mathbf{S}_D$. For $i \in [1, D]$, $\delta \in \{0, 1\}$, let $B^\delta_i = \{I \in B^\delta; i \in I\}$, $\kappa(B) = |B^0|$, $\tilde{\epsilon}^{-1}(B) = (|B^1_i| - |B^0_i| - \underline{|B^0|})(|B^1_i| - |B^0_i| - \underline{|B^0|} + 1)/2 \in \mathbf{F}$, where for $z \in \mathbf{Z}$ we define $\underline{z} \in \{0, 1\}$ by $z = \underline{z} \mod 2$. We define $\tilde{\epsilon}(B) = \sum_{i \in [1,D]} \tilde{\epsilon}^{-1}(B) e_i \in V_D$. By [8, 1.17(b)], $B \mapsto \tilde{\epsilon}(B)$ defines a bijection $\mathbf{S}_D \xrightarrow{\sim} V_D$. Hence to prove 3.3(a) in our case it is enough to show that for any $B \in \mathbf{S}_D$ we have $\tilde{\epsilon}(B) = \epsilon_D(B^*)$ that is $\tilde{\epsilon}^{-1}(B) = \epsilon^{-1}(B^*) + \epsilon^{\mathbf{a}}(B^*)$ for any $i \in [1, D]$, or equivalently that

$$(|B^1_i| - |B^0_i| - \underline{|B^0|})(|B^1_i| - |B^0_i| - \underline{|B^0|} + 1)/2 = n_i(n_i+1)/2 + n_{\mathbf{a}}(n_{\mathbf{a}}+1)/2 \mod 2$$

where we write $n_i, n_{\mathbf{a}}$ instead of $n_i(B^*), n_{\mathbf{a}}(B^*)$. From the definition we have $n_{\mathbf{a}} = |B^0|$,

$$n_i = |B^1_i| + \sharp(I \in B^0, i \notin I) = |B^1_i| + |B^0| - |B^0_i|.$$

Hence

$$n_i(n_i+1) + n_{\mathbf{a}}(n_{\mathbf{a}}+1) = (|B^1_i| + |B^0| - |B^0_i|)(|B^1_i| + |B^0| - |B^0_i| + 1) + |B^0|(|B^0| + 1).$$

Setting $b = |B^0|, p = |B^1_i| - |B^0_i| - \underline{b}$, we see that it is enough to show

$$(p + b + \underline{b})(p + b + \underline{b} + 1) + b(b+1) = p(p+1) \mod 4.$$

This follows from $2p(b+\underline{b}) = 0 \bmod 4$, $2b^2+2b = 0 \bmod 4$, $2b\underline{b}+(\underline{b})^2+\underline{b} = 0 \bmod 4$. This completes the deduction of 3.3(a) (for D even) from [8].

3.5. For $I \in \tilde{\mathcal{I}}_D$ we set $e_I = \sum_{j\in I} e_j \in V_D$ (with e_a as in 3.2.) When $I \in \mathcal{I}_D$ this agrees with the earlier definition. For $B \in \mathbf{S}^*_D$ let $\mathbf{E}_B$ be the subspace of V_D spanned by $\{e_I; I \in B\}$. For $B \in \mathbf{S}_D$ we have

$$\mathbf{E}_B = \mathbf{E}_{B^*}.$$

(It is enough to show that for any $I \in \mathcal{I}_D$ we have $e_{I^*} = e_I$; this is clear from the definition.)

We define $\bar{\epsilon}_D : \boldsymbol{\mathcal{F}}(V_D) \to V_D$ by $\mathbf{E}_B = \mathbf{E}_{B^*} \mapsto \epsilon_D(B)$ for any $B \in \mathbf{S}^*_D$. The following result is a reformulation of 3.3(a) (we use 1.2(g)).

(a) *For any D, the map $\bar{\epsilon}_D : \boldsymbol{\mathcal{F}}(V_D) \to V_D$ is a bijection.*

We have the following result.

(b) *If $\boldsymbol{E} \in \boldsymbol{\mathcal{F}}(V_D)$, then $\bar{\epsilon}_D(\boldsymbol{E}) \in \boldsymbol{E}$; this property characterizes the bijection $\bar{\epsilon}_D$.*

When D is even this can be deduced from [8, 1.22(b)] using the arguments in 3.4. A similar proof applies when D is odd.

When D is odd, we define $\bar{\epsilon}'_D : \boldsymbol{\mathcal{F}}'(V_D) \to V'_D$ by $\mathbf{E}_B \mapsto \epsilon'_D(B)$ for any $B \in \mathbf{S}'^*_D$. The following result is a reformulation of 3.3(b).

(c) *For odd D, the map $\bar{\epsilon}'_D : \boldsymbol{\mathcal{F}}'(V_D) \to V'_D$ is a bijection.*

From (b) we deduce (for D odd):

(d) *If $\boldsymbol{E} \in \boldsymbol{\mathcal{F}}'(V_D)$, then $\bar{\epsilon}'_D(\boldsymbol{E}) \in \pi(\boldsymbol{E})$.*

One can show that

(e) *the property in (d) characterizes the bijection $\bar{\epsilon}'_D$.*

From (e) we can deduce that (for odd D):

(f) *if $\boldsymbol{E}, \boldsymbol{E}'$ in $\boldsymbol{\mathcal{F}}'(V_D)$ satisfy $\pi(\boldsymbol{E}) = \pi(\boldsymbol{E}')$, then $\boldsymbol{E} = \boldsymbol{E}'$.*

Indeed, let $\mathbf{E}, \mathbf{E}'$ in $\boldsymbol{\mathcal{F}}'(V_D)$ be such that $\pi(\mathbf{E}) = \pi(\mathbf{E}')$. We define a new bijection $\epsilon'' : \boldsymbol{\mathcal{F}}'(V_D) \to V'_D$ by $\epsilon''(\mathbf{E}_1) = \bar{\epsilon}'_D(\mathbf{E}_1)$ if $\mathbf{E}_1 \in \boldsymbol{\mathcal{F}}'(V_D)$, $\mathbf{E}_1 \notin \{\mathbf{E}, \mathbf{E}'\}$, $\epsilon''(\mathbf{E}) = \bar{\epsilon}'_D(\mathbf{E}')$, $\epsilon''(\mathbf{E}') = \bar{\epsilon}'_D(\mathbf{E})$. For $\mathbf{E}_1 \in \boldsymbol{\mathcal{F}}'(V_D)$ we have $\epsilon''(\mathbf{E}_1) \in \pi(\mathbf{E}_1)$. (When $\mathbf{E}_1 \notin \{\mathbf{E}, \mathbf{E}'\}$, this follows from (d); when $\mathbf{E}_1 \in \{\mathbf{E}, \mathbf{E}'\}$, this follows

(d) and from $\pi(\mathbf{E}) = \pi(\mathbf{E}')$.) Using (e) we deduce that $\epsilon'' = \vec{\epsilon}'_D$ so that $\vec{\epsilon}'_D(\mathbf{E}) = \vec{\epsilon}'_D(\mathbf{E}')$. Using the injectivity of $\vec{\epsilon}'_D$ we deduce that $\mathbf{E} = \mathbf{E}'$, as desired.

3.6. Note that

(a) *there exists a partial order $\le$ on V_D such that for any $B \in \boldsymbol{S}^*_D$ and any $x \in \boldsymbol{E}_B$ we have $x \le \epsilon_D(B)$.*

When D is even this follows from [8, 1.22(b)]. The proof for D odd is similar.

Assuming that D is odd, one can show that the following analogue of (a) holds:

(b) *there exists a partial order $\le$ on V'_D such that for any $B \in \boldsymbol{S}'^*_D$ and any $x \in \pi(\boldsymbol{E}_B)$ we have $x \le \epsilon'_D(B)$.*

3.7. We now assume that G is as in 0.6(i). We prove 0.6(a) in this case. By [4] we can identify $\mathcal{G}_c$ with $V^\delta_{D'}$ for some even $D' \ge 0$ and some $\delta \in \{0, 1\}$. We can assume that $D' = D$. Then $M(\mathcal{G}_c)$ becomes

$$V^\delta_D \oplus \mathrm{Hom}(V^\delta_D, \mathbf{F}) = V^{1-\delta}_D \oplus V^\delta_D = V_D.$$

By [3, 5, 6], we have $\mathbf{H}_c = \mathcal{C}(V^d_D)$. We define $\breve{\mathbf{H}}_c$ to be the subset $\breve{\mathcal{C}}(V^\delta_D)$ of $\mathcal{C}(V^\delta_D) \times \mathcal{C}(V^\delta_D) = \mathbf{H}_c \times \mathbf{H}_c$. Let $(\mathcal{L}, \tilde{\mathcal{L}}) \in \breve{\mathcal{C}}(V^\delta_D) = \breve{\mathbf{H}}_c$ (we have $\mathcal{L} \subset \tilde{\mathcal{L}}$). Let $E = \mathcal{L} \oplus \tilde{\mathcal{L}}^! \subset V_D$. We have $E \subset E^\perp = \tilde{\mathcal{L}} \oplus \mathcal{L}^!$. Recall that $E \in \mathcal{F}(V_D)$. We have

$$M(\tilde{\mathcal{L}}/\mathcal{L}) = \tilde{\mathcal{L}}/\mathcal{L} \oplus \mathrm{Hom}(\tilde{\mathcal{L}}/\mathcal{L}, \mathbf{F}) = \tilde{\mathcal{L}}/\mathcal{L} \oplus \mathcal{L}^!/\tilde{\mathcal{L}}^! = (\tilde{\mathcal{L}} \oplus \mathcal{L}^!)/(\mathcal{L} \oplus \tilde{\mathcal{L}}^!) = E^\perp/E.$$

Let $Prim(\mathcal{L}, \tilde{\mathcal{L}}) = Prim(E^\perp/E)$ be the subset of $M(\tilde{\mathcal{L}}/\mathcal{L}) = E^\perp/E$ consisting of the subspaces $E(k)/E$ of $E^\perp/E$ for various $k \in [0, D/2 - \dim(E)]$ (notation of 2.6). Since $(\mathcal{L}, \tilde{\mathcal{L}}) \mapsto E$ identifies $\breve{\mathcal{C}}(V^\delta_D)$ with $\mathcal{F}(V_D)$, we see that the map appearing in 0.6(a) can be identified with a map $\oplus_{E \in \mathcal{F}(V_D)} Prim(E^\perp/E) \to V_D$. We define such a map by $(E, E(k)/E) \mapsto \bar{\epsilon}_D(E(k))$. Using 2.10(a) and 3.3 we see that this map is a bijection. From the definitions, if E corresponds to $(\mathcal{L}, \tilde{\mathcal{L}})$ as above, then $\mathbf{s}_{\mathcal{L}, \tilde{\mathcal{L}}; V^\delta_D}(E(k)/E) \in \mathbf{C}[V_D]$ is the characteristic function of $E(k)$ hence its support contains $\bar{\epsilon}_D(E(k))$ (see 3.5(b)). This proves the existence of the bijection Θ in 0.6(a) in our case. The uniqueness of Θ follows from 3.5(b).

