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PROPERTY OF INSTANT INDEPENDENCE AND STOCHASTIC INTEGRATION

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Abstract

In this work, we present property of instant independence and we give a new approach on stochastic integration with respect to fractional Brownian motion for processes not necessarily adapted .

1. Introduction

As is well-known, the classical Brownian motion is a stochastic process which is self-similar of index 1/2 and has stationary increments. It is actually the only continuous Gaussian process (up to a constant factor) to have these two properties that are often observed in the "real life", for instance in the movement of particles suspended in a fluid or in the behavior of the logarithm of the price of a financial asset. More generally, it is natural to wonder whether there exists a stochastic process which would be at the same time Gaussian, with stationary increments and self-similar, but not necessarily with an index 1/2 as in the Brownian motion case. Such a process happens to exist, and was introduced by Kolmogorov [18] in the early 1940s for modeling turbulence in liquids.

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The name fractional Brownian motion (fBm in short), which is the terminology everyone uses nowadays, comes from the paper by Mandelbrot and Van Ness [22]. The law of fBm relies on a single parameter H between 0 and 1, the so-called Hurst parameter or self-similarity index. Fractional Brownian motion is interesting for modeling purposes, as it allows the modeler to adjust the value of H to be as close as possible to its observations. It is worthwhile noting at this stage, however the picture is not as rosy as it seems. Indeed, except when its self-similarity index is 1/2, fBm is neither a semimartingale, nor a Markov process. As a consequence, its toolbox is limited, so that solving problems involving fBm is often a non-trivial task. On the positive side, fBm offers new challenges for the specialists of stochastic calculus!

If $H \neq \frac{1}{2}$ the fBm is not a semimartingale and we cannot apply the stochastic calculus developed by Ito in order to define stochastic integrals with respect to fBm. Different approaches have been used in order to construct a stochastic calculus with respect to fBm and we can mention the following contributions to this problem:

- Lin [21] and Dai and Heyde [10] defined stochastic integrals with respect to the fractional Brownian motion with parameter $H > \frac{1}{2}$ using a pathwise Riemann-Stieltjes method, the integrator must have finite $p-$variation where $\frac{1}{p} + H > 1$.
- The stochastic calculus of variations (see [23]) with respect to the Gaussian process B is a powerful technique that can be used to define stochastic integrals. More precisely, as in the case of the Brownian motion the divergence operator with respect to B can be interpreted as a stochastic integral, this idea has been developed by Decreusefond and Üstünel [11, 12], Carmona and Coutin [6], Alòs, Mazet and Nualart [1, 2], Duncan, Hu and Pasik-Duncan [15] and Hu and Ksendal [16]. The integral constructed by this method has zero mean, and can be obtained as the limit of Riemann sums defined using Wick products.
- Using the notions of fractional integral and derivative, Zähle has introduced in [30] a pathwise stochastic integral with respect to the fBm B with parameter $H \in (0, 1)$. If the integrator has $\lambda-$Hölder continuous paths with $\lambda > 1-H$, then this integral can be interpreted as a Riemann-Stieltjes integral and coincides with the forward and Stratonovich integrals studied in [1] and [3].

There are some representations of the fBm as a Wiener integral (i.e. w.r.t Brownian motion). We would like to have such Levy-Hida representation, we have that the natural filtration of the Brownian motion and of the fBm that it generates coincides, comparing to the Mandelbrot Van-Ness representation.

The results presented in this paper generalized those presented in Ayed and Kuo [4]. Our paper is organized as follows : we recall some necessary preliminaries on the fractional Brownian motion in Section 1, in Section 2 we construct suitable spaces of integrands in order to have a well-defined integral using integral representation. In Section 3 we give new results on stochastic integration w.r.t. fBm for no adapted processes.

2. Preliminaries on fBm

Fractional Brownian motion.

