Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **17** (2022), No. 1, pp. 1-51
DOI: 10.21915/BIMAS.2022101

ASYMPTOTIC EXPANSION OF THE BERGMAN KERNEL VIA SEMI-CLASSICAL SYMBOLIC CALCULUS

YU-CHI HOU

Department of Mathematics, National Taiwan University, Asotronomy-Mathematics Building, No.1, Sec. 4, Roosevelt Road, Taipei, 10617, Taiwan.
E-mail: r08221011@ntu.edu.tw

Abstract

We give a new proof on the pointwise asymptotic expansion for Bergman kernel associated to k-th tensor power of a hermitian holomorphic line bundle on the points where the curvature of the line bundle is positive and satisfies local spectral gap condition. The main point is to introduce a suitable semi-classical symbol space and related symbolic calculus inspired from recent work of Hsiao and Savale. Particularly, we establish the existence of pointwise asymptotic expansion on the positive part for certain semi-positive line bundles.

1. Introduction and the Main Result

Let L be a holomorphic line bundle over a complex manifold X with $\dim_{\mathbb{C}} X = n$. If we endow a positive, smooth $(1,1)$-form ω on X, which induces a Riemannian volume form $d\nu_X = \omega_n := \frac{\omega^n}{n!}$, and a hermitian metric h^L on L given by local weight ϕ, then they give rise to a scalar product on $C_c^\infty(X,L)$, the space of smooth global sections for L with compact supports. We then complete $C_c^\infty(X,L)$ with respect to the scalar product to get a Hilbert space $L^2_{\omega,\phi}(X,L)$. The orthogonal projection $\Pi : L^2_{\omega,\phi}(X,L) \to \mathcal{H}^0(X,L)$ onto the subspace of L^2-integrable holomorphic sections of L is called the *Bergman projection*, and its Schwartz kernel $K(z,w)$ is called the *Bergman kernel*. It is well-known that $K(z,w)$ is a smoothing kernel.

Received February 20, 2022.
AMS Subject Classification: 32L10.
Key words and phrases: Bergman kernel, semi-classical symbol space.

In general, it is difficult to calculate the Bergman kernel explicitly. However, when we replace L by its k-th tensor power $L^k := L^{\otimes k}$ and the hermitian metric ϕ by $k\phi$, the large k-behavior of the Bergman kernel is rather tractable and has important applications such as approximation of Kähler metrics by Fubini-Study metrics via the Kodaira map ([40], [5], and [36]), existence of canonical Kähler metrics (eg. [17], [19], and [9]–[10]), Berezin–Toeplitz quantization (eg. [4], [37], and [34]), and in physics [15].

If L is positive and X is compact, then a well-known asymptotic formula asserts that there exist smooth functions $b_r(x) \in C^\infty(X)$, for $r \in \mathbb{N}_0$, such that for any $N, l \in \mathbb{N}$, there exists a constant $C := C_{N,l} > 0$ independent of k satisfying

$$\left\| K_k(z,z) - \sum_{r=0}^{N} k^{n-r} b_r(x) \right\|_{C^l(X)} \leq C k^{n-(N+1)}, \quad k \gg 1. \tag{1}$$

The existence of formula (1) has been worked out in various generalities and through a variety of methods over the last thirty years. The leading asymptotic was first proved independently by Tian (1990, [40]) using Hörmander's L^2-estimates and by Bouche (1990, [5]) using heat kernel. The full asymptotic was later developed independently by Catlin (1999, [8]) and Zelditch (1998, [43]) using a result in CR geometry due to Boutet de Monvel and Sjörstrand (1975, [2]). Later, Dai, Liu, and Ma (2006, [16]) and Ma and Marienscu (2006, [33]) obtained both diagonal and off-diagonal expansions for generalized Bergman kernels for spin^c-Dirac operators on compact symplectic manifolds based on the analytic localization technique due to Bismut and Lebeau. We refer the book of Ma and Marinescu [32] and the references therein for this approach.

If one drops the positive curvature assumption for L and assumes instead that the curvature is non-degenerate with constant signature (n_+, n_-), then Berman and Sjöstrand (2007, [7]) showed the similar asymptotic expansions holds for orthogonal projection onto the space of harmonic $(0,q)$-form $\mathscr{H}^{0,q}(X, L^k)$ of Kodaira Laplacian if X is compact and $q = n_-$. Independently, Ma and Marinescu (2006, [31]) proved the analogous results in the setting of spin^c-Dirac operators on compact symplectic manifolds. In [27], Hsiao and Marinescu (2014) proved that the spectral function for Kodaira Laplacian always admits local asymptotic expansion for any hermitian holomorphic line bundle on the non-degenerate points of the curvature, and they deduce local asymptotic of Bergman kernel under the additional spectral gap condition (cf. Definition 1).