3.8. We now assume that G is as in 0.6(ii). We prove 0.6(a) in this case. By [4] we can identify $\mathcal{G}_c$ with ${V'_{D'}}^1$ (see 2.11) for some odd $D' \geq 1$. We can assume that $D' = D$. Then $M(\mathcal{G}_c)$ becomes

$${V'_D}^1 \oplus \mathrm{Hom}({V'_D}^1, \mathbf{F}) = {V'_D}^0 \oplus {V'_D}^1 = V'_D.$$

By [3, 5, 6], we have $\mathbf{H}_c = \mathcal{C}({V'_D}^1)$ (see 2.11). We define $\breve{\mathbf{H}}_c$ to be the subset $\breve{\mathcal{C}}({V'_D}^1)$ of $\mathcal{C}({V'_D}^1) \times \mathcal{C}({V'_D}^1) = \mathbf{H}_c \times \mathbf{H}_c$ (see 2.11).

Consider an element of $\breve{\mathcal{C}}({V'_D}^1) = \breve{\mathbf{H}}_c$; it can be written in the form $\gamma_E = (\pi(E^1), \pi((E^0)^!)$ where $E \in \mathcal{F}'(V_D)$ is uniquely determined (see 2.11). We have $E \subset E^\perp = (E^0)^! \oplus (E^1)^!$ and

$$\begin{aligned} &M(\pi((E^0)^!)/\pi(E^1)) = \pi((E^0)^!)/\pi(E^1) \oplus \mathrm{Hom}(\pi((E^0)^!)/\pi(E^1), \mathbf{F}) \\ &= \pi((E^0)^!)/\pi(E^1) \oplus \pi((E^1)^!)/\pi(E^0) \\ &\quad = (\pi((E^0)^!) \oplus \pi((E^1)^!))/(\pi(E^1) \oplus \pi(E^0)) = \pi(E^\perp)/\pi(E). \end{aligned}$$

Let $Prim(\gamma_E)$ be the subset of $M(\pi((E^0)^!)/\pi(E^1)) = \pi(E^\perp)/\pi(E)$ consisting of the subspaces $\pi(E(k))/\pi(E)$ (notation of 2.6) of $\pi(E^\perp)/p(E)$ for various k in

$[1, (D-2\dim(E)-1)/2]^1$ if $E \in \mathcal{F}'(V_D) - \mathcal{F}(V_{D-1})$, $\dim(E) < (D-1)/2$, or in

$\{0\} \cup [1, (D-2\dim(E)-1)/2]^1$ if $E \in \mathcal{F}(V_{D-1})$.

We now see that the map appearing in 0.6(a) can be identified with a map $\oplus_{E \in \mathcal{F}'(V_D)} Prim(\gamma_E) \to V'_D$. We define such a map by $(E, E(k)/E) \mapsto \vec{\epsilon}'_D(E(k))$. Using 2.12(a) and 3.5(c) we see that this map is a bijection. From the definitions, for E, k as above we see that $\mathbf{s}_{\gamma_E; {V'_D}^1}(\pi(E(k))/\pi(E)) \in \mathbf{C}[V'_D]$ is the characteristic function of $\pi(E(k))$ hence its support contains $\vec{\epsilon}'_D(\pi(E(k)))$ (see 3.5(d)). This proves the existence of the bijection Θ in 0.6(a) (in our case). The uniqueness of Θ follows from 3.5(e).

3.9. Following [8, 1.11] we define $u : V_D \to \mathbf{Z}$ by

$$u(x) = \sum_{s \in [1,r]; a_s \neq b_s \bmod 2} (-1)^{a_s}$$

where

$$x = e_{[a_1,b_1]} + e_{[a_2,b_2]} + \cdots + e_{[a_r,b_r]} \in V_D$$

with $1 \le a_1 \le b_1 \lll a_2 \le b_2 \ll \cdots \ll a_r \le b_r \le D$. When D is even let $\tilde{u} : V_D \to \mathbf{N}$ be the function defined by $\tilde{u}(x) = 2u(x)$ if $u(x) \ge 0$, $\tilde{u}(x) = -2u(x) - 1$ if $u(x) < 0$. When D is odd let $\tilde{u} : V_D \to \mathbf{N}$ be the function defined by $\tilde{u}(x) = 2u(x) - 1$ if $u(x) > 0$, $\tilde{u}(x) = -2u(x) + 1$ if $u(x) < 0$, $\tilde{u}(x) = 0$ if $u(x) = 0$.

(a) *Let $k \in \boldsymbol{N}$. Under the bijection $\epsilon_D : \boldsymbol{S}^*_D \to V_D$ (see 3.3(a)), the subset $\tilde{u}^{-1}(k)$ of V_D corresponds to the subset $\boldsymbol{S}^*_D(k)$ of $\boldsymbol{S}^*_D$ consisting of all $B \in \boldsymbol{S}^*_D$ such that the number of intervals $I \in B$ containing $\boldsymbol{a}$ (or equivalently such that $|I^*|$ is even) is equal to k.*

When D is even, this follows from [8, 1.14(c)]; the case where D is odd is similar.

3.10. In this subsection we assume that D is odd. We define $\tilde{u}' : V'_D \to \mathbf{N}$ as the composition $V'_D \to V_{D-1} \to V_D \to \mathbf{N}$ where the first map is the inverse of the bijection $V_{D-1} \to V'_D$ induced by π, the second map is the obvious inclusion and the third map is $\tilde{u}$ as in 3.9. The following result can be deduced from 3.9(a).

(a) *Let $k \in \boldsymbol{N}$. Under the bijection $\epsilon'_D : \boldsymbol{S'}^*_D \to V'_D$ (see 3.3(b)), the subset $\tilde{u}'^{-1}(k)$ of V'_D corresponds to the subset $\boldsymbol{S'}^*_D(k) = \boldsymbol{S'}^*_D \cap \boldsymbol{S}^*_D(k)$ of $\boldsymbol{S'}^*_D$ (see 3.9(a)).*

3.11. In this subsection we assume that D is even. We define a bijection $\kappa : \overline{[1,D]} \to \overline{[1,D]}$ by $\kappa(j) = j+1$ if $j \in [1, D-1]$, $\kappa(D) = \mathbf{a}$, $\kappa(\mathbf{a}) = 1$. This induces a bijection $\kappa : \tilde{\mathcal{I}}_D \to \tilde{\mathcal{I}}_D$, $I \mapsto \kappa(I) = \{\kappa(j); j \in I\}$ and a bijection $\kappa : R_D \to R_D$, $B \mapsto \kappa(B) = \{\kappa(I); I \in B\}$. From [9] it is known that for $B \in R_D$ we have

(a) *$B \in \boldsymbol{S}^*_D$ if and only if $\kappa(B) \in \boldsymbol{S}^*_D$.*

We show:

(b) *Let $B \in R_D$ be such that $|I|$ is odd for all $I \in B$. We have $B \in \boldsymbol{S}^*_D$ if and only if $\kappa^s(B) \in S_D$ for some $s \geq 0$.*

We can assume that $D \geq 2$. If B satisfies $\kappa^s(B) \in S_D$ for some $s \geq 0$ then by (a) we have $B \in \mathbf{S}^*_D$ (since $S_D \subset \mathbf{S}^*_D$). Conversely, assume that $B \in \mathbf{S}^*_D$. From the definition of $\mathbf{S}^*_D$ we see by induction on D that $\overline{[1,D]} - \cup_{I \in B} I \neq \emptyset$; let j be in this last set. For some $s \geq 0$ we have $\kappa^s(j) \neq \mathbf{a}$, so that if $B' = \kappa^s(B)$ we have $\mathbf{a} \notin \overline{[1,D]} - \cup_{I \in B'} I$, that is $\cup_{I \in B'} I \subset [1,D]$ so that $I^* = I$ for any $I \in B'$. It follows that $B'^* = B'$. By (a) we have $B' \in \mathbf{S}^*_D$ hence $B' \in \mathbf{S}_D$. Since $B' \in \mathbf{S}_D$, $B' = B'^1$ we have by definition $B' \in S_D$. This proves (b).

In view of 1.11, (b) provides a non-inductive description of $\mathbf{S}^*_D$ hence also of $\mathbf{S}_D$, which is simpler than that in [8, 1.3(c)].

4. Tables in Classical Types

4.1. Given $\tilde{B} \in \mathbf{S}^*_D$ we can define $B \in S_D$ by $\lambda^{-1}(\tilde{B}) = (B, k)$ (see 2.8) and then $E = \mathbf{E}_B \in \mathcal{F}(V_D)$ and $X = (E^0)^! \in \mathcal{C}(V^1_D)$, $Y = E^1 \in \mathcal{C}(V^1_D)$; we can also form $\epsilon_D(\tilde{B}) \in V_D$. We record the assignments $\tilde{B} \mapsto (Y, X)$ and $\tilde{B} \mapsto \epsilon_D(\tilde{B})$ in the tables of 4.2-4.4 (representing the cases where $D = 2, 4, 6$) and in the tables of 4.5-4.8 (representing the cases where $D = 1, 3, 5, 7$).

The table for V_D consists of several subtables, one for each $X \in \mathcal{C}(V^1_D)$. The subtable indexed by X has the name X in a box, has one row for each $Y \in \mathcal{C}(V^1_D)$ such that $(Y, X) \in \breve{\mathcal{C}}(V^1_D)$ and has a list of the elements $\tilde{B} \in \mathbf{S}^*_D$ which give rise as above to $Y \subset X$; the image $\epsilon_D(\tilde{B}) \in V_D$ of such a $\tilde{B}$ is also given.

Given $\tilde{B} \in \mathbf{S'}^*_D$ with D odd we can define (Y, X) as above; we set $Y' = \pi(Y) \in \mathcal{C}(V'^1_D), X' = \pi(X) \in \mathcal{C}(V'^1_D)$. We record the assignments $\tilde{B} \mapsto (Y', X')$ and $\tilde{B} \mapsto \epsilon'_D(\tilde{B})$ in the tables of 4.9-4.12 (representing the cases where $D = 1, 3, 5, 7$). The table for V'_D consists of several subtables, one for each $X' \in \mathcal{C}(V'^1_D)$. The subtable indexed by X' has the name X' in a box, has one row for each $Y' \in \mathcal{C}(V'^1_D)$ such that $(Y', X') \in \breve{\mathcal{C}}(V'^1_D)$ and has a list of the elements $\tilde{B} \in \mathbf{S'}^*_D$ which give rise as above to $Y' \subset X'$; the image $\epsilon'_D(\tilde{B}) \in V'_D$ of such a $\tilde{B}$ is also given.