Fractional Brownian motion was originally defined and studied by Kolmogorov [18] within a Hilbert space framework. Fractional Brownian motion of Hurst index $H \in (0,1)$ is a centered Gaussian process W^i with covariance

$$\mathbb{E}[B^{(H)}(t)B^{(H)}(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \quad (s,t \geq 0)$$

(for $H = \frac{1}{2}$ we obtain standard Brownian motion).

Fractional Brownian motion has stationary increments

$$\mathbb{E}[B^{(H)}(t)B^{(H)}(s)] = |t-s|^{2H} \quad (s,t \geq 0).$$

and is H-self similar

$$(\frac{1}{c^H}B^{(H)}(ct); t \geq 0) = (B^{(H)}(t); t \geq 0) \ (for\ all\ c > 0).$$

The Hurst parameter H accounts not only for the sign of the correlation of the increments, but also for the regularity of the sample paths. Indeed, for $H > \frac{1}{2}$, the increments are positively correlated, and for $H < \frac{1}{2}$ they are negatively correlated. Furthermore, for every $\beta \in (0,H)$, its sample paths are almost surely Hölder continuous withe index β finally, it is worthy of note that for $H > \frac{1}{2}$, according to Beran's definition , it is a long memory process: the covariance of increments at distance u decrease as u^{2H-2}.

these significant properties make fractional Brownian motion a natural candidate as a model of noise in mathematical finance (see Comte and Renault [9], Rogers [26]), and in communication networks (see for instance, Leland, Taqqu et al. [25].

Recently, there has been numerous attempts at defining a stochastic integral with respect to fractional Brownian motion. Indeed, for $H \neq \frac{1}{2}$, $B^{(H)}$ is not a semi-martingale, and usual Itô stochastic calculus may not be applied. However, the integral

$$\int_0^t a(s)dB^{(H)}(s) \tag{1}$$

may be defined for suitable a. In one hand, since $B^{(H)}$ has almost its sample paths Hölder continuous of index β, for any $\beta < H$, the integral (1) exists in the Riemann-Stieljes sense (path by path) if almost every sample path of a has finite $p-$variation with $\frac{1}{p} + \beta > 1$ (see Young [29]): this is the approach used by Dai and Heyde [10] when $H > \frac{1}{2}$. Let us recall that the $p-$variation of a function f over an interval $[0, t]$ is the least upper bound of sums $\sum_k |f(x_k - f(x_{k-1}))|^p$ over all partitions $0 = x_0 < x_1 < \cdots < x_n = T$. A recent survey of the important properties of Riemann-Stieljes integral is the concentrated advanced course of Dudley and Norvaisa [13]. An extension of Riemann-Stieljes integral has been defined by Zăhle [25], by means of composition formulas, integration by parts formula, Weyl derivative formula concerning fractional integration/differentiation, and the generalized quadratic variation of Russo and Vallois [28, 27].

On the other hand, $B^{(H)}$ is a Gaussian process, and (1) can be defined for deterministic processes a by way of an L^2 isometry: see, for example, Norros, Valkeila and Virtamo or Pipiras and Taqqu [25]. With the help of stochastic calculus of variations (see [25]) this integral may be extended to random processes a. In this case, the stochastic integral (1) is a divergence operator, that is the adjoint of a stochastic gradient operator (see the pioneering paper of Decreusefond and Üstünel [11]). It must be noted that Duncan, Hu and Pasik-Duncan [14] have defined the stochastic integral in a similar way by using Wick product. Ciesielski, Kerkyacharian and Roynette [8] also used the Gaussian property of $B^{(H)}$ to prove that $B^{(H)}$ belongs to suitable function spaces and construct a stochastic integral.

Eventually, Alos, Mazet and Nualart [1] have established, following the ideas introduced in a previous version of this paper, very sharp sufficient conditions that ensures existence of the stochastic integral (1).

