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HIGH DIMENSIONAL STATISTICS: QUADRATIC ERROR IN THE LOCAL LINEAR ESTIMATION OF THE RELATIVE REGRESSION

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Abstract

In this paper, we use the mean squared relative error as a loss function to construct a local linear estimator of the regression operator. More precisely, we consider n pairs of independent random variables (X_i, Y_i) for $i = 1, \ldots, n$ that we assume drawn from the pair (X, Y), which is valued in $(\mathfrak{F}, \mathbb{R})$, where $\mathfrak{F}$ is a semi-metric space equipped with the semi-metric d. Under some standard assumptions, we give the convergence rate in mean square of the constructed estimator. The usefulness of the estimator is highlighted through the exact expression involved in the leading terms of the quadratic error. Notice that this method is useful in analyzing data with positive responses, such as life times.

1. Introduction and Motivations

Nonparametric methods have grown significantly since the work of Bosq [11] and Collomb [12]. Forecasting is one of the most important problems in statistics especially in nonparametric modeling. This problem is interpreted as a study of the covariability between two random variables and there are several ways to explain it. It is well known that regression analysis is one

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of the most commonly used techniques and the popular one of this field of modern statistics. More precisely, we assume that the relationship between a real response variable and a functional explanatory variable is expressed by the following relations

$$Y = R(X) + \epsilon,$$

where R is an operator regression which is defined from a semi-metric space $(\mathfrak{F}, d)$ and ϵ is a random error variable independent to X.

From a historical point of view, the first works in this subject were introduced by Ferraty and Vieu [19]. These authors studied some asymptotic properties of the kernel estimator of the regression operator in the case of fractal-dimensional data. These results have been extended by Ferraty et al. [20] to non-standard regression problems such as forecasting in the context of time series. Moreover, Ferraty and Vieu [21] also provided a new regression model approach for an explanatory variable with values in a semi-metric space, they established asymptotic results using new methods on the Nadaraya-Watson estimator in the dependent case. Next, Dabo-Niang and Rhomari [13] established some asymptotic results and gave upper bounds of the $p-$mean and the almost sure estimation errors under general conditions. Almost complete convergence for the strongly mixing case has been studied by Ferraty et al. [22]. More generally, it should be noted that the contribution of Ferraty and Vieu [24] can be considered as determining in the functional nonparametric area. In the statistical literature, there are several nonparametric methods for the estimation of the regression operator such as the method of the local constant approach (kernel approach), the local polynomial regression, single index model (see, for instance, the following references [27], [15], [17], [16], [1], [3], [29], [30], [2], [4] and [5]).

In our work we used an alternative approach to the Nadaraya-Watson (NW) estimation, which has more advantages over the latter such as: design adaptation, high minimax efficiencies and, in many cases, the smaller bias in the multivariate as well as the functional case (see Fan and Gijbels [18]).

Recall that, the local linear estimator of the regression function in the Hilbertien case has been studied by Baìllo and Grané [8]. However, in the Banachique case, Barrientos et al. [9] proposed the functional local linear estimate the regression operator noted by $R(x)$ which is obtained as the solution for the following minimization problem:

$$\sum_{i=1}^{n}(Y_i - a - b\beta(X_i, x))^2 K(h_K^{-1}\delta(x, X_i)). \tag{1}$$

However, the minimization problem (1), which is considered as a measure of the prediction performance may be unadapted to some situations. (see, for instance, Demongeot et al. [14]). Indeed, using the least square regression is translated as treating all variables. So when Y is a positive response variable it is sometimes the case that relative prediction error, $(Y - R(X))|Y$, is more important than the usual prediction error, $(Y - R(X))$ (see, for instance, park). For this reason, in this contribution, we overcome this drawback by using an alternative loss function based on the relative squared error which is defined, for $Y > 0$, by

$$\mathbb{E}\left[\left(\frac{Y - R(x)}{Y}\right)^2 |Y\right]. \tag{2}$$

In nonparametric functional data analysis (NFDA), the literature on the relative error (RER) is very restricted. In fact, Demongeot et al. [14] provides an estimator of the relative regression based on the minimization of the mean squared relative error. The asymptotic results of this article are the explicit form of the mean squared error (MSE) and the asymptotic normality of the local constant estimator of the relative regression relative. In the case where the observations are associated, we can cite Mechab and Laksaci [28]. This paper focus on proving of the strong consistency and the asymptotic normality of the kernel estimator for the relative regression under weak dependence conditions (see also, Bouhadjera et al. [10]). In the context of incomplete functional data, we can cite for instance, Horrigue and Ould-Said [32], [26] for the censored conditional quantiles estimation. On the other hand, the almost complete convergence and the asymptotic normality of the local constant estimator of the functional relative error for truncated data have been obtained in Altendji et al. [7]. Recently Ibrahim et al. [6] established the uniform almost complete convergence of the kernel estimator for the functional relative error regression.

