Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **17** (2022), No. 1, pp. 105-123
DOI: 10.21915/BIMAS.2022104

SOLVABILITY OF SOLUTION OF SINGULAR AND DEGENERATE FRACTIONAL NONLINEAR PARABOLIC DIRICHLET PROBLEMS

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Abstract

In this paper, we establish the existence and uniqueness of the weak solution in functional weighted Sobolev space for a class of initial-boundary value degenerate and singular fractional semi-linear parabolic problems. The results are established by using a priori estimate and applying an iterative process based on results obtained for the linear problem.

1. Introduction

In recent years, fractional differential equations (FDEs) and its application have gotten extensive attention by researchers, The so-called fractional differential equations are specified by generalizing the standard integer order derivative to arbitrary order. which can be obtained in time and space with a power law memory kernel of the nonlocal relationships. They provide a powerful tool to describing the memory of different substances and the nature of the inheritance. All of these studies have a clear, which open up a new field of scientific research in many areas, including a new theoretical analysis, applications in viscoelasticity, electro-chemistry, signal processing, electromagnetic, porous media, electrical networks, electromagnetic

Received December 26, 2021.

AMS Subject Classification: 35R11, 35K58, 35K67.

Key words and phrases: Partial fractional differential equation, energy inequality, existence, uniqueness, singular and degenerate equation.

theory and probability, signal and image processing, numerical methods for fractional order dynamical systems and many other physical processes are diverse applications of (FDEs).

Newly, there has been a significant development in field the fractional differential equations. This is due to the several recent papers studies in this field, see the monographs of Kilbas et al.[7], Miller and Ross [8], Samko et al.[9] and the papers of Agarwal et al. [10], Anguraj A. and Karthikeyan P. [11], Belmekki et al.[12], Daftardar-Gejji and Jafari [17], Furati and Tatar [24, 25], Kaufmann and Mboumi [26], Kilbas and Marzan [27], Yu and Gao [31], Oussaeif [32], and also the general references in Baleanu et al. [33], and the references therein. However, many phenomena can better be described by Dirichlet boundary conditions . Dirichiet boundary condition the user in solving many complex issues such as porous media, electromagnetic and environmental science.

There are not many works in the fractional field of partial differential equation, this is due to the difficulty of applying classical theories and methods to a field of fractional partial differential equations. Motivated by this, the present paper is devoted to the study the unique solvability of solution of singular and degenerate fractional semi-linear parabolic with Dirichlet condition, which has not been studied so far.

2. Preliminaries and Functional Spaces

Let $\Omega = [0, T]$ be a finite interval of the real numbers $\mathbb{R}$ and $\mathbf{\Gamma}(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Caputo derivative are the Riemann Liouville derivative are, respectively, defined as follows: Let $\Gamma(\cdot)$ denote the gamma function.

(1) The left Caputo derivatives:

$$^{C}D_t^{\alpha} u(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^{\alpha}} d\tau.$$

(2) The left Riemann-Liouville derivatives:

$$^{R}D_t^{\alpha} u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha}} d\tau.$$

(3) The right Caputo derivatives:

$$ {}_{t}^{C}D^{\alpha}u(x,t) := \frac{-1}{\Gamma\left(1-\alpha\right)}\int_{t}^{T}\frac{\partial u(x,\tau)}{\partial\tau}\frac{1}{\left(\tau-t\right)^{\alpha}}d\tau. $$

(4) The right Riemann-Liouville derivatives:

$$ {}_{t}^{R}D^{\alpha}u\left(x,t\right) = \frac{-1}{\Gamma\left(1-\alpha\right)}\frac{\partial}{\partial t}\int\limits_{t}^{T}\frac{u\left(x,\tau\right)}{\left(\tau-t\right)^{\alpha}}d\tau. $$

Many authors think that the Caputo's version is more natural because it allows the handling of inhomogeneous initial conditions in an easier way. Then the two definitions are linked by the following relationship, which can be verified by a direct calculation:

$$ {}^{R}D_{t}^{\alpha}u\left(x,t\right) = {}^{C}D_{t}^{\alpha}u\left(x,t\right) + \frac{u\left(x,0\right)}{\mathbf{\Gamma}\left(1-\alpha\right)t^{\alpha}}. $$