In a similar way, given a Hilbert space $\mathbb{V}$ we denote by $\mathbb{D}^{k,p}(\mathbb{V})$ the corresponding Sobolev space of $\mathcal{V}$-valued random variables. The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship.

$$E(F\delta(u)) = E(DF, u)_{\mathcal{H}},$$

where u is a random variable in $L^2(\Omega;\mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by Dom δ, if the above expression is continuous in the L^2 norm of F. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in Dom δ.

The following are two basic properties of the divergence operator:

1. For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$:

$$E\delta(u)^2 = E\|u\|^2_{\mathcal{H}} + E < D_u, (D_u)^* >_{\mathcal{H}\bigotimes\mathcal{H}}, \tag{2}$$

   where $(D_u)^*$ is the adjoint of (D_u) in the Hilbert space $\mathcal{H}\bigotimes\mathcal{H}$
2. for any F in $\mathbb{D}^{1,2}$ and any u in the domain of δ such that Fu and $F\delta(u) + < DF, u >_{\mathcal{H}}$ are square integrable, then Fu is in the domain of δ and

$$\delta(F_u) = F\delta(u) + < DF, u >_{\mathcal{H}} . \tag{3}$$

We denote by $|\mathcal{H}|\bigotimes|\mathcal{H}|$ the space of measurable functions φ on $[0,T]^2$ such that

$$\|\varphi\|^2_{|\mathcal{H}|\bigotimes|\mathcal{H}|} = \alpha_H^2 \int_{[0,T]^4} |\varphi_{r,\theta}||\varphi_{u,\eta}||r-u|^{2H-2}|\theta-\eta| dr du d\theta d\eta < \infty.$$

As we mentioned before, $|\mathcal{H}|\bigotimes|\mathcal{H}|$ is a Banach space with respect to the norm $\|.\|_{|\mathcal{H}|\bigotimes|\mathcal{H}|}$. Furthermore, equipped with the inner product

$$< \varphi, \psi >_{\mathcal{H}\bigotimes\mathcal{H}} = \alpha_H^2 \int_{[0,T]^4} \varphi_{r,\theta}\varphi_{u,\eta}|r-u|^{2H-2}|\theta-\eta|^{2H-2} dr du d\theta d\eta.$$

The space $|\mathcal{H}|\bigotimes|\mathcal{H}|$ is isometric to a subspace of $\mathcal{H}\bigotimes\mathcal{H}$ and it will be identified with this subspace.

2.1. Non-semimartingale property

We have seen that for $H\neq\frac{1}{2}$ fBm does not have independent increments. In this subsection we will show that for $H\neq\frac{1}{2}$, fBm is not semimartingale. A proof in the case $H>\frac{1}{2}$ can be found in [21]. We will present here the proof given by Rogers in [26] for any $H\neq\frac{1}{2}$. The main arguments of this proof are as follows. For $p>0$ set

$$Y_{n,p}=n^{pH-1}\sum_{i=1}^{n}|B^{(H)}_{i/n}-B^{(H)}_{(i-1)/n}|^{p}.$$

By the self-similar property of fBm, the sequence $\{Y_{n,p},n\geq 1\}$ has the same distribution as $\{\tilde{Y}_{n,p},n\geq 1\}$, where

$$\tilde{Y}_{n,p}=n^{-1}\sum_{i=1}^{n}|B^{(H)}_{i}-B^{(H)}_{i-1}|^{P}.$$

The stationary sequence $\{B^{(H)}(t)B^{(H)}(s),i\geq 1\}$ is mixing. Hence, by the Ergodic Theorem $\tilde{Y}_{n,p}$ converges almost surely and in L^1 to $E(|B^{(H)}_1|^p)$ as n tends to infinity. As a consequence, $Y_{n,p}$ converges in probability as n tends to infinity to $E(|B^{(H)}_1|^p)$. Therefore,

$$V_{n,p}=\sum_{i=1}^{n}|B^{(H)}_{i/n}-B^{(H)}_{(i-1)/n}|^{p}$$

converges in probability to zero as n tends to infinity if $pH>1$, and to infinity if $pH<1$. Consider the following two cases:

(i) If $H<\frac{1}{2}$, we can choose $p>2$ such that $pH<1$, and we obtain that the $p-$variation of fBm (defined as the limit in probability $\lim_{n\to\infty}V_{n,p}$) is infinite. Hence, the quadratic variation ($p=2$) is also infinite.