We now explain the notation in the tables. Any $\tilde{B}$ in $\mathbf{S}^*_D$ (or in $\mathbf{S}'^*_D$ with D odd) is represented as list of intervals. For example $(6, 56\mathbf{a}, 456\mathbf{a})$ (for V_6) represents the set of intervals $\{6\}, \{5, 6, \mathbf{a}\}, \{4, 5, 6, \mathbf{a}\}$. The elements of V_D are written as $i_1 i_2 \dots i_k$ instead of $e_{i_1} + e_{i_2} + \dots + e_{i_k}$. For example 236 represents $e_2 + e_3 + e_6 \in V_6$. The elements of V'_D are written as $i_1 i_2 \dots i_k$ instead of $\pi(e_{i_1} + e_{i_2} + \dots + e_{i_k})$. The subspaces X or Y of V^1_D are represented by a sequence of generating vectors. For example $(13, 5)$ represents the subspace spanned by $e_1 + e_3$ and e_5. The subspaces X' or Y' of V'^1_D are written in the form $\pi(X)$ or $\pi(Y)$ with X or Y written as above.

4.2.

$$\boxed{X = (1)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}) \mapsto 12$; $Y = 0$

$(1) \mapsto 1$; $Y = (1)$

$$\boxed{X = 0}$$

$(2) \mapsto 2$; $Y = 0$

4.3. Table for V_4

$$\boxed{X = (1, 3)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}) \mapsto 1234$; $(\mathbf{a}, 1\mathbf{a}4) \mapsto 23$; $Y = 0$

$(1) \mapsto 1$; $(1, \mathbf{a}12) \mapsto 34$; $Y = (1)$

$(3) \mapsto 3$; $(3, \mathbf{a}) \mapsto 124$; $Y = (3)$

$(1, 3) \mapsto 13$; $Y = (1, 3)$

$$\boxed{X = (13)}$$

$(2) \mapsto 2$; $(2, \mathbf{a}) \mapsto 134$; $Y = 0$

$(2, 123) \mapsto 123$; $Y = (13)$

$$\boxed{X = (1)}$$

$(4) \mapsto 4$; $(4, \mathbf{a}43) \mapsto 12$; $Y = 0$

$(1,4) \mapsto 14$; $Y = (1)$

$$\boxed{X = (3)}$$

$(3,234) \mapsto 234$; $Y = (3)$

$$\boxed{X = 0}$$

$(2,4) \mapsto 24$; $Y = 0$

4.4. Table for V_6

$$\boxed{X = (1,3,5)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}) \mapsto 123456$; $(\mathbf{a}, 1\mathbf{a}6) \mapsto 2345$; $(\mathbf{a}, 1\mathbf{a}6, 21\mathbf{a}65) \mapsto 1256$; $Y = 0$

$(1) \mapsto 1$; $(1, \mathbf{a}12) \mapsto 3456$; $(1, 21\mathbf{a}, 321\mathbf{a}6) \mapsto 145$; $Y = (1)$

$(3) \mapsto 3$; $(3, \mathbf{a}) \mapsto 12456$; $(3, \mathbf{a}, 1\mathbf{a}6) \mapsto 245$; $Y = (3)$

$(5) \mapsto 5$; $(5, \mathbf{a}) \mapsto 12346$; $(5, \mathbf{a}, 1\mathbf{a}654) \mapsto 23$; $Y = (5)$

$(1,3) \mapsto 13$; $(1, 3, \mathbf{a}1234) \mapsto 56$; $Y = (1,3)$

$(1,5) \mapsto 15$; $(1, 5, \mathbf{a}12) \mapsto 346$; $Y = (1,5)$

$(3,5) \mapsto 35$; $(3, 5, \mathbf{a}) \mapsto 1246$; $Y = (3,5)$

$(1,3,5) \mapsto 135$; $Y = (1,3,5)$

$$\boxed{X = (13,5)}$$

$(2) \mapsto 2$; $(2, \mathbf{a}) \mapsto 13456$; $(2, \mathbf{a}, 321\mathbf{a}6) \mapsto 45$; $Y = 0$

$(2,5) \mapsto 25$; $(2, 5, \mathbf{a}) \mapsto 1346$; $Y = (5)$

$(2,123) \mapsto 123$; $(2, 123, \mathbf{a}1234) \mapsto 256$; $Y = (13)$

$(2,5,123) \mapsto 1235$; $Y = (13,5)$

$$\boxed{X = (1,35)}$$

$(4) \mapsto 4$; $(4, \mathbf{a}) \mapsto 12356$; $(4, \mathbf{a}, 1\mathbf{a}6) \mapsto 235$; $Y = 0$

$(1,4) \mapsto 14$; $(1, 4, \mathbf{a}12) \mapsto 356$; $Y = (1)$

$(4,345) \mapsto 345$; $(4, 345, \mathbf{a}) \mapsto 126$; $Y = (35)$

$(1,4,345) \mapsto 1345$; $Y=(1,35)$

$$\boxed{X=(1,3)}$$

$(6) \mapsto 6$; $(6,56\mathbf{a}) \mapsto 1234$; $(6,56\mathbf{a},1\mathbf{a}654) \mapsto 236$; $Y=0$

$(1,6) \mapsto 16$; $(1,6,21\mathbf{a}65) \mapsto 34$; $Y=(1)$

$(3,6) \mapsto 36$; $(3,6,56\mathbf{a}) \mapsto 124$; $Y=(3)$

$(1,3,6) \mapsto 136$; $Y=(1,3)$

$$\boxed{X=(1,5)}$$

$(5,456) \mapsto 456$; $(5,456,3456\mathbf{a}) \mapsto 125$; $Y=(5)$

$(1,5,456) \mapsto 1456$; $Y=(1,5)$

$$\boxed{X=(3,5)}$$

$(3,5,23456) \mapsto 23456$; $Y=(3,5)$

$$\boxed{X=(3,135)}$$

$(3,234) \mapsto 234$; $(3,234,\mathbf{a}) \mapsto 156$; $Y=(3)$

$(3,234,12345) \mapsto 1245$; $Y=(3,135)$

$$\boxed{X=(135)}$$

$(2,4) \mapsto 24$; $(2,4,\mathbf{a}) \mapsto 1356$; $Y=0$

$(2,4,12345) \mapsto 12345$; $Y=(135)$

$$\boxed{X=(1)}$$

$(4,6) \mapsto 46$; $(4,6,3456\mathbf{a}) \mapsto 12$; $Y=0$

$(1,4,6) \mapsto 146$; $Y=(1)$

$$\boxed{X=(3)}$$

$(3, 6, 234) \mapsto 2346$; $Y = (3)$

$$\boxed{X = (5)}$$

$(2, 5, 456) \mapsto 2456$; $Y = (5)$

$$\boxed{X = (13)}$$

$(2, 6) \mapsto 26$; $(2, 6, 56\mathbf{a}) \mapsto 134$; $Y = 0$

$(2, 6, 123) \mapsto 1236$; $Y = (13)$

$$\boxed{X = (35)}$$

$(4, 345, 23456) \mapsto 2356$; $Y = (35)$

$$\boxed{X = 0}$$

$(2, 4, 6) \mapsto 246$; $Y = 0$

4.5. Table for V_1

$$\boxed{X = (1)}$$

$\emptyset \mapsto 0$; $Y = 0$

$(1) \mapsto 1$; $Y = (1)$

4.6. Table for V_3

$$\boxed{X = (1, 3)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}3) \mapsto 12$; $(1\mathbf{a}) \mapsto 23$; $Y = 0$

$(1) \mapsto 1$; $Y = (1)$

$(3) \mapsto 3$; $Y = (3)$

$(1, 3) \mapsto 13$; $Y = (1, 3)$

$$\boxed{X = (13)}$$

$(2) \mapsto 2$; $Y = 0$

$(2,123) \mapsto 123$; $Y = (13)$

4.7. Table for V_5

$$\boxed{X = (1,3,5)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}5) \mapsto 1234$; $(1\mathbf{a}) \mapsto 2345$; $Y = 0$

$(1) \mapsto 1$; $(1, 21\mathbf{a}5) \mapsto 34$; $(1, 321\mathbf{a}) \mapsto 145$; $Y = (1)$

$(3) \mapsto 3$; $(3, \mathbf{a}5) \mapsto 124$; $(3, 1\mathbf{a}) \mapsto 245$; $Y = (3)$

$(5) \mapsto 5$; $(5, 1\mathbf{a}54) \mapsto 23$; $(5, \mathbf{a}543) \mapsto 125$; $Y = (5)$

$(1,3) \mapsto 13$; $Y = (1,3)$

$(1,5) \mapsto 15$; $Y = (1,5)$

$(3,5) \mapsto 35$; $Y = (3,5)$

$(1,3,5) \mapsto 135$; $Y = (1,3,5)$

$$\boxed{X = (13,5)}$$

$(2) \mapsto 2$; $(2, \mathbf{a}5) \mapsto 134$; $(2, 321\mathbf{a}) \mapsto 45$; $Y = 0$

$(2,5) \mapsto 25$; $Y = (5)$

$(2,123) \mapsto 123$; $Y = (13)$

$(2,5,123) \mapsto 1235$; $Y = (13,5)$

$$\boxed{X = (1,35)}$$

$(4) \mapsto 4$;$(4, \mathbf{a}543) \mapsto 12$; $(4, 1\mathbf{a}) \mapsto 235$; $Y = 0$

$(1,4) \mapsto 14$; $Y = (1)$

$(4,345) \mapsto 345$; $Y = (35)$

$(1,4,345) \mapsto 1345$; $Y = (1,35)$

$$\boxed{X = (3,135)}$$

$(3,234) \mapsto 234$; $Y = (3)$

$(3,234,12345)\mapsto 1245;\ Y=(3,135)$

$$\boxed{X=(135)}$$

$(2,4)\mapsto 24;\ Y=0$

$(2,4,12345)\mapsto 12345;\ Y=(135)$

4.8. Table for V_7

$$\boxed{X=(1,3,5,7)}$$

$\emptyset\mapsto 0;\ (\mathbf{a}7)\mapsto 123456;\ (1\mathbf{a})\mapsto 234567;\ (\mathbf{a}7,1\mathbf{a}76,21\mathbf{a}765)\mapsto 1256;$

$(1\mathbf{a},21\mathbf{a}7,321\mathbf{a}76)\mapsto 2367;\ Y=0$

$(1)\mapsto 1;\ (1,21\mathbf{a}7)\mapsto 3456;\ (1,321\mathbf{a})\mapsto 14567;\ Y=(1)$