We now formulate the main result. Let (X, ω) be a hermitian manifold of complex dimension n, where ω is a smooth, positive $(1,1)$-form on X, inducing the hermitian structure on X. We denote $\langle \cdot | \cdot \rangle_\omega$ by the hermitian metric on $T^{1,0}X$ induced by ω. A canonical Riemannian volume form $d\nu_X$ for (X, ω) is given by $\omega_n := \frac{\omega^n}{n!}$. Let L be a holomorphic line bundle on X and set $L^k := L^{\otimes k}$, for $k \in \mathbb{N}$. For any hermitian metric h on L, we can define the Chern connection ∇ on L with respect to h with curvature $R^L(h) = (\nabla^L)^2 \in A^{1,1}(X)$. We identify $R^L(h)$ with the curvature operator $\dot{R}^L(h) \in C^\infty(X, \mathrm{End}(T^{1,0}X))$ by

$$\sqrt{-1} R^L(h)(x)(v \wedge \overline{w}) = \langle \dot{R}^L(h)(x) v | w \rangle_\omega, \tag{2}$$

for any $x \in X$, $v, w \in T^{1,0}_x X$. We denote $n_+(x)$,$n_-(x)$,$n_0(x)$ by the number of positive, negative, and zero eigenvalues of $\dot{R}^L(h)$ at x. For $q = 0, \ldots, n$, we let $X(q) := \{x \in X : n_+(x) = n - q, n_-(x) = q, n_0(x) = 0\}$. Notice that $X(q)$ is an open set of X, for each $q \in \{0, 1 \ldots, n\}$.

Locally, if s is a holomorphic trivialization of L over an open set $U \subset X$, then the hermitian metric h is determined by $|s|_h^2 = e^{-2\phi}$, where $\phi \in C^\infty(U, \mathbb{R})$ is called the *local weight* of h. On the k-th tensor power L^k of L, h induces a natural hermitian metric h^k on L^k with local weight $k\phi$. Let $\langle \cdot | \cdot \rangle_{k\phi}$ be the pointwise scalar product on the bundle L^k and $(\cdot | \cdot)_{\omega, k\phi}$ be the inner product on the space $C_c^\infty(X, L^k)$ of compact supported smooth sections of L^k, induced by ω and h^k. We denote $|\cdot|_{k\phi}$ and $\|\cdot\|_{\omega, k\phi}$ be the pointwise and L^2-norm associated to ω and h^k, and let $L^2(X, L^k)$ be the completion of $C_c^\infty(X, L^k)$ with respect to $\|\cdot\|_{\omega, k\phi}$.

Let $\bar{\partial} : C^\infty(X, L^k) \to A^{0,1}(X, L^k)$ be the Cauchy–Riemann operator acting on smooth sections of L^k, $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$ with respect to $(\cdot | \cdot)_{\omega, k\phi}$, and $\Box^{(0)}_{\omega, k\phi} := \bar{\partial}^* \bar{\partial}$ be the Kodaira Laplacian acting on $C^\infty(X, L^k)$ (cf. (13)). We denote by $\Box^{(0)}_k$ by Gaffney extension of the Kodaira Laplacian (cf. [32, Proposition 3.1.2]). Let $\mathscr{H}^0(X, L^k)$ be the kernel of $\Box^{(0)}_k$ and let $\Pi^{(0)}_k : L^2(X, L^k) \to \mathscr{H}^0(X, L^k)$ be the Bergman kernel for L^k-sections.

To state our result, we first define local spectral gap property.

Definition 1. For $d \in \mathbb{R}$ and an open set $D \subset X$, we say that $\Box^{(0)}_{\omega,k\phi}$ has *local spectral gap condition* of order d on D if there exists $C > 0$ and $k_0 \in \mathbb{N}$ such that for any $u \in C_c^\infty(D, L^k)$, if $k \geq k_0$,

$$\left\| (I - \Pi_k^{(0)})u \right\|_{\omega,k\phi} \leq \frac{1}{Ck^d} \left\| \Box^{(0)}_{\omega,k\phi} u \right\|_{\omega,k\phi}, \tag{3}$$

where $\Pi_k^{(0)}$ is the Bergman projection from $L^2(X, L^k) \to \mathscr{H}^0(X, L^k)$.

Next, we introduce the key ingredients in our approach. Namely, a kind of semi-classical symbol space inspired from the recent work of [28].

Definition 2. For $m \in \mathbb{R}$, a function $a(x, y, k)$ with parameter $k \in \mathbb{N}$ is in $\widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ if

(i) $a(x, y, k) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, for each $k \in \mathbb{N}$, and

(ii) for any$(\alpha, \beta) \in \mathbb{N}_0^{2d}$, there exists $l = l(\alpha, \beta) \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that for any $N \in \mathbb{N}$, there exists a constant $C = C_{\alpha,\beta,N}(a) > 0$ independent of k satisfying

$$\left| \partial_x^\alpha \partial_y^\beta a(x, y, k) \right| \leq Ck^{m + \frac{|\alpha|+|\beta|}{2}} \frac{(1 + |\sqrt{k}x| + |\sqrt{k}y|)^l}{(1 + |\sqrt{k}(x - y)|)^N}, \tag{4}$$

for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, any $k \geq k_0$.