This paper is organized as follows. The local linear relative regression estimator is presented in Section 2 in the general form of the relative error estimation involving independent variables. Then, the asymptotic theory is provided in Section 3 in terms of the quadratic error. Finally, all proofs are deferred to the Appendix.

2. The Estimator

Let (X_i, Y_i) be a sequence of i.i.d. random vectors where the random variable (r.v.) X_i belongs to a semi-metric space $(\mathfrak{F}, d)$ and Y_i is a real-valued r.v. We adopt the fast version proposed by Barrientos et al. [9] and we use the loss function (2) to estimate (a, b) as follows:

$$(\widehat{a}, b) = \arg\min_{(a,b)} \sum_{i=1}^{n} Y_i^{-2}(Y_i - a - b\beta(X_i, x))^2 K(h_K^{-1}\delta(x, X_i)), \qquad (3)$$

where $\beta(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ are known functions from $\mathfrak{F}\times\mathfrak{F}$ into $\mathbb{R}$, K is a kernel, and $h_K := h_{K,n}$ is chosen as a sequence of positive real numbers. However, if the bi-functional operator β is such that, for all $z \in \mathfrak{F}$, $\beta(z, z) = 0$, Then, by some algebra, the estimator $\widehat{R}(x) = a$ can explicitly be rewritten as follows:

$$\widehat{R}(x) = \frac{\sum_{i,j=1}^{n} W_{ij} Y_j^{-1}}{\sum_{i,j=1}^{n} W_{ij}(x) Y_j^{-2}} = \frac{\sum_{j=1}^{n} \Delta_j K_j Y_j^{-1}}{\sum_{j=1}^{n} \Delta_j K_j Y_j^{-2}}, \qquad (4)$$

where $W_{ij} = V_{ij} Y_i^{-2}$, $V_{ij} = \beta_i(\beta_i - \beta_j) K_i K_j$ and

$$\Delta_j = K_j^{-1}\left(\sum_{i=1}^{n} W_{ij}\right) = \sum_{i=1}^{n} \beta_i^2 K_i Y_i^{-2} - \left(\sum_{i=1}^{n} \beta_i K_i Y_i^{-2}\right)\beta_j,$$

2.1. Assumptions and notations

In what follows we denote by x (resp. y) a fixed point in $\mathfrak{F}$ (resp. $\mathbb{R}$), $\mathcal{N}_x$ (resp. $\mathcal{N}_y$) a fixed neighborhood of x (resp. of y) and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$ and let G be the real valued function defined as for any $\lambda \in \{1, 2\}$: $G_\lambda(s) = \mathbb{E}[\mathbf{g}_\lambda(X) - \mathbf{g}_\lambda(x)|\beta(X, x) = s]$ with $\mathbf{g}_\lambda(.) = \mathbb{E}[Y^{-\lambda}|X = .]$ Then, we assume that our nonparametric model satisfies the following conditions

(H1) For any $r > 0$, $\phi_x(r) := \phi_x(-r, r) > 0$, and there exists a function $\chi_x(\cdot)$ such that, for all $t \in [-1, 1]$, $\lim_{h_K \to 0} \dfrac{\phi_x(th_K, h_K)}{\phi_x(h_K)} = \chi_x(t)$.

(H2) For any $\lambda \in \{1, 2\}$, the quantities $G_\lambda^{'}(0)$ and $G_\lambda^{''}(0)$ exist, where $G_\lambda^{'}$ (resp. $G_\lambda^{''}$) denotes the first (resp. the second) derivative of G_λ

(H3) The functions $\delta(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are such that:

(i) for all $z \in \mathfrak{F}, |\delta(x, z)| = d(x, z)$ and $C_1 |\delta(x, z)| \leq |\beta(x, z)| \leq C_2 |\delta(x, z)|$ where C_1 and $C_2 > 0$.