$(3)\mapsto 3;\ (3,\mathbf{a}7)\mapsto 12456;\ (3,1\mathbf{a})\mapsto 24567;\ Y=(3)$

$(5)\mapsto 5;\ (5,\mathbf{a}7)\mapsto 12346;\ (5,1\mathbf{a})\mapsto 23467;\ Y=(5)$

$(7)\mapsto 7;\ (7,1\mathbf{a}76)\mapsto 2345;\ (7,\mathbf{a}765)\mapsto 12347;\ Y=(7)$

$(1,3)\mapsto 13;\ (1,3,4321\mathbf{a}7)\mapsto 56;\ (1,3,54321\mathbf{a})\mapsto 1367;\ Y=(1,3)$

$(1,5)\mapsto 15;\ (1,5,21\mathbf{a}7)\mapsto 346;\ (1,5,321\mathbf{a})\mapsto 1467;\ Y=(1,5)$

$(1,7)\mapsto 17;\ (1,7,321\mathbf{a}76)\mapsto 145;\ (1,7,21\mathbf{a}765)\mapsto 347;\ Y=(1,7)$

$(3,5)\mapsto 35;\ (3,5,\mathbf{a}7)\mapsto 1246;\ (3,5,1\mathbf{a})\mapsto 2467;\ Y=(3,5)$

$(3,7)\mapsto 37;\ (3,7,1\mathbf{a}76)\mapsto 245;\ (3,7,\mathbf{a}765)\mapsto 1247;\ Y=(3,7)$

$(5,7)\mapsto 57;\ (5,7,1\mathbf{a}7654)\mapsto 23;\ (5,7,\mathbf{a}76543)\mapsto 1257;\ Y=(5,7)$

$(1,3,5)\mapsto 135;\ Y=(1,3,5)$

$(1,3,7)\mapsto 137;\ Y=(1,3,7)$

$(1,5,7)\mapsto 157;\ Y=(1,5,7)$

$(3,5,7)\mapsto 357;\ Y=(3,5,7)$

$(1,3,5,7)\mapsto 1357;\ Y=(1,3,5,7)$

$$\boxed{X=(13,5,7)}$$

$(2)\mapsto 2;\ (2,\mathbf{a}7)\mapsto 13456;\ (2,321\mathbf{a})\mapsto 4567;\ Y=0$

$(2,5) \mapsto 25$; $(2,5,\mathbf{a}7) \mapsto 1346$; $(2,5,321\mathbf{a}) \mapsto 467$; $Y=(5)$

$(2,7) \mapsto 27$; $(2,7,321\mathbf{a}76) \mapsto 45$; $(2,7,\mathbf{a}765) \mapsto 1347$; $Y=(7)$

$(2,123) \mapsto 123$; $(2,123,4321\mathbf{a}7) \mapsto 256$; $(2,123,54321\mathbf{a}) \mapsto 12367$; $Y=(13)$

$(2,5,7) \mapsto 257$; $Y=(5,7)$

$(2,5,123) \mapsto 1235$; $Y=(13,5)$

$(2,7,123) \mapsto 1237$; $Y=(13,7)$

$(2,5,7,123) \mapsto 12357$; $Y=(13,5,7)$

$$\boxed{X=(1,35,7)}$$

$(4) \mapsto 4$; $(4,\mathbf{a}7) \mapsto 12356$; $(4,1\mathbf{a}) \mapsto 23567$; $Y=0$

$(1,4) \mapsto 14$; $(1,4,21\mathbf{a}7) \mapsto 356$; $(1,4,54321\mathbf{a}) \mapsto 167$; $Y=(1)$

$(4,7) \mapsto 47$; $(4,7,1\mathbf{a}76) \mapsto 235$; $(4,7,\mathbf{a}76543) \mapsto 127$; $Y=(7)$

$(4,345) \mapsto 345$; $(4,345,\mathbf{a}7) \mapsto 126$; $(4,345,1\mathbf{a}) \mapsto 267$; $Y=(35)$

$(1,4,7) \mapsto 147$; $Y=(1,7)$

$(1,4,345) \mapsto 1345$; $Y=(1,35)$

$(4,7,345) \mapsto 3457$; $Y=(35,7)$

$(1,4,7,345) \mapsto 13457$; $Y=(1,35,7)$

$$\boxed{X=(1,3,57)}$$

$(6) \mapsto 6$; $(6,\mathbf{a}765) \mapsto 1234$; $(6,1\mathbf{a}) \mapsto 23457$; $Y=0$

$(1,6) \mapsto 16$; $(1,6,21\mathbf{a}765) \mapsto 34$; $(1,6,321\mathbf{a}) \mapsto 1457$; $Y=(1)$

$(3,6) \mapsto 36$; $(3,6,\mathbf{a}765) \mapsto 124$; $(3,6,1\mathbf{a}) \mapsto 2457$; $Y=(3)$

$(6,567) \mapsto 567$; $(6,567,1\mathbf{a}7654) \mapsto 236$; $(6,567,\mathbf{a}76543) \mapsto 12567$; $Y=(57)$

$(1,3,6) \mapsto 136$; $Y=(1,3)$

$(1,6,567) \mapsto 1567$; $Y=(1,57)$

$(3,6,567) \mapsto 3567$; $Y=(3,57)$

$(1,3,6,567) \mapsto 13567$; $Y=(1,3,57)$

$$\boxed{X=(3,135,7)}$$

$(3,234) \mapsto 234$; $(3,234,\mathbf{a}7) \mapsto 156$; $(3,234,54321\mathbf{a}) \mapsto 367$; $Y=(3)$

$(3,7,234) \mapsto 2347$; $Y=(3,7)$

$(3,234,12345) \mapsto 1245$; $Y=(3,135)$

$(3,7,234,12345) \mapsto 12457$; $Y=(3,135,7)$

$$\boxed{X=(1,5,357)}$$

$(5,456) \mapsto 456$; $(5,456,\mathbf{a}76543) \mapsto 125$; $(5,456,1\mathbf{a}) \mapsto 237$; $Y=(5)$

$(1,5,456) \mapsto 1456$; $Y=(1,5)$

$(5,456,34567) \mapsto 3467$; $Y=(5,357)$

$(1,5,456,34567) \mapsto 13467$; $Y=(1,5,357)$

$$\boxed{X=(3,5,1357)}$$

$(3,5,23456) \mapsto 23456$; $Y=(3,5)$

$(3,5,23456,1234567) \mapsto 12467$; $Y=(3,5,1357)$

$$\boxed{X=(135,7)}$$

$(2,4) \mapsto 24$; $(2,4,\mathbf{a}7) \mapsto 1356$; $(2,4,54321\mathbf{a}) \mapsto 67$; $Y=0$

$(2,4,7) \mapsto 247$; $Y=(7)$

$(2,4,12345) \mapsto 12345$; $Y=(135)$

$(2,4,7,12345) \mapsto 123457$; $Y=(135,7)$

$$\boxed{X=(13,57)}$$

$(2,6) \mapsto 26$; $(2,6,\mathbf{a}765) \mapsto 134$; $(2,6,321\mathbf{a}) \mapsto 457$; $Y=0$

$(2,6,123) \mapsto 1236$; $Y=(13)$

$(2,6,567) \mapsto 2567$; $Y=(57)$

$(2, 6, 123, 567) \mapsto 123567$; $Y = (13, 57)$

$$\boxed{X = (1, 357)}$$

$(4, 6) \mapsto 46$; $(4, 6, \mathbf{a}76543) \mapsto 12$; $(4, 6, 1\mathbf{a}) \mapsto 2357$; $Y = 0$

$(1, 4, 6) \mapsto 146$; $Y = (1)$

$(4, 6, 34567) \mapsto 34567$; $Y = (357)$

$(1, 4, 6, 34567) \mapsto 134567$; $Y = (1, 357)$

$$\boxed{X = (3, 1357)}$$

$(3, 6, 234) \mapsto 2346$; $Y = (3)$

$(3, 6, 234, 1234567) \mapsto 124567$; $Y = (3, 1357)$

$$\boxed{X = (5, 1537)}$$

$(2, 5, 456) \mapsto 2456$; $Y = (5)$

$(2, 5, 456, 1234567) \mapsto 123467$; $Y = (5, 1357)$

$$\boxed{X = (35, 1357)}$$

$(4, 345, 23456) \mapsto 2356$; $Y = (35)$

$(4, 345, 23456, 1234567) \mapsto 1267$; $Y = (35, 1357)$

$$\boxed{X = (1357)}$$

$(2, 4, 6) \mapsto 246$; $Y = 0$

$(2, 4, 6, 1234567) \mapsto 1234567$; $Y = (1357)$

4.9. Table for V_1'

$$\boxed{X' = \pi(1)}$$

$\emptyset \mapsto 0$; $Y' = 0$

4.10. Table for V_3'

$$\boxed{X' = \pi(1,3)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}3) \mapsto 12$; $Y' = 0$

$(1) \mapsto 1$; $Y' = \pi(1)$

$$\boxed{X' = \pi(13)}$$

$(2) \mapsto 2$; $Y' = 0$

4.11. Table for V_5'

$$\boxed{X' = \pi(1,3,5)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}5) \mapsto 1234$; $Y' = 0$

$(1) \mapsto 1$; $(1, 21\mathbf{a}5) \mapsto 34$; $Y' = \pi(1)$

$(3) \mapsto 3$; $(3, \mathbf{a}5) \mapsto 124$; $Y' = \pi(3)$

$(5, 1\mathbf{a}54) \mapsto 23$; $Y' = \pi(5)$

$(1,3) \mapsto 13$; $Y' = \pi(1,3)$

$$\boxed{X' = \pi(13,5)}$$

$(2) \mapsto 2$; $(2, \mathbf{a}5) \mapsto 134$; $Y' = 0$

$(2, 123) \mapsto 123$; $Y' = \pi(13)$

$$\boxed{X' = \pi(1,35)}$$

$(4) \mapsto 4$; $(4, \mathbf{a}543) \mapsto 12$; $Y' = 0$

$(1,4) \mapsto 14$; $Y' = \pi(1)$

$$\boxed{X' = \pi(3,135)}$$

$(3,234) \mapsto 234$; $Y' = \pi(3)$

$$\boxed{X' = \pi(135)}$$

$(2,4) \mapsto 24$; $Y' = 0$

4.12. Table for V_7'

$$\boxed{X' = \pi(1,3,5,7)}$$

$\emptyset \mapsto 0$; $(\mathbf{a}7) \mapsto 123456$; $(\mathbf{a}7, 1\mathbf{a}76, 21\mathbf{a}765) \mapsto 1256$; $Y' = 0$