Furthermore, we say $a \in \widehat{S}^m_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$ if $a \in \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ and there exists a sequence $a_j \in \widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$ (cf. Definition 4 for the definition of $\widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$), for $j \in \mathbb{N}_0$, so that

$$a(x, y, k) - \sum_{j=0}^{N-1} k^{m-\frac{j}{2}} a_j(\sqrt{k}x, \sqrt{k}y) \in \widehat{S}^{m-\frac{N}{2}}(\mathbb{R}^d \times \mathbb{R}^d), \quad \forall N \in \mathbb{N}. \tag{5}$$

It is convenient to work in an equivalent set-up for which the norm is defined by integral without parameter k. Let s be a local holomorphic trivialization of L over an open set $U \subset X$, we can make the following identification:

$$A^{0,q}(U, L^k) \to A^{0,q}(U), \quad u = s^k \otimes \alpha \mapsto \widetilde{\alpha} := \alpha e^{-k\phi}$$

so that for any $u, v \in L^2_{0,q}(U, L^k) \cap A^{0,q}(U, L^k)$,

$$(u|v)_{\omega,k\phi} = \int_U \langle \alpha|\beta\rangle_\omega e^{-2k\phi}\omega_n = \int_U \langle e^{-k\phi}\alpha|e^{-k\phi}\beta\rangle_\omega \omega_n =: (\widetilde{\alpha}|\widetilde{\beta})_\omega.$$

Let $L^2_{0,q}(U,\omega)$ be the completion of $L^2_{0,q}(U)$ with respect to $(\cdot|\cdot)_\omega$ defined above. Clearly, above identification extends to an isometry $L^2_{0,q}(U, L^k) \cong L^2_{0,q}(U,\omega)$. We define the *localized Bergman kernel* with respect to s by

$$\Pi^{(q)}_{k,s}\alpha = e^{-k\phi}s^{-k}\Pi^{(q)}_k(e^{k\phi}\alpha \otimes s^k),$$

where s^{-k} is the dual section of s^k so that $s^{-k}(s^k) \equiv 1$ on U. We denote $K_{k,s}$ by the Schwartz kernel of the localized Bergman kernel $\Pi_{k,s}$, called the *localized Bergman kernel*.

We now can state the main result.

Theorem 1. *Suppose $X(0) \neq \emptyset$, say $x \in X(0)$. For any $D \subset X(0)$ of x satisfying the spectral gap condition (cf. Definition 1), there exists a trivializing open set $U \Subset D$ and a holomorphic coordinate z on U centered at x so that on U, we have*

$$\rho(z)K_{k,s}(z,w)\chi_k(w) \in \widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n \times \mathbb{C}^n),$$

where $\rho \in C^\infty_c(U)$, $\chi_k(z) := \chi(8k^{1/2-\epsilon}z)$, $\chi \in C^\infty_c(\mathbb{C}^n)$ satisfying

$$\operatorname{supp}\chi \subset B_1(0), \quad \chi = 1 \text{ on } B_{1/2}(0), \quad \rho = 1 \text{ near } 0,$$

and $\epsilon \in (0, \frac{1}{6})$.

From (5), there exists a sequence $a_j \in \widehat{S}(\mathbb{C}^n \times \mathbb{C}^n)$ such that for any $N \in \mathbb{N}$,

$$\rho(z)K_{k,s}(z,w)\chi_k(w) - \sum_{j=0}^{N} k^{n-j/2}a_j(\sqrt{k}z, \sqrt{k}w) \in \widehat{S}^{n-\frac{N+1}{2}}(\mathbb{C}^n \times \mathbb{C}^n). \quad (6)$$

Furthermore, we can calculate the first coefficient of the expansion.

Theorem 2. *In the setting of Theorem 1, under the choice of holomorphic* *coordinates and holomorphic trivializations on* U *(cf. Fact* 3) *such that*

$$\phi(z) = \phi_0(z) + O(|z|^3), \quad \phi_0(z) := \sum_{i=1}^{n} \lambda_{i,x} |z^i|^2, \quad \lambda_{i,x} > 0,$$

$$\omega = \omega_0(z) + O(|z|), \quad \omega_0(z) := \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j,$$

the first coefficient in (6) *is given by*

$$a_0(z, w) = \frac{2^n \lambda_{1,x} \cdots \lambda_{n,x}}{\pi^n} e^{\sum_{j=1}^{n} \lambda_{j,x}(2z^j \overline{w}^j - |z^j|^2 - |w^j|^2)},$$

where $4\lambda_{1,x}, \ldots, 4\lambda_{n,x}$ *are the eigenvalues of curvature operator* $\dot{R}^L(h)$ *at* $x \in X$*.*

By (4), this means that given $N \in \mathbb{N}$, there exists $l = l(N) \in \mathbb{N}$ such that for any $M > 0$, there exists a constant $C = C(N, M) > 0$ satisfying

$$\left| \rho(z) K_{k,s}(z, 0) - \sum_{j=0}^{N} k^{n-j/2} a_j(z, 0) \right| \leq C k^{n - \frac{N+1}{2}} (1 + |z|)^{l-M}, \quad |z| < \frac{1}{2} k^{\epsilon - 1/2}.$$

If we further put $z = 0$, then we obtain a pointwise asymptotic for

$$K_{k,s}(z) \sim \sum_{j=0}^{\infty} k^{n-j/2} a_j(z)$$

in the sense that for any $N \in \mathbb{N}$,

$$\left| K_{k,s}(0, 0) - \sum_{j=0}^{N} k^{n-j/2} a_j(0, 0) \right| \leq C k^{n - \frac{N+1}{2}}.$$

This establishes the local pointwise asymptotic of Bergman kernel function on $X(0)$ with local spectral gap condition.