Definition 1 ([35, 36])**.** For any real $\sigma > 0$ and finite interval $[a,b]$ of the real axis $\mathbb{R}$, we define the semi-norm:

$$ |u|_{{}^{l}H^{\sigma}(\Omega)}^{2} := \left\|{}^{R}D_{t}^{\sigma}u\right\|_{L_{2}(\Omega)}^{2}, $$

and norm:

$$ \|u\|_{{}^{l}H^{\sigma}(\Omega)} := \left(\|u\|_{L_{2}(\Omega)}^{2} + |u|_{{}^{l}H^{\sigma}(\Omega)}^{2}\right)^{\frac{1}{2}}, $$

we then define ${}^{l}H_{0}^{\sigma}\left(\Omega\right)$ as the closure of $C_{0}^{\infty}\left(\Omega\right)$ with respect to the norm $\|\cdot\|_{{}^{l}H^{\sigma}(\Omega)}$.

Definition 2 ([35, 36])**.** For any real $\sigma > 0$, we define the semi-norm:

$$ |u|_{{}^{r}H^{\sigma}(\Omega)}^{2} := \left\|{}_{t}^{R}D^{\sigma}u\right\|_{L_{2}(\Omega)}^{2}, $$

and norm:

$$ \|u\|_{{}^{r}H^{\sigma}(\Omega)} := \left(\|u\|_{L_{2}(\Omega)}^{2} + |u|_{{}^{r}H^{\sigma}(\Omega)}^{2}\right)^{\frac{1}{2}}, $$

we then define ${}^{R}H_{0}^{\sigma}\left(\Omega\right)$ as the closure of $C_{0}^{\infty}\left(\Omega\right)$ with respect to the norm $\|\cdot\|_{{}^{r}H^{\sigma}(\Omega)}$.

Definition 3. For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{^cH^\sigma(\Omega)} = \left| \frac{\left({}^RD_t^\sigma u, \ {}_t^RD^\sigma u \right)_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{1/2}$$

and norm:

$$\|u\|_{^cH^\sigma(\Omega)} = (\|u\|^2_{L^2(\Omega)} + |u|^2_{^cH^\sigma(\Omega)})^{1/2}.$$

Lemma 1 ([35, 36])**.** *For any real $\sigma \in \mathbb{R}_+$, if $u \in {}^lH^\alpha(\Omega)$ and $v \in C_0^\infty(\Omega)$, then*

$$({}^RD_t^\sigma u(t), v(t))_{L^2(\Omega)} = (u(t), \ {}_t^RD^\sigma v(t))_{L^2(\Omega)}.$$

Lemma 2 ([35, 36])**.** *For $0 < \sigma < 2$, $\sigma \neq 1$, $u \in H_0^{\frac{\sigma}{2}}(\Omega)$, on a :*

$${}^RD_t^\sigma u(t) = \ {}^RD_t^{\frac{\sigma}{2}} \, {}^RD_t^{\frac{\sigma}{2}} u(t).$$

Lemma 3 ([35, 36])**.** *For $\sigma \in \mathbb{R}_+$, $\sigma \neq n+\frac{1}{2}$, the semi- norms $|.|_{^lH^\sigma(\Omega)}$, $|.|_{^rH^\sigma(\Omega)}$ and $|.|_{^cH^\sigma(\Omega)}$ are equivalent. Then we pose*

$$|.|_{^lH^\sigma(\Omega)} \cong |.|_{^rH^\sigma(\Omega)} \cong |.|_{^cH^\sigma(\Omega)} .$$

Lemma 4 ([35, 36])**.** *For any real $\sigma > 0$, the space ${}^RH_0^\sigma(\Omega)$ with respect to the norm in Definition 2 is complete.*