$(1) \mapsto 1$; $(1, 21\mathbf{a}7) \mapsto 3456$; $Y' = \pi(1)$

$(3) \mapsto 3$; $(3, \mathbf{a}7) \mapsto 12456$; $Y' = \pi(3)$

$(5) \mapsto 5$; $(5, \mathbf{a}7) \mapsto 12346$; $Y' = \pi(5)$

$(7, 1\mathbf{a}76) \mapsto 2345$; $Y' = \pi(7)$

$(1,3) \mapsto 13$; $(1, 3, 4321\mathbf{a}7) \mapsto 56$; $Y' = \pi(1,3)$

$(1,5) \mapsto 15$; $(1, 5, 21\mathbf{a}7) \mapsto 346$; $Y' = \pi(1,5)$

$(1, 7, 321\mathbf{a}76) \mapsto 145$; $Y' = \pi(1,7)$

$(3,5) \mapsto 35$; $(3, 5, \mathbf{a}7) \mapsto 1246$; $Y' = \pi(3,5)$

$(3, 7, 1\mathbf{a}76) \mapsto 245$; $Y' = \pi(3,7)$

$(5, 7, 1\mathbf{a}7654) \mapsto 23$; $Y' = \pi(5,7)$

$(1,3,5) \mapsto 135$; $Y' = \pi(1,3,5)$

$$\boxed{X' = \pi(13,5,7)}$$

$(2) \mapsto 2$; $(2, \mathbf{a}7) \mapsto 13456$; $Y' = 0$

$(2,5) \mapsto 25$; $(2, 5, \mathbf{a}7) \mapsto 1346$; $Y' = \pi(5)$

$(2, 7, 321\mathbf{a}76) \mapsto 45$; $Y' = \pi(7)$

$(2,123) \mapsto 123$; $(2, 123, 4321\mathbf{a}7) \mapsto 256$; $Y' = \pi(13)$

$(2, 5, 123) \mapsto 1235$; $Y' = \pi(13,5)$

$$\boxed{X' = \pi(1,35,7)}$$

$(4) \mapsto 4$; $(4, \mathbf{a}7) \mapsto 12356$; $Y' = 0$

$(1, 4) \mapsto 14$; $(1, 4, 21\mathbf{a}7) \mapsto 356$; $Y' = \pi(1)$

$(4, 7, 1\mathbf{a}76) \mapsto 235$; $Y' = \pi(7)$

$(4, 345) \mapsto 345$; $(4, 345, \mathbf{a}7) \mapsto 126$; $Y' = \pi(35)$

$(1, 4, 345) \mapsto 1345$; $Y' = \pi(1, 35)$

$$\boxed{X' = \pi(1, 3, 57)}$$

$(6) \mapsto 6$; $(6, \mathbf{a}765) \mapsto 1234$; $Y' = 0$

$(1, 6) \mapsto 16$; $(1, 6, 21\mathbf{a}765) \mapsto 34$; $Y' = \pi(1)$

$(3, 6) \mapsto 36$; $(3, 6, \mathbf{a}765) \mapsto 124$; $Y' = \pi(3)$

$(6, 567, 1\mathbf{a}7654) \mapsto 236$; $Y' = \pi(57)$

$(1, 3, 6) \mapsto 136$; $Y' = \pi(1, 3)$

$$\boxed{X' = \pi(3, 135, 7)}$$

$(3, 234) \mapsto 234$; $(3, 234, \mathbf{a}7) \mapsto 156$; $Y' = \pi(3)$

$(3, 234, 12345) \mapsto 1245$; $Y' = \pi(3, 135)$

$$\boxed{X' = \pi(1, 5, 357)}$$

$(5, 456) \mapsto 456$; $(5, 456, \mathbf{a}76543) \mapsto 125$; $Y' = \pi(5)$

$(1, 5, 456) \mapsto 1456$; $Y' = \pi(1, 5)$

$$\boxed{X' = \pi(3, 5, 1357)}$$

$(3, 5, 23456) \mapsto 23456$; $Y' = \pi(3, 5)$

$$\boxed{X' = \pi(135, 7)}$$

$(2, 4) \mapsto 24$; $(2, 4, \mathbf{a}7) \mapsto 1356$; $Y' = 0$

$(2,4,12345) \mapsto 12345$; $Y' = \pi(135)$

$$\boxed{X' = \pi(13,57)}$$

$(2,6) \mapsto 26$; $(2,6,\mathbf{a}765) \mapsto 134$; $Y' = 0$

$(2,6,123) \mapsto 1236$; $Y' = \pi(13)$

$$\boxed{X' = \pi(1,357)}$$

$(4,6) \mapsto 46$; $(4,6,\mathbf{a}76543) \mapsto 12$; $Y' = 0$

$(1,4,6) \mapsto 146$; $Y' = \pi(1)$

$$\boxed{X' = \pi(3,1357)}$$

$(3,6,234) \mapsto 2346$; $Y' = \pi(3)$

$$\boxed{X' = \pi(5,1537)}$$

$(2,5,456) \mapsto 2456$; $Y' = \pi(5)$

$$\boxed{X' = \pi(35,1357)}$$

$(4,345,23456) \mapsto 2356$; $Y' = \pi(35)$

$$\boxed{X' = \pi(1357)}$$

$(2,4,6) \mapsto 246$; $Y' = 0$

5. Exceptional types

5.1. We now return to the setup in 0.1 and assume that G is of exceptional type. Recall that we have fixed a family c in $\mathrm{Irr}(W)$. We must be in one of the following cases.

(i) $|c| = 1$, $\mathcal{G}_c = S_1$.

(ii) $|c| = 2$ (with W of type E_7 or E_8), $\mathcal{G}_c = S_2$.

(iii) $|c| = 3$, $\mathcal{G}_c = S_2$.

(iv) $|c| = 4$ (with W of type G_2), $\mathcal{G}_c = S_3$.

(v) $|c| = 5$ (with W of type E_6, E_7 or E_8), $\mathcal{G}_c = S_3$.

(vi) $|c| = 11$ (with W of type F_4), $\mathcal{G}_c = S_4$.

(vii) $|c| = 17$ (with W of type E_8), $\mathcal{G}_c = S_5$.

Here for $n \in [1, 5]$, S_n denotes the group of permutations of $[1, n]$.

5.2. In this subsection we assume that $\mathcal{G}_c = S_5$. We write $H_{5!} = S_5$. Let $H_{4!}$ be the group of all $\sigma \in S_5$ which map $[1, 4]$ to itself and 5 to itself. Let $H_{2!3!}$ be the group of all $\sigma \in S_5$ which map $[1, 2]$ to itself and $[3, 5]$ to itself. Let H_8 be the group of all $\sigma \in S_5$ which commute with the permutation $1 \mapsto 2 \mapsto 1, 3 \mapsto 4 \mapsto 3,\ 5 \mapsto 5$. Let $H_{3!}$ be the group of all $\sigma \in S_5$ which map $[3, 5]$ to itself and 1 to 1, 2 to 2. Let $H_{2!2!}$ be the group of all $\sigma \in S_5$ which map $[1, 2]$ to itself, $[3, 4]$ to itself and 5 to 5. Let $H_{2!}$ be the subgroup of all $\sigma \in S_5$ which map $[1, 2]$ to $[1, 2]$, 3 to 3, 4 to 4 and 5 to 5. Let $H_{1!} = \{1\} \subset S_5$. Let

$$\mathbf{H}_c = \{H_{5!}, H_{4!}, H_{2!3!}, H_8, H_{2!2!}, H_{3!}, H_{2!}, H_{1!}\}.$$

We define $\breve{\mathbf{H}}_c$ as the set of all pairs (H_i, H_j) where $i \le j$ are such that H_i is a normal subgroup of H_j but $(i, j) \ne (1, 8)$.

5.3. In this subsection we assume that $\mathcal{G}_c = S_4$. We write $H_{4!} = S_4$. Let H_8 be the group of all $\sigma \in S_4$ which commute with the permutation $1 \mapsto 2 \mapsto 1, 3 \mapsto 4 \mapsto 3$. Let $H_{3!}$ be the group of all $\sigma \in S_4$ which map $[1, 3]$ to itself and 4 to 4. Let $H_{2!2!}$ be the group of all $\sigma \in S_4$ which map $[1, 2]$ to itself, $[3, 4]$ to itself. Let $H_{2!}$ be the subgroup of all $\sigma \in S_4$ which map $[1, 2]$ to $[1, 2]$, 3 to 3, 4 to 4. Let $H_{1!} = \{1\} \subset S_4$. Let

$$\mathbf{H}_c = \{H_{4!}, H_8, H_{2!2!}, H_{3!}, H_{2!}, H_{1!}.\}$$

We define $\breve{\mathbf{H}}_c$ as the set of all pairs (H_i, H_j) where $i \le j$ are such that H_i is a normal subgroup of H_j but $(i, j) \ne (1, 8)$.

5.4. In this subsection we assume that $\mathcal{G}_c = S_3$. We write $H_{3!} = S_3$. Let $H_{2!}$ be the subgroup of all $\sigma \in S_3$ which map $[1, 2]$ to $[1, 2]$, 3 to 3. Let

$H_{1!} = \{1\} \subset S_3$. Let

$$\mathbf{H}_c = \{H_{3!}, H_{2!}, H_{1!}.\}$$

We define $\breve{\mathbf{H}}_c$ as the set of all pairs (H_i, H_j) where $i \le j$ are such that H_i is a normal subgroup of H_j.

5.5. If $\mathcal{G}_c = S_2$ let $H_{2!} = S_2, H_{1!} = \{1\} \subset S_2$. Let $\mathbf{H}_c = \{H_{2!}, H_{1!}\}$. We define $\breve{\mathbf{H}}_c$ as the set of all pairs $H_i \subset H_j$ where $i \le j$. If $\mathcal{G}_c = S_1$ let $H_{1!} = \{1\} = S_1$. Let $\mathbf{H}_c = \{H_{1!}\}$. We define $\breve{\mathbf{H}}_c$ as the set consisting of $(H_{1!}, H_{1!})$.

5.6. In each of the cases in 5.2-5.5, for any (H_i, H_j) in $\breve{\mathbf{H}}_c$ we can describe H_j/H_i as follows.

If $i = j$ then $H_j/H_i = \{1\}$.

If $i = k!$ then $H_i/\{1\} = S_k$ canonically.

If $i = 2!3!$ then $H_i/H_{2!} = S_3, H_i/H_{3!} = S_2, H_i/\{1\} = S_2 \times S_3$ canonically.

If $i = 2!2!$ then $H_i/H_{2!} = S_2, H_i/\{1\} = S_2 \times S_2$ canonically.

If $i = 8$ then $H_i/H_{2!2!} = S_2$ canonically.