(ii) If $H<\frac{1}{2}$, we can choose p such that $\frac{1}{H}<p<2$. Then the $p-$variation is zero, and as a consequence, the quadratic variation is also zero. On the other hand, if we choose p such that $1 < p < \frac{1}{H}$ we deduce that the total variation is infinite.

Therefore, we have proved that for $H \neq \frac{1}{2}$ fBm cannot be a semimartingale. In a recent paper [7] Cheridito has introduced the notion of weak semimartingale as a stochastic process $\{X_t, t \geq 0\}$ such that for each $T > 0$, the set of random variables

$$\left\{ \sum_{j=1}^{n} f_i(B_{t_i}^{(H)} - B_{t_{i-1}}^{(H)}), n \geq 1, 0 \leq t_0 < \cdots < t_n \leq T, \right.$$
$$\left. |f_i| \leq 1, f_i \text{ is } \mathcal{F}_{t_{i-1}}^X\text{-mesurable} \right\}$$

is bounded in L^0, where for each $t \geq 0$, $\mathcal{F}_t^X$ is the σ-field generated by the random variables $\{X_s, 0 \leq s \leq t\}$. It is important to remark that this σ-field is not completed with the null sets. Then, in [7] it is proved that fBm is not a weak semimartingale if $H \neq \frac{1}{2}$.

Let us mention the following surprising result also proved in [7]. Suppose that $\{B_t^{(H)}, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0, 1)$, and $\{B_t, t \geq 0\}$ ordinary Brownian motion. Assume they are independent. Set

$$M_t^{(H)} = B_t^{(H)} + B_t.$$

Then $\{M_t^{(H)}, t \geq 0\}$ is not a weak semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$, and it is a semimartingale, equivalent in law to Brownian motion on any finite time interval $[0, T]$, if $H \in (\frac{3}{4}, 1)$.

3. General Construction of the Space of Integrands using Integral Representation

In this section, we will explain the reasoning we adopt to construct suitable spaces of integrands in order to have a well-defined integral. Note that it is a heuristic approach, recall that we can be represent a fBm by an integral over $\mathbf{T}$ of a kernel with respect to the Brownian motion. Since the fBm is a particular case of the so-called Volterra process, where we say that

X_t is a Volterra process, if we can write

$$X_t = \int_0^t K(t,s)dW_s,$$

where K is the Volterra kernel and W is a Brownian motion.(see [5] and [17]). Now let us focus on the fBm, which can be represented by

$$(B_t^{(H)})_{t\in\mathbf{T}} \equiv (\int_{\mathbf{T}} k_H(t,s)dB_s)_{t\in\mathbf{T}},$$

with $k_H(t,s) = \mathbf{k}_H \mathbf{1}_{[0,t)}(s)$ where the kernel is in fact the image of the indicator function through the operator $\mathbf{k}_H$. Without going deeply in the theory of operator, it is in fact the Hilbert-Schmidt operator. Thus, heuristically,

$$I^H(f) \equiv \int_{\mathbf{T}} \mathbf{k}_H f(s)dB_s,$$

So, to be well defined, we must have, as a space of integrands

$$\mathcal{S}^H = \{f : \int_{\mathbf{T}} (\mathbf{k}_H f(s))^2 ds < \infty\},$$

with an inner product satisfying,

$$\langle f,g\rangle_{\mathcal{S}^H} = \mathbb{E}(I^H(f)I^H(f))$$

This is the general construction in [25] for the case $\mathbf{T} = \mathbb{R}$ and [24] for the case $\mathbf{T} = [0,T]$. Besides, as we shall see, for example in Subsection 10.5.4 in [20], even if the approach is different, will use this idea to construct the integral.