3. Formulation of the Problem and Functional Space

Let $T > 0, a \in \mathbb{R}_+^*$; $\Omega = (0, l)$ and

$$Q = \Omega \times (0, T) = \left\{(x, t) \in \mathbb{R}^{n+1} : x \in \Omega, 0 < t < T\right\}.$$

We consider the linear fractional parabolic problem:

$$\begin{cases} {}^CD_t^\alpha v(x,t) - a\frac{\partial}{\partial x}\left(x^\beta \frac{\partial v(x,t)}{\partial x}\right) + bv = g(x,t,v) & \text{in } Q_T, \\ v(x,0) = \varphi(x), & \forall\ x \in (0,l), \\ v(0,t) = v(l,t) = 0, & \forall\ t \in (0,T). \end{cases} \quad . \qquad (\mathrm{P}_1)$$

where g, φ are known functions. We shall assume that the function φ satisfies a compatibility conditions:

$$\varphi(0) = \varphi(l) = 0.$$

the function g is Lipchitzian, there is a positive constant k such that:

$$\| g(x,t,v_1,) - g(x,t,v_2) \|_{L^2(Q)} \le k \left(\|v_1 - v_2\|_{L^2(0,T,H^1(0,1))} \right).$$

For applying our method, we must introduced a new function $u(x,t) = v(x,t) - \varphi(x)$. Then the problem can be formulated as:

$$\begin{cases} {}^C D_t^\alpha u(x,t) - a\frac{\partial}{\partial x}\left(x^\beta \frac{\partial u(x,t)}{\partial x}\right) + bu & \\ = g(x,t,u) + \frac{\partial}{\partial x}\left(x^\beta \frac{\partial \varphi(x,t)}{\partial x}\right) - b\varphi = f(x,t,u) & \text{in } Q_T, \\ u(x,0) = 0, & \forall\, x \in (0,l), \\ u(0,t) = u(l,t) = 0 & \forall\, t \in (0,T). \end{cases} \tag{P$_1$}$$

Whose linear fractional parabolic equation is given as follows

$$\mathcal{L}u = {}^C D_t^\alpha u - \frac{\partial}{\partial x}\left(x^\beta \frac{\partial u(x,t)}{\partial x}\right) + bu = f(x,t) \tag{3.1}$$

with the initial condition

$$lu = u(x,0) = 0 \quad \forall x \in [0,l] \tag{3.2}$$

the Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \forall t \in [0,T] \tag{3.3}$$

where are given functions. and α and β satisfy the following assumptions:

1. $0 \le \alpha \le 1$, $0 \le \beta < 1$, $(x,t) \in \bar{Q}$.

We establish a priori bound and prove the existence of a solution to the problems (3.1)−(3.3). With $Lu = F$, where $L = (\mathcal{L}, l)$, and $F = (f, u_0)$ be the operator equation corresponding to problems (3.1)−(3.3). The operator L acts from E to F defined as follows. The Banach space E consists of all functions $u(x,t)$ with the finite norm

$$\|u\|_E^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| x^{\frac{\beta}{2}} \nabla v \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2. \tag{3.4}$$

The Hilbert space F consists of the vector valued functions f with the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 \tag{3.5}$$

4. A Priori Bound of Linear Case

Theorem 1. *If the assumptions $A1$ are satisfied then for any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\left\| {}^C D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| x^{\frac{\beta}{2}} \nabla u \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \leq c \left(\|f\|_{L^2(Q)}^2 \right), \tag{4.1}$$

and $D(L)$ is the domain of definition of the operator L defined by

$$D(L) = \{u \in L^2(Q) \ / \ {}^C D_t^{\frac{\alpha}{2}} u, \ x^{\frac{\beta}{2}} \nabla u \in L^2(Q)\},$$

satisfying conditions (3.3).