1.1. Application of main results

We now give a digression on spectral gap condition given in Definition 1 and demonstrate that Theorem 1 guarantees the existence of pointwise asymptotic in many situations.

First of all, it is clear that if (3) holds on D, it holds on any open subset of D. Now, we say $\Box^{(0)}_{\omega,k\phi}$ satisfies *global spectral gap condition* if (3) holds for $D = X$. When X is compact, by Hodge theorem, global spectral gap condition is equivalent to

$$\lambda_1(X, L^k) := \inf\{\lambda \in \operatorname{Spec}\Box^{(0)}_{\omega,k\phi} : \lambda \neq 0\} \geq Ck^d.$$

In other words, it is equivalent to $\operatorname{Spec}\Box^{(0)}_{\omega,k\phi} \subset \{0\} \cup [Ck^d, \infty)$. We now give some known examples for spectral gap condition.

Example 1 (cf. [32], Theorem 1.5.5)**.** Given a compact complex manifold X, a positive line bundle L with respect to a hermitian metric h, by Nakano inequality, there exists constants $C_0, C_1 > 0$ such that for any $k \in \mathbb{N}$,

$$\operatorname{Spec}\Box^{(0)}_{\omega,k\phi} \subset \{0\} \cup (C_0 k - C_1, \infty).$$

Hence, $\Box^{(0)}_{\omega,k\phi}$ satisfies global spectral gap condition of order 1.

Example 2. In [39], Siu conjectured the following "eigenvalue conjecture": if X is compact and L is quasi-positive, then

$$\inf_{k\in\mathbb{N}} \lambda_1(X, L^k) > 0. \tag{7}$$

Particularly, (7) implies that $\Box^{(0)}_{\omega,k\phi}$ satisfies global spectral gap of order N, for some $N > 0$. However, Donnelly [18] demonstrated that Siu's conjecture is false in general. Moreover, let $S \to X$ be the unit circle bundle of L, which is a CR manifold, Donnelly also showed that (7) is true if the tangential Cauchy–Riemann operator $\bar{\partial}_b$ has closed range. From this, one can deduce that if L is a positive line bundle with semi-positive metric, then (7) is true (with respect to the semi-positive metric). This particularly implies that (7) is true for any quasi-positive line bundle on compact Riemann surfaces.

Example 3. Let (L, h^L) be a semi-positive holomorphic line bundle over a compact hermitian manifold (X, ω) with $\dim_{\mathbb{C}} X = n$. If we arrange the eigenvalue of $\dot{R}^L$ at x as $0 \leq \mu_1(x) \leq \cdots \leq \mu_n(x)$, then $\mu_1(x)$ is a continuous function on X. Bouche [6] showed that if $\int_X \mu_1^{-6n} d\nu_X < \infty$, then $\lambda_1(X, L^k) \geq k^{\frac{10n+1}{12n+1}}$. Hence, Bouche condition implies that $\Box^{(0)}_{\omega,k\phi}$ satisfies global spectral gap condition of order $s = \frac{10n+1}{12n+1}$.

Now, we consider non-compact examples for spectral gap condition.

Example 4. Let (L, h^L) be a semi-positive holomorphic line bundle over a complete Kähler manifold (X, ω) with $\dim_{\mathbb{C}} = X = n$, ω is a Kähler metric which is not necessarily complete. Then Demailly's L^2-estimate [13] implies the following. If $g \in L^2_{0,1}(X, K_X \otimes L)$ satisfying $\bar{\partial} g = 0$ and $\int_X |g|^2_{R^L} d\nu_X < \infty$, where $|g|_{R^L}(x) := \inf_{g' \in \bigwedge^{n,1} T^*X \otimes L} \frac{\langle \sqrt{-1} R^L \Lambda g'.g' \rangle}{\langle g, g' \rangle^2(x)}$, then there exists $f \in L^2(X, L \otimes K_M)$ with $\bar{\partial} f = g$ and

$$\int_X |f|^2_{h^L} d\nu_X \leq \int_X |g|^2_{R^L} d\nu_X.$$

From Demailly's result, Hsiao and Marinescu in [27] proved that for any precompact open set $D \Subset X(0)$, $\Box^{(0)}_{\omega,k\phi}$ has spectral gap of order 1 on D.