(ii) $\sup_{u \in B(x,r)} |\beta(u, x) - \delta(x, u)| = o(r)$, where $B(x, r) = \{z \in \mathfrak{F} / |\delta(z, x)| \leq r\}$ denotes the closed-ball centered at x and of radius r.

(H4) The kernel K is positive, differentiable function with support $[-1, 1]$.

(H5) The inverse moments of the response variable are continuous in a neighborhood of x and should verify

$$\text{for all } m \geq 1, \;\; \mathbb{E}[Y^{-m} | X = .] < C < \infty,$$

and

$$\forall i \neq j, \;\; \mathbb{E}[Y_i^{-\lambda} Y_j^{-l} | (X_i, X_j)] < C, \;\; \text{for } l \in \{3, 4\}, \quad \lambda \in \{1, 2\}.$$

(H6) The bandwidths are such that: there exists a positive integer n_0 for which

$$-\frac{1}{\phi_x(h_K)} \int_{-1}^{1} \phi_x(zh_K, h_K) \frac{d}{dz} \left(z^2 K(z)\right) dz > C_3 > 0 \text{ for } n > n_0.$$

$$\lim_{n \to \infty} h_K = 0 \text{ and } \lim_{n \to \infty} n h_K \phi_x(h_K) = \infty,$$

and

$$h_K \int_{B(x, h_K)} \beta(u, x) dP(u) = o\left(\int_{B(x, h_K)} \beta^2(u, x) \, dP(u) \right),$$

where $dP(u)$ is the cumulative distribution of X.

Remark 1.

- Notice that in the case $b = 0$, the mean Square Convergence of this estimation has been studied by Demongeot et al. [14].
- When $b = 0$, the solution to (3) is the local constant estimator can be explicitly expressed by the ratio of first two conditional inverse moments of Y given X:

$$\widehat{r}(x) = \frac{\widehat{\mathbf{g}}_{\mathbf{1}}(\mathbf{x})}{\widehat{\mathbf{g}}_{\mathbf{2}}(\mathbf{x})} = \frac{\sum_{i=1}^{n} Y_i^{-1} K(h_k^{-1} d(x, X_i))}{\sum_{i=1}^{n} Y_i^{-2} K(h_k^{-1} d(x, X_i))}.$$

2.2. Comments on the assumptions

Obviously, all these assumptions are very standard in the FDA context. In particular, assumption (H1) is an adaptation of assumption H1 in Ferraty et al. [25], when one replaces the semi-metric d by some bi-functional operator δ. The assumption (H1) characterizes the concentration property of the probability measure of the functional variable X, which permits to control the effect of the topological structure in the asymptotic results (see Ferraty et al. [23]). Assumption (H2) is a regularity condition which characterizes the functional space of our model and is needed to explicit the bias term. Then, assumption (H3) has been introduced and commented, first, in Barrientos et al. [9] and it plays an important role in our methodology, particularly when we will compute the exact constant terms involved in the asymptotic result. Concerning (H4), similar conditions have already been imposed in the literature to deal with the regression estimation problem: see for example assumptions(M2) in Demongeot et al. [14]. Moreover, the first and the third parts of the assumption (H6) are common in the setting of functional local linear fitting (see for instance Laksaci et al. [27] and Rachdi et al. [31]). The remainder of the hypotheses are imposed for a sake of brevity of our result's proofs.

3. Main Results

Before enouncing our main results, we introduce the quantities M_j and $N(a,b)$, which will appear in the bias and variance dominant terms:

$$M_j \ = \ K^j(1) - \int_{-1}^{1} (K^j(u))' \chi_x(u) du \text{ where } j = 1, 2,$$

and for all $a > 0$ and $b = 2, 4$, $\quad N(a,b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \chi_x(u) du.$

Moreover, let $\widehat{R}(x) = \dfrac{\mu_1(x)}{\mu_2(x)}$, where $\mu_\lambda(x) = \dfrac{1}{n\,(n-1)\,\mathbb{E}(W_{12})} \sum_{i \neq j} W_{ij} Y_j^{-\lambda}$. Then, we have the following theorem.