4. Stochastic Integration w.r.t. fBm for no Adapted Processes

Definition 4.1. A stochastic process $X(t)$ is said to be instantly independent with respect to a filtration $\{\mathcal{F}_t\}$ if $X(t)$ and $\mathcal{F}_t$ are independent for each t.

Lemma 4.1. *If a stochastic process $X(t)$ is both adapted and instantly independent with respect to a filtration $\{\mathcal{F}_t\}$, then $X(t)$ is a deterministic function.*

Proof. Since $X(t)$ is adapted, we have $\mathbb{E}(X(t)|\mathcal{F}_t) = X(t)$. On the other hand, $X(t)$ is instantly independent, we also have $\mathbb{E}(X(t)|\mathcal{F}_t) = \mathbb{E}(X(t))$. Hence $X(t) = \mathbb{E}(X(t))$, which shows that $X(t)$ is a deterministic function.□

4.1. Levy-Hida representation

Note that the fBm is a particular case of Volterra processes. Following Decreusefond and Üstünel, we have this kernel :

$$K_H(t,s) = \frac{(t-s)_+^{H-1/2}}{\Gamma(H+1/2)} F(1/2-H, H-1/2, H+1/2, 1-\frac{t}{s}),\ 0<s<t<\infty,$$

where F is the Gauss hypergeometric function.

For the case $H \in (1/2,1)$, we have that the kernel is

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} du, \quad t>s,$$

where

$$c_H = (\frac{H(2H-1)}{\mathbf{B}(2-2H, H-1/2)})^{1/2}$$

with $\mathbf{B}$ the Beta function, i.e. $\mathbf{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$.

We have

$$\frac{\partial K_H(t,s)}{\partial t} = c_H (\frac{t}{s})^{H-1/2} (t-s)^{H-3/2}.$$

Now, we introduce a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$, defined by

$$(K_H^* \phi)(s) = \int_s^T \phi(t) \frac{\partial K_H(t,s)}{\partial t} dt, \tag{4}$$

where $\phi \in \mathcal{E}$.

For the case $H \in (0,1/2)$, we have that the kernel is given by

$$K_H(t,s) = b_H \Big((\frac{t}{s})^{H-1/2} (t-s)^{H-1/2} - (H-1/2) s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} du \Big),$$

where

$$b_H = (\frac{2H}{(1-2H)\mathbf{B}(1-2H, H+1/2)})^{1/2}.$$

We have

$$\frac{\partial K_H(t,s)}{\partial t} = c_H(H - 1/2\frac{t}{s})^{H-1/2}(t-s)^{H-3/2}.$$

Now, we introduce a linear operator $K_H^* : \mathcal{E} \to L^2([0,T])$, defined by

$$(K_H^*\phi)(s) := K_H(T,s)\phi(s) + \int_s^T (\phi(t) - \phi(s))\frac{\partial K_H(t,s)}{\partial t} dt. \qquad (5)$$

Case $H = 1/2$.

It is obvious that $K_{1/2}(t,s) = 1_{[0,t]}(s)$. Indeed, we obtain

$$B_t^{1/2} = \int_0^t K_{1/2}(t,s)dW_s = \int_0^t 1_{[0,t]}(s)dW_s = W_t$$

We have

$$(K_H^* 1_{[0,t]})(s) = K_H(t,s)1_{[0,t]}(s).$$

Thus the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ that can be extended to an isometry between the closure of $\mathcal{E}$, namely the Hilbert space $S^{(H)}$ and $L^2([0,T])$. Indeed, we have

$$\begin{aligned} < 1_{[0,t]}, 1_{[0,s]} >_{S^{(H)}} &= R_H(t,s) \\ &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du \\ &= < K_H(t,.)1_{[0,t]}, K_H(s,.)1_{[0,s]} >_{L^2([0,T])} \\ &= < K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]} >_{L^2([0,T])} \end{aligned}$$

4.2. Our approach for stochastic integration

Using previous theory, we could define the Wiener integration using the operator K_H^* as

$$\int_0^T \phi(s)dB_s^H = \int_0^T (K_H^*\phi)(s)dB_s,$$

for $\phi \in S^{(H)}$. But, for the right-hand side of equation to be well-defined we must have that $K_H^*\phi \in L^2([0,T])$.