Proof. Taking the scalar product in $L^2(Q)$ of Eequation (3.1) and the operator

$$Mu = u,$$

where $Q^\tau = \Omega \times (0, T)$, we have

$$\begin{aligned} (\mathcal{L}u, Mu)_{L^2(Q^\tau)} &= \left({}^C D_t^{\alpha} u, u\right)_{L^2(Q^\tau)} - a \left(\frac{\partial}{\partial x} \left(x^\beta \frac{\partial u(x,t)}{\partial x} \right), u \right)_{L^2(Q^\tau)} \\ &\quad + b\,(u,u)_{L^2(Q^\tau)} \\ &= \left(\tilde{f}, u\right)_{L^2(Q^\tau)}, \end{aligned} \tag{4.2}$$

The successive integration by parts of integrals on the right-hand side of (4.2), yields

$$\begin{aligned} \left({}^C D_t^{\alpha} u, u\right)_{L^2(Q^\tau)} &= \left({}^C D_t^{\frac{\alpha}{2}} u, {}^C_t D^{\frac{\alpha}{2}} u\right)_{L^2(Q^\tau)} \\ &= \cos\left(\frac{\alpha\pi}{2}\right) |u|_{{}^C H^{\frac{\alpha}{2}}(\Omega)}^2 \\ &= \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q^\tau)}^2, \end{aligned} \tag{4.3}$$

and

$$-a\left(\frac{\partial}{\partial x}\left(x^{\beta}\frac{\partial u(x,t)}{\partial x}\right),u\right)_{L^2(Q^{\tau})}=a\left\|x^{\frac{\beta}{2}}\nabla u\right\|^2_{L^2(Q^{\tau})}, \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we obtain

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^{C}D_t^{\frac{\alpha}{2}}u\right\|^2_{L^2(Q^{\tau})}+a\left\|x^{\frac{\beta}{2}}\nabla u\right\|^2_{L^2(Q^{\tau})}\leq\left(\tilde{f},u\right), \tag{4.5}$$

estimate the last term on the right-hand side of (4.5) by applying Cauchy inequality with ε, ($|ab|\leq\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2}$), then we get

$$\begin{aligned}&\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^{C}D_t^{\frac{\alpha}{2}}u\right\|^2_{L^2(Q^{\tau})}+a\left\|x^{\frac{\beta}{2}}\nabla u\right\|^2_{L^2(Q^{\tau})}+b\left\|u\right\|^2_{L^2(Q^{\tau})}\\&\leq\frac{1}{2\varepsilon}\left\|\tilde{f}\right\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|u\right\|^2_{L^2(Q)},\end{aligned} \tag{4.6}$$

Then the estimate (4.6) becomes

$$\begin{aligned}&\left\|{}^{C}D_t^{\frac{\alpha}{2}}u\right\|^2_{L^2(Q)}+\left\|x^{\frac{\beta}{2}}\nabla u\right\|^2_{L^2(Q^{\tau})}+\left\|u\right\|^2_{L^2(Q^{\tau})}\\&\leq\frac{1}{2\varepsilon\min(\cos\left(\frac{\alpha\pi}{2}\right),a,\left(b-\frac{\varepsilon}{2}\right))}\left\|\tilde{f}\right\|^2_{L^2(Q)},\end{aligned}$$

So, finally we get

$$\left\|{}^{C}D_t^{\frac{\alpha}{2}}u\right\|^2_{L^2(Q)}+\left\|x^{\frac{\beta}{2}}\nabla u\right\|^2_{L^2(Q^{\tau})}+\left\|u\right\|^2_{L^2(Q^{\tau})}\leq\delta\left\|\tilde{f}\right\|^2_{L^2(Q)},$$

Where

$$\delta=\frac{1}{2\varepsilon\min(\cos\left(\frac{\alpha\pi}{2}\right),a,\left(b-\frac{\varepsilon}{2}\right))}.$$

So, we have

$$\left\|u\right\|_E\leq\delta\left\|Lu\right\|_F \tag{4.7}$$

Let $R(L)$ be the range of the operator L. However, since we do not have any information about $R(L)$, except that $R(L)\subset F$, we must extend L, so that estimate (??) holds for the extension and its range is the whole space F. We first state the following proposition. □