Example 5. Let (X, ω) be a compact hermitian manifold. Assume $(L, h^L) \to X$ is a smooth quasi-positive line bundle. Then by the solution of Grauert–Riemenschneider conjecture (cf. [32, Chapter2]), we know that X is a Moischezon manifold and L is a big line bundle. From [32, Lemma 2.3.6], L admits a singular hermitian metric h^L_{sing} which is smooth outside an analytic set Σ and whose curvature is strictly positive current. Hsiao and Marinescu in [27, Lemma 8.1, Theorem 8.2] proved that for any open set $D \Subset X(0) \cap (X \setminus \Sigma)$, $\Box^{(0)}_{\omega,k\phi}$ for the open manifold $X \setminus \Sigma$ has spectral gap of order $N = -\sup_{x \in D} 2(\phi(x) - \phi_{\text{sing}}(x))$, where ϕ and ϕ_{sing} are local weights of h^L and h^L_{sing}, respectively.

These examples illustrates that Theorem 1 asserts the existence of pointwise asymptotic of Bergman kernel in more general situation.

2. Preliminaries

2.1. Standard notations

We denote $\mathbb{N} := \{1, 2, \dots\}$ by the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We adopt the following two multi-indices notations. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we denote $|\alpha| := \sum_{i=1}^n \alpha_i$. We adopt standard notations such as $\alpha! = \alpha_1! \cdots \alpha_n!$, x^α, and ∂_x^α. On the other hand, a n-tuple $J = (j_1, \dots, j_q) \in \{1, \dots, n\}^q$ is called a *strictly increasing multi-index of* *length* q if $1 \leq j_1 < \cdots < j_q \leq n$. For a differential q-form α, the local expression in local coordinate $x = (x^1, \ldots, x^n)$ is given by

$$\alpha = \sum\nolimits'_{|I|=q} \alpha_I dx^I,$$

where $\sum'_{|I|=q}$ means that summation is over strictly increasing multi-indices I of length q. Also, we denote dm by the standard Lebesgue measure on Euclidean spaces and $B_r(z)$ by the open ball with radius $r > 0$ and center $z \in \mathbb{C}^n$

Let X be a complex manifold. We introduce some standard notations of various function spaces. For any open subset $U \subset X$, we denote $\mathcal{O}_X(U)$ by the space of holomorphic functions on U. In case of $X = \mathbb{C}^n$, we denote $\mathcal{O}_{\mathbb{C}^n}(U)$ by $\mathcal{O}(U)$. We also denote $C^\infty(U)$ and $C_c^\infty(U)$ by the space of smooth functions and the test functions on U, respectively. If $E \to X$ is a complex vector bundle, we denote $C^\infty(U, E)$ and $C_c^\infty(U, E)$ by the space of smooth sections and its subspace whose elements having compact supports in an open subset $U \subset X$. Similarly, we denote $\mathcal{D}'(U, E)$ and $\mathcal{E}'(U, E)$ by the space of distribution sections of E over U and its subspace whose elements having compact supports. For $t \in \mathbb{R}$, we denote $W^t(U, E)$ by the Sobolev space[1] of order t of sections of E over U,

$$W^t_{\mathrm{loc}}(U, E) := \{u \in \mathcal{D}'(U, E) : \phi u \in W^t(U, E), \forall \phi \in C_c^\infty(U)\}, \text{ and}$$
$$W^t_{\mathrm{comp}}(U, E) := W^t_{\mathrm{loc}}(U, E) \cap \mathcal{E}'(U, E).$$

2.2. Backgrounds in hermitian geometry

For a complex manifold X, we have a natural almost complex structure $J : TX \to TX$ from multiplication by $\sqrt{-1}$. Hence, J induces an eigenspace decomposition $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$, where $T^{1,0}X$ is the $\sqrt{-1}$-eigenspace of J and $T^{0,1}X$ is the $-\sqrt{-1}$-eigenspace of J. Also, J induces an almost complex structure on T^*X. Hence, we also have the eigenspace decomposition for complexified cotangent bundle $T^*X \otimes_{\mathbb{R}} \mathbb{C} =$

[1] The usual notation for L^2-Sobolev space is H^s. However, to avoid the confusion with cohomology group, we denote it by W^s.

$\bigwedge^{1,0} X \oplus \bigwedge^{0,1} X$. Moreover, this extends to exterior algebra of complexified cotangent bundle:

$$\bigwedge^{r} T^*X \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=r} \bigwedge^{p,q} X,$$

where $\bigwedge^{p,q} X$ is locally spanned by $dz^I \wedge d\bar{z}^J$, for any strictly increasing multi-indices $I \in \{1, \dots, n\}^p$, $J \in \{1, \dots, n\}^q$. We denote $A^r(U) = C^\infty(U, \bigwedge^r T^*X)$ and $A^{p,q}(U) = C^\infty(U, \bigwedge^{p,q} X)$ by the space of smooth r-forms and smooth (p,q)-forms on U, respectively.