5.7. When $n > 1$ let μ'_n be the subset of $\mathbf{C}^*$ consisting of the primitive n-th roots of 1.

Let Γ be one of the groups $S_n, n = 1, 2, 3, 4, 5$ or $S_2 \times S_2$ or $S_3 \times S_2$. We define a subset $Prim(\Gamma)$ of $\mathbf{C}[M(\Gamma)]$ as follows. If $\Gamma = S_1$ then $Prim(\Gamma)$ consists of $(1, 1)$.

If $\Gamma = S_n$, $n = 2, 3, 4, 5$, then $Prim(\Gamma)$ consists of $(1, 1)$ and of Λ_e, $e \in \mu'_n$ (as in [8, 3.2]).

If $\Gamma = S_2 \times S_3$, then $Prim(\Gamma)$ consists of the five elements $\Lambda_{-1} \boxtimes (1, 1)$, $\Lambda_{-1} \boxtimes \Lambda_e, (1, 1) \boxtimes \Lambda_e$ $(e \in \mu'_3)$ of $\mathbf{C}[M(S_2)] \otimes \mathbf{C}[M(S_3)] = \mathbf{C}[M(\Gamma)]$ and of $(1, 1) \in \mathbf{C}[M(\Gamma)]$. (Note that in this case we have $Prim(\Gamma) = Prim(S_2) \otimes Prim(S_3) \subset \mathbf{C}[M(S_2)] \otimes \mathbf{C}[M(S_3)] = \mathbf{C}[M(\Gamma)]$.)

If $\Gamma = S_2 \times S_2$, then $Prim(\Gamma)$ consists of the two elements $\Lambda_{-1} \boxtimes \Lambda_{-1}, \Lambda_{-1} \boxtimes (1, 1)$ of $\mathbf{C}[M(S_2)] \otimes \mathbf{C}[M(S_2)] = \mathbf{C}[M(\Gamma)]$ and of $(1, 1) \in \mathbf{C}[M(\Gamma)]$. (Note that the two factors in $S_2 \times S_2$ play an asymmetric role: in fact, $S_2 \times S_2$ can be viewed as a two-dimensional vector space with a given ordered basis.)

5.8. In each of the cases in 5.2-5.5, for any (H_i, H_j) in $\breve{\mathbf{H}}_c$ we set

$$Prim(H_i, H_j) = Prim(H_j/H_i);$$

we use the identifications in 5.6 and the definitions in 5.7. Therefore both the source and target of Θ in 0.6(a) are defined. The existence and uniqueness of such a Θ has been already verified in [7], [8]. (Note that in [8] the definition of $Prim(S_5)$ and that of $Prim(S_3)$ for G of type G_2 is different from the present one, but this does not affect the proofs.) Thus the statements in 0.6 and hence those in 0.2 are established when G is of exceptional type.

In 5.9-5.13 we describe the bijection 0.6(a) in each of the cases in 5.2-5.5. We give a table for each type of $\mathcal{G}_c$. Each table consists of several subtables, one for each $H' \in \mathbf{H}_c$. The subtable indexed by $H' \in \mathbf{H}_c$ has the name $X = H'$ (in a box), has one row for each $Y = H \in \mathbf{H}_c$ such that $(H, H') \in \breve{\mathbf{H}}_c$ and has a list of the elements of $M(\mathcal{G}_c)$ (in the notation of [4, §4]) which are in the fibre of α_c (in 0.3) at (H, H').

5.9. Table for $\mathcal{G}_c = S_1$

$$\boxed{X = H_{1!}}$$

$(1, 1)$; $Y = 1$

5.10. Table for $\mathcal{G}_c = S_2$

$$\boxed{X = H_{2!}}$$

$(1, 1)$, (g_2, ϵ); $Y = 1$

$(g_2, 1)$; $Y = H_{2!}$

$$\boxed{X = 1}$$

$(1, \epsilon)$; $Y = 1$

5.11. Table for $\mathcal{G}_c = S_3$

$$\boxed{X = H_{3!}}$$

$(1,1)$, $(g_3,\theta),(g_3,\theta^2)$; $Y=1$

$(g_3,1)$; $Y=H_{3!}$

$$\boxed{X=H_{2!}}$$

$(1,r)$, (g_2,ϵ); $Y=1$

$(g_2,1)$; $Y=H_{2!}$

$$\boxed{X=1}$$

$(1,\epsilon)$; $Y=1$

5.12. Table for $\mathcal{G}_c=S_4$

$$\boxed{X=H_{4!}}$$

$(1,1)$; $(g_4,i),g_4(-i)$; $Y=1$

$(g_4,1)$; $Y=H_{4!}$

$$\boxed{X=H_8}$$

$(g_2',1),(g_4,-1)$ $Y=H_{2!2!}$

(g_2',ϵ'); $Y=H_8$

$$\boxed{X=H_{2!2!}}$$

$(1,\sigma)$, $(g_2,\epsilon'),(g_2',\epsilon)$; $Y=1$

$(g_2,1)$, (g_2',r); $Y=H_{2!}$

(g_2',ϵ''); $Y=H_{2!2!}$

$$\boxed{X=H_{3!}}$$

$(1,\lambda^1)$, $(g_3,\theta),(g_3,\theta^2)$; $Y=1$

$(g_3,1)$; $Y=H_{3!}$

$$\boxed{X=H_{2!}}$$

$(1,\lambda^2)$, (g_2,ϵ); $Y=1$

$(g_2, \epsilon''); \ Y = H_{2!}$

$$\boxed{X = 1}$$

$(1, \lambda^3); \ Y = 1$

5.13. Table for $\mathcal{G}_c = S_5$

$$\boxed{X = H_{5!}}$$

$(1, 1); \ (g_5, \zeta), (g_5, \zeta^2), (g_5, \zeta^3), (g_5, \zeta^4); \ Y = 1$

$(g_5, 1); \ Y = H_{5!}$

$$\boxed{X = H_{4!}}$$

$(1, \lambda^1); \ (g_4, i), g_4(-i); \ Y = 1$

$(g_4, 1); \ Y = H_{4!}$

$$\boxed{X = H_{2!3!}}$$

$(1, \nu), \ (g_2, -1), (g_3, \theta), (g_3, \theta^2), (g_6, -\theta), (g_6, -\theta^2); \ Y = 1$

$(g_2, 1), (g_6, \theta), (g_6, \theta^2); \ Y = H_{2!}$

$(g_3, 1), (g_6, -1); \ Y = H_{3!}$

$(g_6, 1); \ Y = H_{2!3!}$

$$\boxed{X = H_8}$$

$(g_2', 1), (g_4, -1) \ Y = H_{2!2!}$

$(g_2', \epsilon'); \ Y = H_8$

$$\boxed{X = H_{2!2!}}$$

$(1, \nu'), \ (g_2, -r), (g_2', \epsilon); \ Y = 1$

$(g_2, r), \ (g_2', r); \ Y = H_{2!}$

$(g_2', \epsilon''); \ Y = H_{2!2!}$

$$\boxed{X = H_{3!}}$$

$(1, \lambda^2), \ (g_3, \epsilon\theta), (g_3, \epsilon\theta^2); \ Y = 1$

(g_3,ϵ); $Y=H_{3!}$

$$\boxed{X=H_{2!}}$$

$(1,\lambda^3)$, $(g_2,-\epsilon)$; $Y=1$

(g_2,ϵ); $Y=H_{2!}$

$$\boxed{X=1}$$

$(1,\lambda^4)$; $Y=1$

6. The partition $\mathcal{U}_c=\sqcup_{\mathbf{g}\in\underline{\mathbf{P}}}\mathcal{U}_c^{\mathbf{g}}$

6.1. Assume that G is as in 0.6(i). The partition 0.2(a) of $\mathcal{U}_c$ (or equivalently of $M(\mathcal{G}_c)=V_D$, D even) corresponds under the bijection ϵ_D in 3.3(a) to the partition

$$\mathbf{S}^*_D=\sqcup_{k\in\mathbf{N}}\mathbf{S}^*_D(k)$$

where $\mathbf{S}^*_D(k)$ is as in 3.9. From the proof in 3.7 we see that 0.6(b) and 0.7(a) hold in our case and that if $\mathbf{g}\in\underline{\mathbf{P}}$ corresponds to k as above, then

(a) $\breve{\mathbf{H}}_c^{\mathbf{g}}=\{(Y\subset X)\in\breve{\mathcal{C}}(V_D^1);k\le\dim(X/Y)\}$.

6.2. Assume that G is as in 0.6(ii). The partition 0.2(a) of $\mathcal{U}_c$ (or equivalently of $M(\mathcal{G}_c)=V'_D$, D odd) corresponds under the bijection ϵ'_D in 3.3(b) to the partition

$$\mathbf{S}'^*_D=\sqcup_{k\in\{0,1,3,5,\dots\}}\mathbf{S}'^*_D(k)$$

where $\mathbf{S}'^*_D(k)$ is as in 3.10. From the proof in 3.8 we see that 0.6(b) and 0.7(a) hold in our case and that if $\mathbf{g}\in\underline{\mathbf{P}}$ corresponds to k as above then

(a) $\breve{\mathbf{H}}_c^{\mathbf{g}}=\{(Y'\subset X')\in\breve{\mathcal{C}}(V'^1_D);k\le\dim(X'/Y')\}$ if $k>0$;
(b) $\breve{\mathbf{H}}_c^{\mathbf{g}}=\{(\pi(Y)\subset\pi(X));(Y\subset X)\in\breve{\mathcal{C}}(V_D{}^1),Y\subset V_{D-1}\}$ if $k=0$

6.3. If G is of type A, then 0.6(b) and 0.7(a) are obvious.

Until the end of 6.9 we assume that G is as in 0.6(iii). In this case, 0.6(b) and 0.7(a) can be verified using the tables in §5. We will describe the sets $\breve{\mathbf{H}}_c^{\mathbf{g}}$ in several examples.

6.4. Assume first that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ were $P = G$ hence R is unipotent cuspidal. We will specify $\mathbf{g}$ by writing the corresponding element of $M(\mathcal{G}_c)$. In the following tables we describe the one element sets $\breve{\mathbf{H}}_c^{\mathbf{g}}$ for various such $\mathbf{g}$.

Assume that $|c| = 17$ so that G is of type E_8. The table is:

$\mathbf{g} \leftrightarrow (g_5, \zeta^j)$: $(1, H_{5!})$ $(j = 1, 2, 3, 4)$;

$\mathbf{g} \leftrightarrow (g_4, i^j)$: $(1, H_{4!})$ $(j = 1, 3)$;

$\mathbf{g} \leftrightarrow (g_6, -\theta^j)$: $(1, H_{2!3!})$ $(j = 1, 2)$;

$\mathbf{g} \leftrightarrow (g_3, \epsilon\theta^j)$: $(1, H_{3!})$ $(j = 1, 2)$;

$\mathbf{g} \leftrightarrow (g_2', \epsilon)$: $(1, H_{2!2!})$;

$\mathbf{g} \leftrightarrow (g_2, -\epsilon)$: $(1, H_{2!})$;

$\mathbf{g} \leftrightarrow (1, \lambda^4)$: $(1, 1)$.