Proposition 1. *The operator $L:E\longrightarrow F$ has a closure*

Proof. let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ a sequence where :

$$u_n \longrightarrow 0 \qquad \text{in } E$$

and

$$Lu_n \longrightarrow \left(\tilde{f};0\right) \quad \text{in } F \tag{4.8}$$

we must proof that

$$f \equiv 0\ .$$

The convergence of u_n to 0 in E drives :

$$u_n \longrightarrow 0 \quad \text{in } D'(Q)\ . \tag{4.9}$$

According to the continuity of the derivation of $D'(Q)$ in $D'(Q)$. The relation (4.9) involved

$$\mathcal{L}u_n \longrightarrow 0 \quad \text{in } D'(Q)\ , \tag{4.10}$$

Moreover, the convergence of $\mathcal{L}u_n$ to f in $L^2(Q)$ gives

$$\mathcal{L}u_n \longrightarrow f \quad \text{in } D'(Q) \tag{4.11}$$

As we have the the uniqueness of the limit in $D'(Q)$, we conclude from (4.10) and (4.11) that

$$f = 0.$$

Then, we get L is closable of this operator, with domain of definition $D(L)$.

Definition 4. A solution of the operator equation $\bar{L}u = \mathcal{F}$ is called a strong solution to problems (3.1)−(3.3).

Definition 5. The priori estimate (4.1) can be extended to strong solutions, i.e., we have the estimate

$$\left\|{}^C D_t^{\frac{\alpha}{2}} u\right\|^2_{L^2(Q)} + \left\|x^{\frac{\beta}{2}} \nabla u\right\|^2_{L^2(Q^\tau)} + \|u\|^2_{L^2(Q^\tau)} \leq \delta \left\|\tilde{f}\right\|^2_{L^2(Q)}, \tag{4.12}$$

We deduce from the estimate (4.12).

Corollary 1. *The range $R(\bar{L})$ of the operator $\bar{L}$ is closed in F and is equal to the closure $\overline{R(L)}$ of $R(L)$, that is $R(\bar{L}) = \overline{R(L)}$.*

Proof. Let $z \in \overline{R(L)}$, so there is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$in F constituted of the elements of the set $R(L)$ such as

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

There is then a corresponding sequence $u_n \in D(L)$ such as

$$z_n = Lu_n.$$

The estimate (4.7) , we get

$$\|u_p - u_q\|_E \leq C\,\|Lu_p - Lu_q\|_F \to 0,$$

Where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$is a Cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \text{ in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E ; If $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z$, then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = z$ as like $\bar{L}$ and is closed, so $\bar{L}u = z$), the function u check :

$$v \in D\left(\bar{L}\right),\ \bar{L}v = z.$$

Then $z \in R(\bar{L})$, so

$$\overline{R(L)} \subset R(\bar{L})$$

Also we conclude here that $R(\bar{L})$ is closed becauce it is Banach (any complete subspace of a metric space (not necessarily complete) is closed). It remains to show the reverse inclusion. Either $z \in R(\bar{L})$ then it exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$in F constituted of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z$$

or $z \in R(\bar{L})$, because $R(\bar{L})$ is a closed subset a completed F, so $R(\bar{L})$ is complete. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = z_n.$$

We get from (4.7):

$$\|u_p - u_q\|_E \leq C \left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0,$$

Where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$is a Cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } E.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such as

$$\bar{L}u_n = Lu_n \text{ on } R\left(L\right), \forall n \in \mathbb{N}.$$

So

$$\lim_{n\longrightarrow+\infty} Lu_n = z,$$

Consequently $z \in \overline{R\left(L\right)}$, then we conclude that

$$R\left(\bar{L}\right) \subset \overline{R\left(L\right)}. \qquad \square$$

5. Existence of Solution of the Linear Case

Theorem 2. *Let the assumptions A_1 be satisfied. Then for all $F = (f, 0) \in F$, there exists a unique strong solution $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of the problem* (3.1)−(3.3).