Recall that a *hermitian form* on a complex manifold X is a smooth $(1,1)$-form $\omega \in A^{1,1}(X)$ such that in a local holomorphic coordinate $\mathbf{z} = (z^1, \dots, z^n)$ on a chart U of X,

$$\omega\big|_U = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} H_{ij} dz^i \wedge d\bar{z}^j, \tag{8}$$

where $H(x) = (H_{ij}(x))_{i,j=1}^n$ is a positive-definite hermitian matrix for any $x \in U$. It is well-known that a hermitian form ω is equivalent to a Riemannian metric g on the underlying real manifold X which the complex structure is an isometry. We then extend g to a hermitian metric on $TX \otimes_{\mathbb{R}} \mathbb{C}$, still denoted by g,

$$g(v \otimes \lambda, w \otimes \mu) := \frac{1}{2} \lambda \overline{\mu} g(v, w),$$

where $v, w \in T_x X$ and $\lambda, \mu \in \mathbb{C}$. Thus, we can define a pointwise hermitian inner product $\langle \cdot | \cdot \rangle_\omega$ on $A^{p,q}(X)$ induced from ω.

Now, let $\alpha, \beta \in A^{p,q}(X, L^k) := C^\infty(X, \bigwedge^{p,q} X \otimes L^k)$ be two L^k-valued (p,q)-forms. Under a choice of trivialization $s : U \to L$ of L, we can write $\alpha = f \otimes s^k$, $\beta = g \otimes s^k$. We define

$$\langle \alpha | \beta \rangle_{\omega, \phi} := \langle f | g \rangle_\omega e^{-2\phi}, \tag{9}$$

where ϕ is the local weight of h associated to s. We then define a L^2-hermitian inner product $(\cdot|\cdot)$ on $A_c^{p,q}(X, L^k)$, the space of compact supported (p,q)-forms valued in L^k, by

$$(\alpha | \beta)_{\omega, k\phi} := \int_X \langle \alpha | \beta \rangle_{\omega, k\phi} d\nu_X. \tag{10}$$

We write $\|\alpha\|^2_{\omega,k\phi} := (\alpha|\alpha)_{\omega,k\phi}$ and denote $L^2_{p,q}(X, L^k)$ by the completion of $A^{p,q}_c(X, L^k)$ with respect to the norm $\|\cdot\|_{\omega,k\phi}$.

Notation. We sometimes denote $L^2_{p,q}(X, L^k)$ by $L^2_{\omega,k\phi}(\bigwedge^{p,q} X \otimes L^k)$ if we wish to stress the choice of ω and $k\phi$.

Given a holomorphic vector bundle E over a complex manifold X, let $\bar{\partial}^E : A^{p,q}(X, E) \to A^{p,q+1}(X, E)$ be the *Cauchy–Riemann operator.* We always choose the Chern connection ∇^E on E which is compatible with a given hermitian metric h^E on E. For a holomorphic line bundle $L \to X$ with a hermitian metric h on it, if (s, U) is holomorphic trivialization of L over U and ϕ is the local weight of h determined by s. In this case, the curvature form $R^L(h)$ of the Chern connection $\nabla := \nabla^L$ is locally given by

$$R^L(h) = -\partial\bar{\partial}\log e^{-2\phi} = 2\partial\bar{\partial}\phi = 2\sum_{j,l=1}^{n} \frac{\partial^2\phi}{\partial z^j \partial \bar{z}^l} dz^j \wedge d\bar{z}^l. \tag{11}$$

In particular, $\sqrt{-1}R^L(h)$ is a closed, real $(1,1)$-form on X. We define the *curvature operator* $\dot{R}^L \in \mathrm{End}(\bigwedge^{1,0} X)$ as in (2).

Under simple change of coordinates and trivialized sections, one can always make the local weight and hermitian form in a normal form.

Fact 3 (cf. [41], Lemma III,2.3)**.** *Let X be a complex manifold with hermitian form ω, L be a holomorphic line bundle on X with a hermitian metric h. Fix a point $x \in X$, we can choose a local complex coordinate $(z^1, \dots, z^n)$ on an open neighborhood $U \subset X$ of x and a holomorphic trivializing section $s \in H^0(U, L)$ such that*

(i) $z^i(x) = 0$, *for* $i = 1, \dots, n$,

(ii) $\omega(z) = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^n H_{ij}(z)dz^i \wedge d\bar{z}_j$ *with* $H_{ij}(0) = \delta_{ij}$, *and*

(iii) $|s(z)|^2_h = e^{-2\phi(z)}$ *with local weight*

$$\phi(z) = \sum_{i=1}^{n} \lambda_{i,x}|z^i|^2 + O(|z|^3),$$

where $4\lambda_{1,x}, \dots, 4\lambda_{n,x}$ *are eigenvalues of* $\dot{R}^L(x)$.

We usually denote $\phi_0(z) = \sum_{i=1}^n \lambda_{i,x}|z^i|^2$.

We denote $\bar\partial^* := \bar\partial^{L^k,*,\omega,k\phi} : A^{0,q}(X,L^k) \to A^{0,q-1}(X,L^k)$ by the formal adjoint of $\bar\partial$ with respect to $(\cdot|\cdot)_{\omega,k\phi}$ which is characterized by

$$(\bar\partial\alpha|\beta)_{\omega,k\phi} = (\alpha|\bar\partial^*\beta)_{\omega,k\phi}, \quad \alpha \in A_c^{0,q}(X,L^k), \beta \in A^{0,q+1}(X,L^k). \tag{12}$$

Kodaira Laplacian for (L,h) is defined by

$$\Box^{(q)}_{\omega,k\phi} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : A^{0,q}(X,L^k) \to A^{0,q}(X,L^k). \tag{13}$$

3. Asymptotic Expansion of Bergman Kernel

3.1. Localized set-up

As stated in the introduction, it is convenient to work in an equivalent set-up for which the norms are defined by integrals without depending on k.