Assume that $|c| = 11$ so that G is of type F_4. The table is:

$\mathbf{g} \leftrightarrow (g_4, i^j)$: $(1, H_{4!})$ $(j = 1, 3)$;

$\mathbf{g} \leftrightarrow (g_3, \theta^j)$: $(1, H_{3!})$ $(j = 1, 2)$;

$\mathbf{g} \leftrightarrow (g_2', \epsilon)$: $(1, H_{2!2!})$;

$\mathbf{g} \leftrightarrow (g_2, \epsilon)$: $(1, H_{2!})$;

$\mathbf{g} \leftrightarrow (1, \lambda^3)$: $(1, 1)$.

Assume that $|c| = 4$ so that G is of type G_2. The table is:

$\mathbf{g} \leftrightarrow (g_3, \theta^j)$: $(1, H_{3!})$ $(j = 1, 2)$;

$\mathbf{g} \leftrightarrow (g_2, \epsilon)$: $(1, H_{2!})$;

$\mathbf{g} \leftrightarrow (1, \epsilon)$: $(1, 1)$.

6.5. Assume now that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ where the adjoint group of $\bar{P}$ is of type E_6 and R is one of the two unipotent cuspidal representations of $\bar{P}(F_q)$. In this case G is of type E_6, E_7 or E_8.

If $|c| = 17$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(1, H_{2!3!}), (H_{2!}, H_{2!3!})\}.$$

If $|c| = 5$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(1, H_{3!})\}.$$

6.6. Assume now that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ where the adjoint group of $\bar{P}$ is of type D_4 and R is unipotent cuspidal. In this case G is of type E_6, E_7 or E_8.

If $|c| = 17$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(1, H_{2!2!}), (H_{2!}, H_{2!2!}), (H_{2!2!}, H_8), (H_{3!}, H_{2!3!}), (1, H_{2!3!})\}.$$

If $|c| = 3$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(1, H_{2!})\}.$$

6.27. Assume now that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ where the adjoint group of $\bar{P}$ is of type B_2 and R is unipotent cuspidal. In this case G is of type F_4.

If $|c| = 11$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(H_{2!}, H_{2!2!}), (H_{2!2!}, H_8), (1, H_{2!2!})\}.$$

If $|c| = 3$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}} = \{(1, H_{2!})\}.$$

6.8. Assume now that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ where the adjoint group of $\bar{P}$ is of type E_7 and R is one of the two unipotent cuspidal representations of $\bar{P}(F_q)$. In this case G is of type E_7 or E_8.

If $|c| = 2$ then $\breve{\mathbf{H}}_c^{\mathbf{g}}$ is either $\{(1, H_{2!})\}$ or $\{(1, 1)\}$.

6.9. Assume now that $\mathbf{g}$ is the image in $\underline{\mathbf{P}}$ of $(P, R) \in \mathbf{P}$ where $\bar{P}$ is a torus and R is the unit representations of $\bar{P}(F_q)$.

If $|c|=17$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c-\{(1,1)\}$$
$$=\{(1,H_{2!}),(H_{2!},H_{2!}),(1,H_{2!2!}),(H_{2!},H_{2!2!}),(H_{2!2!},H_{2!2!}),(1,H_{3!}),$$
$$(H_{3!},H_{3!}),(H_{2!2!},H_8),(H_8,H_8),(1,H_{2!3!}),(H_{2!},H_{2!3!}),(H_{3!},H_{2!3!}),$$
$$(H_{2!3!},H_{2!3!}),(1,H_{4!}),(H_{4!},H_{4!}),(1,H_{5!}),(H_{5!},H_{5!})\}.$$

If $|c|=11$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c-\{(1,1)\}$$
$$=\{(1,H_{2!}),(H_{2!},H_{2!}),(1,H_{2!2!}),(H_{2!},H_{2!2!}),(H_{2!2!},H_{2!2!}),(1,H_{3!}),$$
$$(H_{3!},H_{3!}),(H_{2!2!},H_8),(H_8,H_8),(1,H_{4!}),(H_{4!},H_{4!})\}.$$

If $|c|=4$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c-\{(1,1)\}=\{(1,H_{2!}),(H_{2!},H_{2!}),(1,H_{3!}),(H_{3!},H_{3!})\}.$$

If $|c|=2$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c-\{(1,1)\}=\{(1,H_{2!}),(H_{2!},H_{2!})\}.$$

If $|c|=5$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c=\{(1,1),(1,H_{2!}),(H_{2!},H_{2!}),(1,H_{3!}),(H_{3!},H_{3!})\}.$$

If $|c|=3$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c=\{(1,1),(1,H_{2!}),(H_{2!},H_{2!})\}.$$

If $|c|=1$ then

$$\breve{\mathbf{H}}_c^{\mathbf{g}}=\breve{\mathbf{H}}_c=\{(1,1)\}.$$

7. The Third Basis of $\mathbf{C}[M(\mathcal{G}_c)]$

7.1. Until the end of 7.5 we assume that D is even. Let $E\in\mathcal{F}(V_D)$. Let

$\Delta = D - 2\dim(E)$. For $k \in [0, \Delta/2]$ let $\mathfrak{T}_E^k$ be as in 2.6 (a subspace of $\mathfrak{T}_E$, see 2.4). This subspace is in $\boldsymbol{\mathcal{F}}(\mathfrak{T}_E)$ hence we can consider its image $v_E^k \in \mathfrak{T}_E$ under the bijection $\boldsymbol{\mathcal{F}}(\mathfrak{T}_E) \to \mathfrak{T}_E$ obtained from 3.5(a) by replacing V_D by $\mathfrak{T}_E$. By [9, 1.9] (for $\mathfrak{T}_E$ instead of V_D), we have

$v_E^k = 0$ if $k = 0$,

$v_E^k = e_{I_{[1,M]}}$ if $k = 1$,

$v_E^k = e_{I_{[2,M-1]}}$ if $k = 2$,

$v_E^k = e_{I_{[1,2]}} + e_{I_{[M-1,M]}}$ if $k = 3$,

$v_E^k = e_{I_{[2,3]}} + e_{I_{[M-2,M-1]}}$ if $k = 4$,

$v_E^k = e_{I_{[1,2]}} + e_{I_{[5,M-4]}} + e_{I_{[M-1,M]}}$ if $k = 5$,

$v_E^k = e_{I_{[2,3]}} + e_{I_{[6,M-5]}} + e_{I_{[M-2,M-1]}}$ if $k = 6$, etc. (Notation of 2.6)

Assuming that $D \geq 2$ and that $i \in [1, D]$, $E' \in \mathcal{F}(V_{D-2})$ are such that $E = T_i(E') \oplus \mathbf{F}e_i$, we note that (by 2.5(a)) we have for $k \in [0, \Delta/2]$

(a) $T_i(v_{E'}^k) = v_E^k$.

Let $\tilde{u} : V_D \to \mathbf{N}$ be as in 3.9; when $D \geq 2$ we denote by $\underline{\tilde{u}} : V_{D-2} \to \mathbf{N}$ the analogous function. We show:

(b) *Let $v \in E$, $k \in [0, \Delta/2]$, $z \in \mathfrak{T}_E^k$. If $z = v_E^k$, then $\tilde{u}(v+z) = k$. If $z \neq v_E^k$, then $\tilde{u}(v + z) < k$.*

We argue by induction on $\dim(E)$. If $E = 0$ then $v = 0$ and the result follows from [8, 1.15(a)]. Assume now that $E \neq 0$. Then $D \geq 2$ and we can find $i \in [1, D]$, $E' \in \mathcal{F}(V')$ such that $E = T_i(E') \oplus \mathbf{F}e_i$. We have $v = T_i(v') + ce_i$ where $v' \in E', c \in \mathbf{F}$. We have $z = T_i(z')$ where $z' \in \mathfrak{T}_{E'}^k$; moreover we have $z = v_E^k$ if and only if $z' = v_{E'}^k$ (see (a)).

We have $\tilde{u}(v + z) = \tilde{u}(T_i(v' + z') + ce_i) = \underline{\tilde{u}}(v' + z')$ (we have used [8, 1.11(b)]). By the induction hypothesis we have $\underline{\tilde{u}}(v' + z') = k$ if $z' = v_{E'}^k$ (that is, if $z = v_E^k$) and $\underline{\tilde{u}}(v' + z') < k$ if $z' \neq v_{E'}^k$ (that is, if $z \neq v_E^k$). Now (b) follows.

7.2. The following is a reformulation of 3.9(a):

(a) *Let $k \in \boldsymbol{N}$. Then $v \in V_D$ satisfies $v \in \tilde{u}^{-1}(k)$ if and only if $v = \bar{\epsilon}_D(\boldsymbol{E})$ where $\boldsymbol{E} = E(k)$ with $(E, k) \in \underline{\boldsymbol{\mathcal{F}}}(V_D)$.*

Let $\mathbf{C}[V_D]$ be the vector space of formal $\mathbf{C}$-linear combinations of elements of V_D. For $f \in \mathbf{C}[V_D]$ we write $f = \sum_{v \in V_D} f(v)v \in \mathbf{C}[V_D]$ where $f(v) \in \mathbf{C}$; let $supp(f) = \{v \in V_D; f(v) \neq 0\}$. For $(E,k) \in \underline{\boldsymbol{\mathcal{F}}}(V_D)$ we set $f_E^k = \sum_{v \in E + v_E^k} v \in \mathbf{C}[V_D]$. For $k \in \mathbf{N}$ let $\mathbf{C}[V_D]_k$ (resp. $\mathbf{C}[V_D]_{\le k}$ be the subspace $\{f \in \mathbf{C}[V_D];\ supp(f) \subset \tilde{u}^{-1}(k)\}$ (resp. $\{f \in \mathbf{C}[V_D];\ supp(f) \subset \cup_{k' \in [0,k]} \tilde{u}^{-1}(k')\}$) of $\mathbf{C}[V_D]$. We show:

(b) *For any $k \in \mathbf{N}$, $\{f_E^k; E \in \mathcal{F}(V_D)\}$ is a $\mathbf{C}$-basis of $\mathbf{C}[V_D]_k$.*

(c) *$\{f_E^k; (E,k) \in \underline{\boldsymbol{\mathcal{F}}}(V_D)\}$ is a $\mathbf{C}$-basis of $\mathbf{C}[V_D]$.*