Remark 4.1. Note that the Brownian motion $(W_t)_{t\in[0,T]}$ and the fBm $(B_t^H)_{t\in[0,T]}$ generate the same filtration. More precisely, the natural filtration of the Brownian motion and of the fBm that it generate through the Levy-Hida representation coincide.

Theorem 4.1 (Definition). *Let $\triangle = \{0 = t_0, t_1, t_2, \cdots, t_n = T\}$ be a partition of the interval $[0,T]$. On the subinterval $[t_{i-1}, t_i]$, we take the " right endpoint "t_i as the evaluation point for the integrand. For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $g(t)$, we define the stochastic integral of $f(t)g(t)$ to be the limit*

$$\int_0^T f(t)g(t)dB^H(t) = \lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}\psi_1^H(f)(t_{i-1})\psi_2^H(g)(t_i)(B(t_i) - B(t_{i-1}))$$

***Proof* 4.1.** Write $K_H^*(f.g)$ like in (4) and in (5) then develop a sequence of calculus (based on the results obtained by Joachim [19]) applied to the kernel in two cases $H < 1/2$ and $H > 1/2$. □

Namely, when one wants to compute $\int_0^1 wvdx$ with $v(x) = \int_0^x v'(y)dy$. We obtain by a classical integration by parts (including the trace terms in the integral) or by Fubini's theorem,

$$\int_0^1 wvdx = \int v'(x)\int_x^1 w(y)dydx,$$

For $\mathbf{H > \frac{1}{2}}$:

$$\int_0^T f(t)g(t)dB_t^H$$
$$= \int_0^T (K_H^*(f.g)(t))dB_t$$
$$= \int_0^T \int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t}dudB_t$$

$$=C_H\int_0^T\int_t^T (f.g)(u)(\frac{u}{t})^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t$$
$$=C_H\int_0^T t^{\frac{1}{2}-H}\int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t$$
$$=C_H\int_0^T t^{\frac{1}{2}-H}\left[\int_t^T\left(g(u)u^{H-\frac{1}{2}}\right)'\int_u^T f(y)(y-t)^{H-\frac{3}{2}}dydu\right]dB_t$$
$$=C_H\int_0^T t^{\frac{1}{2}-H}\Bigg[\Gamma(H-\frac{1}{2})\int_t^T\left(g(u)u^{H-\frac{1}{2}}\right)'\Big[-\frac{1}{\Gamma(H-\frac{1}{2})}\int_t^u f(y)(y-t)^{H-\frac{3}{2}}dy$$
$$+\frac{1}{\Gamma(H-\frac{1}{2})}\int_t^T f(y)(y-t)^{H-\frac{3}{2}}dy\Big]du\Bigg]dB_t$$
$$=C_H\int_0^T t^{\frac{1}{2}-H}\left[\Gamma(H-\frac{1}{2})\int_t^T\left(g(u)u^{H-\frac{1}{2}}\right)'(-(I_{u^-}^{H-\frac{1}{2}}f)(t)+(I_{T^-}^{H-\frac{1}{2}}f)(t))du\right]dB_t.$$

Let

$$J=\Gamma(H-\frac{1}{2})\int_t^T\left(g(u)u^{H-\frac{1}{2}}\right)'(-(I_{u^-}^{H-\frac{1}{2}}f)(t)+(I_{T^-}^{H-\frac{1}{2}}f)(t))du$$
$$=\int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}du.$$