Proof. We have

$$(Lu, W)_F = \int_Q \mathcal{L}u.wdxdt \tag{5.1}$$

Where

$$W = (w, 0)\,.$$

Si for $w \in L^2\left(Q\right)$ and for all $u \in D_0(L) = \{u,\ u \in D\left(L\right) : \ell u = 0\}$, we have

$$\int_Q \mathcal{L}u.wdxdt = 0$$

By putting $w = u$, and using the same estimate of section 1, we obtain

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^{C}D_{t}^{\frac{\alpha}{2}}u\right\|_{L^{2}(Q)}^{2}+a\left\|x^{\frac{\beta}{2}}\nabla u\right\|+b\left\|u\right\|_{L^{2}(Q)}^{2}=0,$$

we get

$$\|u\| \leq 0 \Rightarrow u = 0\ .$$

So, it gives $u = w = 0$. □

Corollary 2. *If for any function $u \in D(L)$, we have the following estimate:*

$$\|u\|_{E} \leq C\,\|\mathcal{F}\|_{F}\,,$$

Then the solution of the problem (P_1) if it exists, it is unique.

Proof. Let u_1 and u_2 be two solutions to the problem (P_1)

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

and as L is linear we then obtain

$$L\left(u_1 - u_2\right) = 0,$$

according to(4.7)

$$\|u_1 - u_2\|_{E}^{2} \leq c\,\|0\|_{F}^{2} = 0,$$

Which give

$$u_1 = u_2.$$ □

6. Solvability of the Weak Solution of the Semi-Linear Problem

This section is devoted to the proof of the existence and the uniqueness of the solution of the semi-linear problem (Pr) :

$$\begin{cases} {}^{C}D_{t}^{\alpha}u(x,t) - \left(x^{\alpha}u_{x}\right)_{x} + bu = f\left(x,t,u\right), & \forall\left(x,t\right)\in Q, \\ u\left(x,0\right) = 0, & \forall x\in\left(0,1\right), \\ u(0,t) = u(l,t) = 0, \quad . & \forall t\in\left(0,T\right). \end{cases} \qquad (P_1)$$

Putting

$$u = y$$

such that y is a solution to the following nonlocal linear problem:

And the solution satisfies the following nonlocal nonlinear problem

$$\mathcal{L}y =^{C} D_t^{\alpha} y(x,t) - (x^{\alpha} y_x)_x + by(x,t) = f(x,t,y), \tag{6.1}$$
$$y(x,0) = 0, \qquad \forall x \in (0,1), \tag{6.2}$$
$$y(0,t) = y(1,t) = 0 \qquad \forall t \in (0,t). \tag{6.3}$$

the function f is also Lipchitzian, there is a positive constant k such that :

$$\| f(x,t,u_1,) - f(x,t,u_2) \|_{L^2(Q)} \leq k \left(\|u_1 - u_2\|_{L^2(0,T,H^1(0,1))} \right). \tag{6.4}$$

Building a recurring sequence starting with $y^{(0)} = 0$.

The sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows : given the element $y^{(n-1)}$, then for $n = 1, 2, 3, ...,$ we will solve the following problem :

$$\begin{cases} {}^{C}D_t^{\alpha} y^{(n)} - (x^{\alpha} y_x^{(n)})_x + by^{(n)} = f\left(x,t,y^{(n-1)}\right) \\ y^{(n)}(x,0) = 0 \\ y^{(n)}(0,t) = y^{(n)}(1,t) = 0. \end{cases}, \tag{P_5}$$

According to the study of the previous linear problem each time we fix the n, the problem (P_5) admits a unique solution $y^{(n)}(x,t)$.

Now by supposing

$$z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t),$$

so we get a new problem :

$$\begin{cases} {}^{C}D_t^{\alpha} z^{(n)} - (x^{\alpha} z_x^{(n)})_x + bz^{(n)} = p^{(n-1)}(x,t), \\ z^{(n)}(x.0) = 0, \\ z^{(n)}(0,t) = z^{(n)}(1,t) = 0. \end{cases} \tag{P_6}$$