Theorem 1. *Under assumptions* (H1)-(H6), *we obtain:*

$$MSE(\widehat{R}(x), R(x)) = \mathbb{E}\left[\widehat{R}(x) - R(x)\right]^2 = Bias(\widehat{R}(x))^2 + Var(\widehat{R}(x)),$$

with

$$Bias(\widehat{R}(x)) = \frac{\mathbf{g_1}(x) - R(x)\mathbf{g_2}(x)}{\mathbf{g_2}(x)} + \frac{h_k^2}{2} \frac{\left(G_1^{(1)}(0) - R(x) G_2^{(2)}(0)\right) N(1,2)}{M_1 \mathbf{g_2}(x)},$$

and

$$\begin{aligned} Var(\widehat{R}(x)) \ &= \ \frac{r(x)\, M_1}{n\, \phi_x(h_K) (\mathbf{g}_2(x))^4 M_2^2} \left(\mathbf{g}_2(x) \mathbb{E}[Y^{-4}|X=x] - 4\mathbb{E}[Y^{-3}|X=x]\right) \\ &+ \frac{r(x)\, M_1}{n\, \phi_x(h_K) (\mathbf{g}_2(x))^4 M_2^2} \left(3r(x) \mathbb{E}^2[Y^{-4}|X=x]\right). \end{aligned}$$

3.1. Proof of Theorem 1

By using same arguments as those used by Demongeot et al. [14] for the regression operator estimation and by Sarda and Vieu [32] we have:

$$Bias(\widehat{R}(x)) = \frac{\mathbb{E}\left[\mu_1(x)\right]}{\mathbb{E}\left[\mu_2(x)\right]} - R(x) + O\left(\frac{1}{n\, \phi_x(h_K)}\right),$$

and

$$Var[\widehat{R}(x))] = \frac{Var[\mu_1(x)]}{\mathbb{E}[\mu_2(x)]^2} - 4\frac{A_1}{(\mathbb{E}[\mu_2(x)])^3} + 3\frac{A_2}{(\mathbb{E}[\mu_2(x)])^4} + O\left(\frac{1}{n\,\phi_x(h_K)}\right),$$

where

$$A_1 = \mathbb{E}[\mu_1(x)]Cov(\mu_1(x), \mu_2(x)),$$

and

$$A_2 = Var[\mu_2(x)]\mathbb{E}[\mu_1(x)]^2.$$

The proof of Theorem 1 will be completed from the above expression and the following results for which proofs are given in the appendix.

Lemma 1. *Under the assumptions of Theorem* 1*, we have for* $\lambda = 1, 2$*:*

$$\mathbb{E}[\mu_\lambda(x)] = \mathbf{g}_\lambda(x) + \frac{h_K^2}{2}G_\lambda^{(2)}(0)\,\frac{N(1,2)}{M_1} \;+\; o(h_K^2).$$

Lemma 2. *Under the assumptions of Theorem* 1*, we have for* $\lambda = 1, 2$

$$Var[\mu_\lambda(x)] = V_\lambda(x,y)\left[\frac{N(2,4)}{N^2(1,2)} + \frac{2\,N(2,2)}{N(1,2)M_1}\right] + \frac{V_\lambda(x,y)\mathbb{E}[Y^{-2\lambda}|X=x]}{(\mathbf{g}_\lambda(x))^2}\,\frac{M_2}{M_1^2},$$

where

$$V_\lambda(x,y) = \frac{\mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_\lambda(x))^2}{n\,\phi_x(h_K)\,(\mathbf{g}_2(x))^2},$$

Furthermore

$$\begin{aligned} Cov(\mu_1(x), \mu_2(x)) = {} & Q(x,y).\left[\frac{N(2,4)}{N^2(1,2)} + \frac{2\,N(2,2)}{N(1,2)M_1}\right] \\ & + \frac{Q(x,y)\,\mathbb{E}[Y^{-3}|X=x]}{\mathbf{g}_1(x)\,\mathbf{g}_2(x)}\,\frac{M_2}{M_1^2}, \end{aligned}$$

where

$$Q(x,y) = \frac{\mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_1(x)\,\mathbf{g}_2(x)}{n\,\phi_x(h_K)\,(\mathbf{g}_2(x))^2}.$$

4. Appendix

4.1. Proof of Lemma 1

Following the ideas used in regression operator estimation (Ferraty et al. [25]) we obtain:

$$\mathbb{E}[\mu_\lambda(x)] = \frac{1}{\mathbb{E}[W_{12}]}\,\mathbb{E}\left[W_{12}\;\mathbb{E}[Y_2^{-\lambda}|X_2]\right].$$