For $\mathbf{E} \in \boldsymbol{\mathcal{F}}(V)$ let $g_{\mathbf{E}} = \sum_{v \in \mathbf{E}} v \in \mathbf{C}[V_D]$. By 7.1(b), if $\mathbf{E}, E, k$ are as in (a), then $g_{\mathbf{E}} \in \mathbf{C}[V_D]_{\le k}$. By [8, 1.17(a)], $\{g_{\mathbf{E}}; \mathbf{E} \in \boldsymbol{\mathcal{F}}(V_D)\}$ is a $\mathbf{C}$-basis of $\mathbf{C}[V_D]$. Hence for any $k \in \mathbf{N}$,

(d) $\{g_{\boldsymbol{E}}; \boldsymbol{E} \in \boldsymbol{\mathcal{F}}(V_D), \boldsymbol{E} = E(k'), (E,k') \in \underline{\boldsymbol{\mathcal{F}}}(V_D), k' \le k\}$

is a linearly independent subset of $\mathbf{C}[V_D]_{\le k}$ of cardinal equal to $|\cup_{k' \in [0,k]} \tilde{u}^{-1}(k')|$ (we use (a)). It follows that

(e) *the elements (d) form a $\mathbf{C}$-basis of $\mathbf{C}[V_D]_{\le k}$.*

We show by induction on $k \in \mathbf{N}$ that

(f) *the elements $\{f_E^{k'}; (E,k') \in \underline{\boldsymbol{\mathcal{F}}}(V_D), k' \in [0,k]\}$ form a $\mathbf{C}$-basis of $\mathbf{C}[V_D]_{\le k}$.*

Assume first that $k = 0$. If $E \in \mathcal{F}(V_D)$, we have $g_E = f_E^0$ so that (f) follows from (e). Next we assume that $k \ge 1$. Let $(E,k') \in \underline{\boldsymbol{\mathcal{F}}}(V_D), k' = k$. We have $g_{E(k)} - f_E^k = \sum_{v \in E(k) - (E + v_E^k)} v \in \mathbf{C}[V_D]$; moreover, by 7.1(b), $E(k) - (E + v_E^k)$ is contained in $\cup_{k' \le k-1} \tilde{u}^{-1}(k')$. Thus, $g_{E(k)} - f_E^k \in \mathbf{C}[V_D]_{\le k-1}$. Using the induction hypothesis we see that $g_{E(k)} - f_E^k$ is a linear combination of elements $f_{E'}^{k'}$ with $(E',k') \in \underline{\boldsymbol{\mathcal{F}}}(V_D), k' \le k-1$. Thus $g_{E(k)}$ is equal to f_E^k plus a linear combination of elements $f_{E'}^{k'}$ with $(E',k') \in \underline{\boldsymbol{\mathcal{F}}}(V_D), k' < k$. The same is true if here k is replaced by $k_1 \in [0, k-1]$ (we use again the induction hypothesis). It follows that the elements $f_E^{k'}$ are related to the elements $g_{E(k')}$ be an upper triangular matrix with 1 on diagonal. Hence (f) follows from (e). By 7.1(b), for any $(E,k) \in \underline{\boldsymbol{\mathcal{F}}}(V_D)$ we have $f_E^k \in \mathbf{C}[V_D]_k$. Hence (b),(c) follow from (f).

Note that we have now proved 0.6(c) in the case 0.6(i).

7.3. Let $\mathbf{E}, E, k$ be as in 7.2(a). We write $\mathbf{E} = E(k)$ with $(E,k) \in \underline{\boldsymbol{\mathcal{F}}}(V_D)$. From 7.1(b) we see that $\mathbf{E} \subset \cup_{k' \in [0,k]} \tilde{u}^{-1}(k')$ and setting $\mathbf{E}_* = \{v \in \mathbf{E}; \tilde{u}(v) = k\}$, we have

(a) $\mathbf{E}_* = E + v^k_E$.

7.4. Let $k \in [0, D/2]$ and let $\mathbf{E} = \langle e_{[1,D]}, e_{[2,D-1]}, \dots, e_{[k,D+1-k]}\rangle$ (a primitive subspace of V_D). Let $v^k_{\{0\}} = \bar{\epsilon}_D(\mathbf{E}) \in V_D$ (a special case of a notation in 7.1). If $k < D/2$ let $E \in \mathcal{F}(V_D)$ be the subspace with basis $e_s, e_{s+2}, e_{s+4}, \dots, e_{D-s}$ where $s = D/2 - k$. If $k = D/2$ let $E = 0$. Then $v^k_E \in \mathfrak{T}_E \subset V_D$ is defined as in 7.1. We have the following result.

(a) $v^k_{\{0\}} = v^k_E$.

Now v^k_E is computed in 7.1 in terms of I_* associated to E; $v^k_{\{0\}}$ is also computed in 7.1, this time in terms of $I_* = (\{1\}, \{2\}, \dots, \{D\})$. The two computations give the same result; (a) follows.

7.5. Assume that $v \in V_D, \tilde{u}(v) = k \in [0, D/2]$. We show:

(a) *There exists* $E' \in \mathcal{F}(V)$ *such that* $\dim(E') = D/2 - k$ *and* $v \in E' + v^k_{E'}$.

We argue by induction on D. Let $\mathbf{E}, E$ be associated to v as in 7.2(a). By 3.5(b) we have $v \in \mathbf{E}$. By the definition of $\mathbf{E}_*$ (see 3.3) we have $v \in \mathbf{E}_*$ hence by 7.3(a) we have $v \in E + v^k_E$. Assume first that $E = 0$. By definition (see 7.1) we have $v = v^k_{\{0\}}$. Using 7.4(a) we deduce that $v = v^k_{E'}$ where $E' \in \mathcal{F}(V_D)$ satisfies $\dim(E') = D/2 - k$. Thus (a) holds in this case.

Next we assume that $E \neq 0$. Then $D \geq 2$, $k \leq (D-2)/2$ and we can find $\mathbf{E}' \in \boldsymbol{\mathcal{F}}(V_{D-2})$ and $i \in [1, D]$ such that $\mathbf{E} = T_i(\mathbf{E}') \oplus \mathbf{F}e_i$. Since $v \in \mathbf{E}$, we have $v = T_i(v') + ce_i$ where $v' \in V_{D-2}$ and $c \in \mathbf{F}$. From [8, 1.11(b)] we have $\tilde{u}(v) = \underline{\tilde{u}}(v')$ (with $\underline{\tilde{u}}$ as in 7.1). Thus we have $\underline{\tilde{u}}(v') = k$. By the induction hypothesis we can find $E'' \in \mathcal{F}(V_{D-2})$ such that $\dim(E'') = (D-2)/2 - k$ and $v' \in E'' + v^k_{E''}$. Let $E' = T_i(E'') + \mathbf{F}e_i$. We have $E' \in \mathcal{F}(V_D)$, $\dim(E') = \dim(E'') + 1 = D/2 - k$. Using 7.1(a) we have

$v = T_i(v') + ce_i \in T_i(E'' + v^k_{E''}) + \mathbf{F}e_i \subset E' + v^k_{E'}$. Thus E' is as desired. This completes the proof of (a).

We see that

(b) $\tilde{u}^{-1}(k)$ *is a union of affine subspaces of the form* $E + v^k_E$ *(which are translates of linear subspaces* $E \in \mathcal{F}(V_D)$ *of dimension* $D/2 - k$*).*

When $k=0$, $\tilde{u}^{-1}(k)$ is the union of the linear subspaces E in $\mathcal{F}_*(V_D)$ corresponding to the various left cell representations. For $k\ge 0$, the affine subspaces in (b) can be regarded as higher left cell representations (generalizing the left cell representations). When $D=4$ these affine subspaces are as follows:

$(0,1,3,13),(0,2,13,123),(0,1,4,14),(0,3,24,234),(0,2,4,24)$ (if $k=0$),

$(12,124),(124,1234),(1234,134),(134,34)$ (if $k=1$),

(23) (if $k=2$).

(We specify a subset of V_D by a list of its vectors; we write $i_1i_2\dots i_t$ instead of $e_{i_1}+e_{i_2}+\dots+e_{i_t}$).

7.6. We now assume that D is odd. Most of the results in 7.1-7.5 have analogues for V'_D. Let $\tilde{u}':V'_D\to\mathbf{N}$ be as in 3.10. Then for $k\in\{0,1,3,5,7,\dots\}$, $\tilde{u}'^{-1}(k)$ is a union of affine subspaces of V'_D which are translates of linear subspaces $E\in\mathcal{F}(V'_D)$ of dimension $(D-1)/2-k'$ where $k'=0$ if $k=0$ and $k'=(k+1)/2$ if $k=1,3,5,\dots$. When $D=5$ these affine subspaces are:

$(0,1,3,13),(0,2,13,123),(0,1,4,14),(0,3,24,234),(0,2,4,24)$ (if $k=0$),

$(12,124),(124,1234),(1234,134),(134,34),(23,12)$ (if $k=1$).

(We specify a subset of V'_D by a list of its vectors; we write $i_1i_2\dots i_t$ instead of $\pi(e_{i_1}+e_{i_2}+\dots+e_{i_t})$).

7.7. Most of the results in 7.1-7.5 have analogues for $M(\mathcal{G}_c)$ in the setup of 0.1. Assume for example that $\mathcal{G}_c=S_5$. The elements $f_{x,\rho}$ in 0.6(c) are as follows.

If $(x,\rho)\in M(\mathcal{G}_c)^{\mathbf{g}}$ with $\mathbf{g}$ as in 6.4 then $f_{x,\rho}=(x,\rho)$.

If $(x,\rho)\in M(\mathcal{G}_c)^{\mathbf{g}}$ with $\mathbf{g}$ as in 6.5 that is, $(x,\rho)\in\{(g_3,\theta^j),(g_6,\theta^j)\}$ (with $j=1$ or 2) then $f_{x,\rho}$ is $(g_3,\theta^j),(g_6,\theta^j)+(g_3,\theta^j)$ respectively. (The last of these is an analogue of a higher left cell in 7.5.)

If $(x,\rho)\in M(\mathcal{G}_c)^{\mathbf{g}}$ with $\mathbf{g}$ as in 6.6 that is,

$(x,\rho)\in\{(g_2,-1),(g_2,-r),(g'_2,r),(g_4,-1)\}$ then $f_{x,\rho}$ is

$(g_2,-1),(g_2,-r)+(g_2,-1),(g'_2,r)+(g_2,-r)+(g_2,-1),(g_4,-1)+(g'_2,r)$,

$(g_6,-1)+(g'_2,r)+(g_2,-1)$ respectively. (The last three of these are analogues of the higher left cells in 7.5.)

If $(x,\rho) \in M(\mathcal{G}_c)^{\mathbf{g}}$ with $\mathbf{g}$ as in 6.9 then $f_{x,\rho} = g_{x,\rho}$ (notation of 0.6).

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