First of all, let η be a $(0,1)$-form. We denote $\epsilon(\eta) := \eta \wedge \cdot : \bigwedge^{0,q} T_x^*X \to \bigwedge^{0,q+1} T_x^*X$ be the wedging η from the left and $\iota(\eta)$ be its adjoint with respect to $\langle\cdot|\cdot\rangle_\omega$. Hence, for $\eta_1, \eta_2 \in A^{0,1}(X)$, $\epsilon(\eta_1)\iota(\eta_2) + \iota(\eta_2)\epsilon(\eta_1) = \langle\eta_1|\eta_2\rangle_\omega id$. Let $e^1(z), \dots, e^n(z)$ be an orthonormal frame for $\bigwedge^{0,1} X$ over U, $Z_1, \dots, Z_n$ be its dual basis for $T^{0,1}X$. We can write Cauchy–Riemann operator $\bar\partial$ on $A^{0,q}(U,L^k)$ as

$$\bar\partial(s^k \otimes \alpha) = s^k \otimes \sum_{j=1}^n \left(\epsilon(e^j)Z_j + \epsilon(\bar\partial e^j)\iota(e^j)\right)\alpha. \tag{14}$$

Its formal adjoint with respect to the scalar product $(\cdot|\cdot)_{\omega,k\phi}$ is given by

$$\bar\partial^*(s^k \otimes \alpha) = s^k \otimes \sum_{j=1}^n \left(\iota(e^j)(Z_j^* + 2k\overline{Z_j}(\phi)) + \epsilon(e_j)\iota(\bar\partial e^j)\right)\alpha, \tag{15}$$

where Z_j^* is the formal adjoint of Z_j with respect to the inner product $(\alpha|\beta)_\omega := \int_X \langle\alpha|\beta\rangle_\omega \omega_n$ on $A_c^{0,q}(X)$. To put $\bar\partial$ and $\bar\partial^*$ in more symmetric form, we make the following identification.

$$\begin{aligned} &A^{0,q}(U,L^k) \to A^{0,q}(U), \quad u = s^k \otimes \alpha \mapsto \widetilde{\alpha} := \alpha e^{-k\phi} \\ &A^{0,q}(U) \to A^{0,q}(U,L^k), \quad \beta \mapsto s^k \otimes e^{k\phi}\beta. \end{aligned} \tag{16}$$

This is a local unitary identification since for $u, v \in L^2_{0,q}(X, L^k) \cap \mathcal{E}'(X, \bigwedge^{0,q} X \otimes L^k)$,

$$(u|v)_{\omega,k\phi} = \int_U \langle \alpha|\beta\rangle_\omega e^{-2k\phi}\omega_n = \int_U \langle e^{-k\phi}\alpha|e^{-k\phi}\beta\rangle_\omega = (\widetilde{\alpha}|\widetilde{\beta})_\omega,$$

where $u = s^k \otimes \alpha, v = s^k \otimes \beta$. Then under this unitary identification, we get

$$\bar{\partial}(s^k \otimes e^{k\phi}\alpha) = s^k \otimes e^{k\phi}\bar{\partial}_{k,s}\alpha, \tag{17}$$

where $\alpha \in A^{0,q}(U)$ and

$$\bar{\partial}_{k,s} = \sum_{j=1}^{n} \big(\epsilon(e^j) \otimes (Z_j + kZ_j(\phi)) + \epsilon(\bar{\partial}e^j)\iota(e^j)\big) = \bar{\partial} + k\epsilon(\bar{\partial}\phi). \tag{18}$$

The formal adjoint $\bar{\partial}^*_{k,s}$ with respect to the local scalar product $(\cdot|\cdot)_\omega$ is given by

$$\bar{\partial}^*_{k,s} = \sum_{j=1}^{n} \Big(\iota(e^j) \otimes (Z^*_j + k\overline{Z_j(\phi)}) + \epsilon(e^j)\iota(\bar{\partial}e^j)\Big) = \bar{\partial}^* + k\iota(\bar{\partial}\phi) \tag{19}$$

and satisfies

$$\bar{\partial}^*(s^k \otimes e^{k\phi}\beta) = s^k \otimes e^{k\phi}\bar{\partial}^*_{k,s}\beta, \quad \beta \in A^{0,q+1}(U). \tag{20}$$

We call $\bar{\partial}_{k,s}$ the *localized Cauchy-Riemann operator* with respect to s. The *localized Kodaira Laplacian* with respect to s is then defined by

$$\Box^{(q)}_{k,s} := \bar{\partial}^*_{k,s}\bar{\partial}_{k,s} + \bar{\partial}_{k,s}\bar{\partial}^*_{k,s}. \tag{21}$$