Since $\mathbb{E}[\beta_2 W_{12}] = 0$ and $G_\lambda(0) = 0$, for $\lambda \in \{1.2\}$, and under assumption (H6), we obtain:

$$\begin{aligned}
\mathbb{E}\left[W_{12}\;\mathbb{E}[Y_2^{-\lambda}|X_2]\right] &= \mathbb{E}[W_{12}\mathbf{g}_\lambda(X_2)],\\
&= \mathbf{g}_\lambda(x)\mathbb{E}[W_{12}] \;+\; \mathbb{E}[W_{12}\mathbb{E}[\mathbf{g}_\lambda(X_2) - \mathbf{g}_\lambda(x)]/\beta(X_2,x)],\\
&= \mathbf{g}_\lambda(x)\mathbb{E}[W_{12}] \;+\; \mathbb{E}[W_{12}G_\lambda(\beta(X_2,x))]\\
&= \mathbf{g}_\lambda(x)\mathbb{E}[W_{12}] \;+\; \frac{1}{2}G_\lambda^{(2)}(0)\;\mathbb{E}[\beta^2(X_2,x)W_{12}]\\
&\quad + o(\mathbb{E}[\beta^2(X_2,x)W_{12}]) \qquad (5)
\end{aligned}$$

So, it suffices to evaluate $\dfrac{\mathbb{E}[\beta^2(X_2,x)W_{12}]}{\mathbb{E}[W_{12}]}$. According to the definition of W_{12} and by conditioning on X_2, we get

$$\frac{\mathbb{E}[\beta^2(X_2,x)W_{12}]}{\mathbb{E}[W_{12}]} = \frac{\mathbb{E}[\beta^2(X_2,x)V_{12}Y_1^{-2}]}{\mathbb{E}[V_{12}Y_1^{-2}]} = \frac{\mathbb{E}[\beta^2(X_2,x)V_{12}\mathbf{g}_1(X_2)]}{\mathbb{E}[V_{12}\mathbf{g}_1(X_2)]}. \qquad (6)$$

By following the same steps as in the proof of (5), we show that:

$$\mathbb{E}[\beta^l(X_2,x)\,V_{12}\,\mathbf{g}_1(X_2)] = \mathbb{E}[\beta^l(X_2,x)\,V_{12}]\,\mathbf{g}_1(X_2) + o(\mathbb{E}[\beta(X_2,x)W_{12}]), \quad \text{for } l \in \{0,2\}$$

Now, by the application of Lemma 3.2 [31], we have:

$$\frac{\mathbb{E}[\beta^2(X_2,x)V_{12}]}{\mathbb{E}[V_{12}]} = h_K^2\,\frac{N(1,2)}{M_1} \;+\; o(h_K^2). \qquad (7)$$

Finally, by combining (3), (6) and (7) to get the claimed result.

4.2. Proof of Lemma 2

It is clear that for $\lambda \in \{1,2\}$ we have

$$var\,(\mu_\lambda(x)) = \frac{1}{(n(n-1)\mathbb{E}[W_{12}])^2}\, Var\Big(\sum_{i\neq j=1}^{n} W_{ij}Y_j^{-\lambda}\Big)$$

$$\begin{aligned}
= \ & \frac{1}{(n(n-1)\mathbb{E}[W_{12}])^2}\Big[n(n-1)\mathbb{E}[W_{12}^2Y_2^{-2\lambda}] + n(n-1)\mathbb{E}[W_{12}W_{21}Y_1^{-1}Y_2^{-\lambda}] \\
& + n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}Y_2^{-\lambda}Y_3^{-\lambda}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}Y_2^{-\lambda}Y_3^{-\lambda}] \\
& + n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}Y_2^{-1}Y_1^{-\lambda}] + n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}Y_2^{-2\lambda}] \\
& - n(n-1)(4n-6)\Big(\mathbb{E}[W_{12}Y_2^{-\lambda}]\Big)^2\Big].
\end{aligned}$$