Then ,

$$J=-\Gamma(H-\frac{1}{2})\left[g(t)t^{H-\frac{1}{2}}(I_{T^-}^{H-\frac{1}{2}}f)(t))\right]$$
$$-\Gamma(H-\frac{1}{2})\int_t^T g(u)u^{H-\frac{1}{2}}.f(u)(u-t)^{H-\frac{3}{2}}du$$
$$=\int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}du.$$

It means that $J=\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\left[g(t)t^{H-\frac{1}{2}}(I_{T^-}^{H-\frac{1}{2}}f)(t)\right]$. Then,

$$\int_0^T f(t)g(t)dB_t^H$$
$$=\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H\int_0^T g(t)(I_{T^-}^{H-\frac{1}{2}}f)(t)dB_t$$

$$=\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H \lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}(I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})g(t_i)(B(t_i)-B(t_{i-1}))$$
$$=\lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}(I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H g(t_i)(B(t_i)-B(t_{i-1}))$$
$$=\lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}\psi_1^H(f)(t_{i-1})\psi_2^H(g)(t_i)(B(t_i)-B(t_{i-1})),$$

where $\psi_1^H(f)(t_{i-1})=(I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})$ and $\psi_2^H(g)(t_i)=\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H g(t_i)$.

For $\mathbf{H<\frac{1}{2}}$:

$$\int_0^T f(t)g(t)dB_t^H$$
$$=\int_0^T (K_H^*(f.g)(t))dB_t$$
$$=\int_0^T\int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t}dudB_t$$
$$=\int_0^T\int_t^T (f.g)(u)\frac{\partial K_H(u,t)}{\partial t}dudB_t$$
$$=C_H(H-\frac{1}{2})\int_0^T\int_t^T (f.g)(u)(\frac{u}{t})^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t$$
$$=C_H(H-\frac{1}{2})\int_0^T t^{\frac{1}{2}-H}\int_t^T f(u).g(u)u^{H-\frac{1}{2}}(u-t)^{H-\frac{3}{2}}dudB_t$$
$$=C_H(H-\frac{1}{2})\int_0^T t^{\frac{1}{2}-H}\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\left[g(t)t^{H-\frac{1}{2}}(I_{T^-}^{H-\frac{1}{2}}f)(t)\right]dB_t$$
$$=C_H(H-\frac{1}{2})\int_0^T \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\left[g(t)(I_{T^-}^{H-\frac{1}{2}}f)(t)\right]dB_t$$
$$=C_H(H-\frac{1}{2})\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}\int_0^T g(t)(I_{T^-}^{H-\frac{1}{2}}f)(t)dB_t$$
$$=\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H(H-\frac{1}{2})\lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}(I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})g(t_i)(B(t_i)-B(t_{i-1}))$$
$$=\lim_{\|\triangle\|\to 0}\sum_{i=1}^{n}(I_{T^-}^{H-\frac{1}{2}}f)(t_{i-1})\frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})}C_H(H-\frac{1}{2})g(t_i)(B(t_i)-B(t_{i-1}))$$

$$= \lim_{\|\triangle\|\to 0} \sum_{i=1}^{n} \psi_1^H(f)(t_{i-1})\psi_2^H(g)(t_i)(B(t_i) - B(t_{i-1})),$$

where

$$\psi_1^H(f)(t_{i-1}) = (I_{T^-}^{H-\frac{1}{2}} f)(t_{i-1}) \text{ and } \psi_2^H(g)(t_i) = \frac{-\Gamma(H-\frac{1}{2})}{1+\Gamma(H-\frac{1}{2})} C_H(H-\frac{1}{2})g(t_i).$$

5. Conclusion

In this paper, we have introduced a new approach on stochastic integration for non-adapted processes with respect to processes having irregular trajectories, based on Levy-Hida representation. Our approach is used to solve stochastic differential equations driven by a fractional Brownian motion for integrants not necessarily adapted. Hoping that these results will serve to other processes such as sub-fractional Brownian motion, mixed fractional Brownian motion or Gaussian processes in general.

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