Or

$$p^{(n-1)}(x,t) = f\left(x,t,y^{(n)}\right) - f\left(x,t,y^{(n-1)}\right).$$

Multiply

$$^{C}D_t^{\alpha}z^{(n)} - (x^{\alpha}z_x^{(n)})_x + bz^{(n)} = p^{(n-1)}(x,t)$$

by $z^{(n)}$,and integrate it on Q_τ, we get :

$$\int_{Q_\tau}^{C} D_t^{\alpha}z^{(n)}(x,t)\cdot z^{(n)}(x,,t)dxdt - \int_{Q_\tau}(x^{\alpha}z_x^{(n)})_x\cdot z^{(n)}\,(x,t)\,dxdt$$
$$+b\int_{Q_\tau}(z^{(n)}(x,t))^2dxdt$$
$$= \int_{Q_\tau}p^{(n-1)}(x,t)\cdot z^{(n)}\,(x,t)\,dxdt.$$

Use an integration by parts by taking account of the initial condition and the boundary conditions, we find :

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^{C}D_t^{\left(\frac{\alpha}{2}\right)}z^{(n)}\right\| + \left\|z_x^{(n)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)} + b\left\|z^{(n)}\right\|_{L^2(Q)}$$
$$= \int_{Q_\tau}p^{(n-1)}(x,t)\cdot z^{(n)}\,(x,t)\,dxdt.$$

We apply the Cauchy-Schwarz inequality on the second part of the equation, we get:

$$\int_{Q_\tau}p^{(n-1)}(x,t)\cdot z^{(n)}\,(x,t)\,dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau}\mid p^{(n-1)}(x,t)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}\,(x,t)\right)^2 dxdt,$$
$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau}\mid f\left(x,t,y^{(n)}\right) - f\left(x,t,y^{(n-1)}\right)\mid^2 dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}\,(x,t)\right)^2 dxdt,$$

Like G Lipschtizienne, we find :

$$\int_{Q_\tau}p^{(n-1)}(x,t)\cdot z^{(n)}\,(x,t)\,dxdt$$
$$\leqslant \frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid y^{(n)} - y^{(n-1)}\mid)^2dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}\,(x,t)\right)^2 dxdt,$$
$$\leqslant \frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid)^2dxdt + \frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}\,(x,t)\right)^2 dxdt,$$

$$\leqslant \frac{k^2}{\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid^2)dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(z^{(n)}\left(x,t\right)\right)^2dxdt,$$
$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|z^{(n)}\right\|^2_{L^2(Q)},$$

we get :

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^CD_t^{\left(\frac{\alpha}{2}\right)}z^{(n)}\right\|+\left\|z_x^{(n)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)}+b\left\|z^{(n)}\right\|^2_{L^2(Q)}$$
$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|z^{(n)}\right\|^2_{L^2(Q)},$$

We integrate on t, we obtain :

$$\cos\left(\frac{\alpha\pi}{2}\right)\left\|{}^CD_t^{\left(\frac{\alpha}{2}\right)}z^{(n)}\right\|+\left\|z_x^{(n)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)}+(b-\frac{\varepsilon}{2})\left\|z^{(n)}\right\|^2_{L^2(Q)}\leqslant\frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2_{(Q)}}$$

Then, we obtain

$$\left\|{}^CD_t^{\left(\frac{\alpha}{2}\right)}z^{(n)}\right\|+\left\|z_x^{(n)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)}+\left\|z^{(n)}\right\|^2_{L^2(Q)}$$
$$\leqslant\frac{k^2}{\varepsilon\min(1,\cos\left(\frac{\alpha\pi}{2}\right),(b-\frac{\varepsilon}{2}))}\left\|z^{(n-1)}\right\|_{L^2_{(Q)}}$$

So, we obtain

$$\left\|{}^CD_t^{\left(\frac{\alpha}{2}\right)}z^{(n)}\right\|+\left\|z_x^{(n)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)}+\left\|z^{(n)}\right\|^2_{L^2(Q)}$$
$$\leqslant\frac{k^2}{\varepsilon\min(1,\cos\left(\frac{\alpha\pi}{2}\right),(b-\frac{\varepsilon}{2}))}\left\|z^{(n-1)}\right\|^2_{L^2(Q)}+\left\|{}^CD_t^{\left(\frac{\alpha}{2}\right)}z^{(n-1)}\right\|+\left\|z_x^{(n-1)}\right\|^2_{L^2_{\sqrt{x^\beta}}(Q)}$$