Observe that under the assumption (H5), we have $\dfrac{\mathbb{E}[W_{12}Y_2^{-\lambda}]}{\mathbb{E}[W_{12}]} = O(1)$. By some simple manipulations and by using our assumptions, we get

$$\begin{cases}
\mathbb{E}[W_{12}^2Y_2^{-2\lambda}] = O(h_K^4\phi_x^2(h_K)), \\
\mathbb{E}[W_{12}W_{21}Y_1^{-\lambda}Y_2^{-\lambda}] = O(h_K^4\phi_x^2(h_K)), \\
\mathbb{E}[W_{12}W_{13}Y_2^{-\lambda}Y_3^{-\lambda}] = \mathbb{E}[Y^{-4}|X=x]\ \mathbb{E}[\beta_1^4K_1^2]\ (\mathbf{g}_\lambda(x)\mathbb{E}[K_1])^2 \\
\qquad + O(h_K^4\phi_x^3(h_K)), \\
\mathbb{E}[W_{12}W_{23}Y_2^{-\lambda}Y_3^{-\lambda}] = \mathbb{E}[Y^{-4}|X=x]\ \mathbb{E}[\beta_1^2K_1]\ (\mathbf{g}_\lambda(x))^2(\mathbb{E}[\beta_1^2K_1^2]\ \mathbb{E}[K_1]) \\
\qquad + O(h_K^4\phi_x^3(h_K)), \\
\mathbb{E}[W_{12}W_{31}Y_2^{-\lambda}Y_1^{-\lambda}] = \mathbb{E}[Y^{-4}|X=x]\ \mathbb{E}[\beta_1^2K_1]\ (\mathbf{g}_\lambda(x))^2(\mathbb{E}[\beta_1^2K_1^2]\ \mathbb{E}[K_1]) \\
\qquad + O(h_K^4\phi_x^3(h_K)), \\
\mathbb{E}[W_{12}W_{32}Y_2^{-2\lambda}] = \mathbb{E}[Y^{-4}|X=x]\ (\mathbb{E}[\beta_1^2K_1])^2\mathbb{E}[Y^{-2\lambda}|X=x](\mathbb{E}[K_1^2]) \\
\qquad + O(h_K^4\phi_x^3(h_K)).
\end{cases}$$

Therefore, the leading term in the expression of $var\,(\mu_1(x))$

$$\begin{aligned}
& var\,(\mu_\lambda(x)) \\
& = \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2}\,\mathbb{E}[Y^{-4}|X=x]\,(\mathbf{g}_\lambda(x))^2\ \big(\mathbb{E}[\beta_1^4K_1^2]\ (\mathbb{E}[K_1])^2\big), \\
& \quad + \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2} 2\mathbb{E}[\beta_1^2K_1]\ (\mathbb{E}[\beta_1^2K_1^2]\ \mathbb{E}[K_1])(\mathbf{g}_\lambda(x))^2 \\
& \quad + \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2}(\mathbb{E}[\beta_1^2K_1])^2\mathbb{E}[Y^{-4}|X=x]\mathbb{E}[Y^{-2\lambda}|X=x](\mathbb{E}[K_1^2])
\end{aligned}$$

$$+O(h_K^4\phi_x^3(h_K)). \tag{8}$$

According to Lemma A.1 [33] and formula (6), we deduce the following result

$$\begin{aligned} var\,(\mu_\lambda(x)) &= \frac{\mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_\lambda(x))^2}{n\,\phi_x(h_K)\,(\mathbf{g}_2(x))^2}.\left[\frac{N(2,4)}{N^2(1,2)}+\frac{2\,N(2,2)}{N(1,2)M_1}\right] \\ &+\frac{\mathbb{E}[Y^{-4}|X=x]\,\mathbb{E}[Y^{-2\lambda}|X=x]}{n\,\phi_x(h_K)\,(\mathbf{g}_2(x))^2}\,\frac{M_2}{M_1^2}. \end{aligned}$$

Concerning the variance term, we use the same arguments as in (8), we get:

$$\begin{aligned} &cov(\mu_1(x),\mu_2(x)) \\ &= \frac{1}{(n(n-1)\,\mathbb{E}[W_{12}])^2}\,Cov\Big(\sum_{i\neq j=1}^{n} W_{ij}Y_j^{-1},\sum_{i'\neq j'=1}^{n} W_{i'j'}Y_{j'}^{-2}\Big), \\ &= \frac{1}{(n(n-1)\mathbb{E}[W_{12}])^2}\,[n(n-1)\mathbb{E}[W_{12}^2Y_2^{-3}]+n(n-1)\mathbb{E}[W_{12}W_{21}Y_2^{-1}Y_1^{-2}] \\ &\quad +n(n-1)(n-2)\mathbb{E}[W_{12}W_{13}Y_2^{-1}Y_3^{-2}]+n(n-1)(n-2)\mathbb{E}[W_{12}W_{23}Y_2^{-1}Y_3^{-2}] \\ &\quad +n(n-1)(n-2)\mathbb{E}[W_{12}W_{31}Y_2^{-1}Y_1^{-2}]+n(n-1)(n-2)\mathbb{E}[W_{12}W_{32}Y_2^{-3}] \\ &\quad -n(n-1)(4n-6)\big(\mathbb{E}[W_{12}Y_2^{-1}]\,\mathbb{E}[W_{12}Y_2^{-2}]\big)]. \end{aligned}$$

with

$$\begin{cases} \mathbb{E}[W_{12}^2Y_2^{-3}] = O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[W_{12}W_{21}Y_2^{-1}Y_1^{-2}] = O(h_K^4\phi_x^2(h_K)), \\ \mathbb{E}[W_{12}W_{13}Y_2^{-1}Y_3^{-2}] = \mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_1(x)\,\mathbf{g}_2(x)\;\mathbb{E}[\beta_1^4K_1^2]\,(\mathbb{E}[K_1])^2 \\ \qquad\qquad + \;O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{23}Y_2^{-1}Y_3^{-2}] = \mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_1(x)\,\mathbf{g}_2(x)\mathbb{E}[\beta_1^2K_1]\,(\mathbb{E}[\beta_1^2K_1^2]\,\mathbb{E}[K_1]) \\ \qquad\qquad + \;O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{31}Y_2^{-1}Y_1^{-2}] = \mathbb{E}[Y^{-4}|X=x]\,\mathbf{g}_1(x)\,\mathbf{g}_2(x)\;\mathbb{E}[\beta_1^2K_1]\,(\mathbb{E}[\beta_1^2K_1^2]\,\mathbb{E}[K_1]) \\ \qquad\qquad + \;O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[W_{12}W_{32}Y_2^{-3}] = \mathbb{E}[Y^{-4}|X=x]\,\mathbb{E}[Y^{-3}|X=x]\;(\mathbb{E}[\beta_1^2K_1])^2(\mathbb{E}[K_1^2]) \\ \qquad\qquad +O(h_K^4\phi_x^3(h_K)), \\ \mathbb{E}[W_{12}Y_2^{-1}]\mathbb{E}[W_{12}Y_2^{-2}] = O(h_K^2\phi_x^2(h_K)). \end{cases}$$

Therefore, the leading term in the expression of $Cov(\mu_1(x), \mu_2(x))$ is

$$\begin{aligned}
&Cov(\mu_1(x), \mu_2(x)) \\
&= \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2} \mathbb{E}[Y^{-4}|X=x]\, \mathbf{g}_1(x)\, \mathbf{g}_2(x) \,\left(\mathbb{E}[\beta_1^4 K_1^2]\, (\mathbb{E}[K_1])^2\right), \\
&\quad + \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2} 2\, \mathbf{g}_1(x)\, \mathbf{g}_2(x) \mathbb{E}[\beta_1^2 K_1]\, (\mathbb{E}[\beta_1^2 K_1^2]\, \mathbb{E}[K_1]) \\
&\quad + \frac{n(n-1)(n-2)}{(n(n-1)\mathbb{E}[W_{12}])^2} (\mathbb{E}[\beta_1^2 K_1])^2 \mathbb{E}[Y^{-4}|X=x] \mathbb{E}[Y^{-2\lambda}|X=x] (\mathbb{E}[K_1^2]) \\
&\quad + O(h_K^4 \phi_x^3(h_K))
\end{aligned}$$

Again, by using Lemma A.1 of [33], we have

$$\begin{aligned}
cov(\mu_1(x), \mu_2(x)) \;=\; & \frac{\mathbb{E}[Y^{-4}|X=x]\, \mathbf{g}_1(x)\, \mathbf{g}_2(x)}{n\, \phi_x(h_K)\, (\mathbf{g}_2(x))^2} . \left[\frac{N(2,4)}{N^2(1,2)} + \frac{2\, N(2,2)}{N(1,2) M_1}\right] \\
& + \frac{\mathbb{E}[Y^{-4}|X=x]\, \mathbb{E}[Y^{-3\lambda}|X=x]}{n\, \phi_x(h_K)\, (\mathbf{g}_2(x))^2} \frac{M_2}{M_1^2}
\end{aligned}$$

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