Putting :

$$c=\max\left\{1,\frac{k^2}{\varepsilon\min(1,\cos\left(\frac{\alpha\pi}{2}\right),(b-\frac{\varepsilon}{2}))}\right\}.$$

So, we get :

$$\left\|z^{(n)}\right\|^2_V\leqslant c\left\|z^{(n-1)}\right\|^2_V$$

where

$$V = \left\{ y,\ y \in L^2(Q), y_x \in L^2_{\sqrt{x^\beta}}(Q) \right\}$$

As we have :

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}$$

According to the convergence criterion of the series, gives that the series $\sum_{n=1}^{\infty} z^{(n)}$ converges if $|c| < 1$, which implies :

$$\left| \frac{k^2}{\varepsilon \min(1, \cos\left(\frac{\alpha\pi}{2}\right), (b - \frac{\varepsilon}{2}))} \right| < 1$$

$$k\sqrt{\frac{k^2}{\varepsilon \min(1, \cos\left(\frac{\alpha\pi}{2}\right), (b - \frac{\varepsilon}{2}))}} < 1$$

$$k < \sqrt{\frac{k^2}{\varepsilon \min(1, \cos\left(\frac{\alpha\pi}{2}\right), (b - \frac{\varepsilon}{2}))}}.$$

then $y^{(n)}$ converges on an element of V , we call y. We will show that in V :

$$\lim_{n \longrightarrow \infty} y^{(n)}(x,t) = y(x,t)$$

is a solution to the problem (P_4) showing that y chacket :

$$A(y,v) = \int_{Q_\tau} f(x,t,y) \cdot v(x,t) dxdt \quad \forall\ v \in O.$$

Where

$$O = \left\{ v \in C^1(Q), v(0.t) = v(1.t) = 0, \forall t \in (0.T) \right\},$$

and

$$A\left(y^{(n)}, v\right) = ({}^C D_t^{\frac{\alpha}{\acute{e}}} y^n(x,t), {}_t^C D^{\frac{\alpha}{2}} v(x,t))_{L^2(Q)} + \int_{Q_\tau} (x^\alpha y_x^{\prime n)}(x,t) \cdot v_x(x,t) dxdt$$
$$+ \int_{Q_\tau} y^n(x,t) \cdot v_x(x,t) dxdt$$

we have :

$$\begin{aligned} A\left(y^{(n)}-y,v\right) &= ({}^{C}D_{t}^{\frac{\alpha}{\acute{e}}}(y^{n}-y)(x,t),{}_{t}^{C}D^{\frac{\alpha}{2}}v(x,t))_{L^{2}(Q)} \\ &\quad +\int_{Q_{\tau}}(x^{\alpha}(y_{x}^{\prime n)}-y)(x,t)\cdot v_{x}(x,t)dxdt \\ &\quad +\int_{Q_{\tau}}(y^{n}-y)(x,t)\cdot v_{x}(x,t)dxdt \end{aligned}$$

We apply the Cauchy Schwartz inequality, we find :

$$A\left(y^{(n)}-y,v\right)\leqslant\| v\|_{V}\left\|(y^{(n)}-y)_{t}\right\|_{V}+\| v\|_{V}\left\|(y^{(n)}-y)_{x}\right\|_{V}$$

On the other hand, as

$$y^{(n)}\longrightarrow y \qquad \text{in } V,$$

so

$$\begin{aligned} y^{(n)} &\longrightarrow y \quad \text{in } L^{2}(Q), \\ y_{t}^{(n)} &\longrightarrow y_{t} \quad \text{in } L^{2}(Q), \\ y_{x}^{(n)} &\longrightarrow y_{x} \quad \text{in } L^{2}_{\sqrt{x^{\alpha}}}(Q), \end{aligned}$$

Let's go to the limit when $n\longrightarrow+\infty$, we get :$\lim_{n\longrightarrow+\infty}A(y^{(n)}-y,v)=0$

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