Of course, from (17), (20), we have

$$\Box^{(q)}_{\omega,k\phi}(s^k \otimes e^{k\phi}\alpha) = s^k \otimes e^{k\phi}\Box^{(q)}_{k,s}\alpha, \quad \alpha \in A^{0,q}(U). \tag{22}$$

The *localized Bergman projection* $\Pi^{(q)}_{k,s} : L^2_{0,q}(U, \omega) \cap \mathcal{E}'(U, \bigwedge^{0,q} X) \to A^{0,q}(U)$ is defined by

$$\Pi^{(q)}_{k,s}\alpha = e^{-k\phi}s^{-k}\Pi^{(q)}_k(e^{k\phi}\alpha \otimes s^k). \tag{23}$$

Thus, we see that $\Pi^{(q)}_{k,s} : L^2_{0,q}(U, \omega) \cap \mathcal{E}'(U, \bigwedge^{0,q} X) \to \ker \Box^{(q)}_{k,s}$. Let $K^{(q)}_{k,s}$ be the Schwartz kernel of $\Pi^{(q)}_{k,s}$, i.e.,

$$(\Pi^{(q)}_{k,s}\alpha)(z) = \int_U K^{(q)}_{k,s}(z,w)(\alpha(w))\omega_n(w). \tag{24}$$

We now give a local expression for localized Kodaira Laplacian.

Proposition 4. *The localized Kodaira Laplacian* $\Box^{(q)}_{k,s}$ *satisfies*

$$\begin{aligned}\Box^{(q)}_{k,s} &= \bar\partial^*_{k,s}\bar\partial_{k,s} + \bar\partial_{k,s}\bar\partial^*_{k,s}\\ &= \sum_{j=1}^n 1\otimes (Z_j^* + k\overline{Z_j(\phi)})(Z_j + kZ_j(\phi))\\ &\quad + \sum_{j,l=1}^n \epsilon(e^j)\iota(e^l)\otimes\left[Z_j + kZ_j(\phi), Z_l^* + k\overline{Z_l(\phi)}\right]\\ &\quad + O(1)(Z+kZ(\phi)) + O(1)(Z^* + k\overline{Z(\phi)}) + O(1),\end{aligned} \tag{25}$$

where $Z + kZ(\phi)$ *indicates a remainder term of the form* $\sum_{j=1}^n a_j(z)(Z_j + kZ_j(\phi))$ *and* $a_j(z)$ *are some* k*-independent smooth functions, and similarly for* $Z^* + \overline{Z(\phi)}$*. Also,* $O(1)$ *indicates some zero order differential operators which are independent of* k*.*

Proof. By direct computation,

$$\begin{aligned}\Box^{(q)}_{k,s} &= \bar\partial^*_{k,s}\bar\partial_{k,s} + \bar\partial_{k,s}\bar\partial^*_{k,s}\\ &= \sum_{j,l=1}^n (\epsilon(e^j)\otimes(Z_j + kZ_j(\phi))(\iota(e^l)\otimes(Z_l^* + k\overline{Z_l(\phi)})))\\ &\quad + (\iota(e^l)\otimes(Z_l + kZ_l(\phi)))(\epsilon(e^j)\otimes(Z_j + kZ_j(\phi)))\\ &\quad + \epsilon(\bar\partial e^j)\iota(e^j)\iota(e^l)\otimes(Z_l^* + k\overline{Z_l(\phi)})\\ &\quad + \iota(e^l)\epsilon(\bar\partial e^j)\iota(e^j)\otimes(Z_j + k\overline{Z_j(\phi)})\\ &\quad + \epsilon(\bar\partial e^j)\iota(e^j)\epsilon(e^l)\iota(\bar\partial e^l) + \epsilon(e^l)\iota(\bar\partial e^l)\epsilon(\bar\partial e^j)\iota(e^j).\end{aligned} \tag{26}$$

Now, we combine the first two terms as

$$\begin{aligned}&(\epsilon(e^j)\iota(e^l) + \iota(e^l)\epsilon(e^j))((Z_j + kZ_j(\phi))(Z_l^* + k\overline{Z_l(\phi)}))\\ &+\epsilon(e^j)\iota(e^l)[Z_j + kZ_j(\phi), Z_l^* + k\overline{Z_l(\phi)}]\\ &=\langle e^j|e^l\rangle((Z_j + kZ_j(\phi))(Z_l^* + k\overline{Z_l(\phi)}) + \epsilon(e^j)\iota(e^l)[Z_j + kZ_j(\phi), Z_l^* + k\overline{Z_l(\phi)}]\end{aligned}$$

$$=\delta_{jl}(Z_j + kZ_j(\phi))(Z_l^* + k\overline{Z_l(\phi)}) + \epsilon(e^j)\iota(e^l)[Z_j + kZ_j(\phi), Z_l^* + k\overline{Z_l(\phi)}]. \qquad \square$$

We define the concept of k-negligible kernels or k-negligible operators.

Definition 3. A k-dependent continuous linear operator $\mathcal{A}_k : L^2_{0,q}(X, L^k) \to L^2_{0,q}(X, L^k)$ is *k-negligible* if it is smoothing for sufficiently large k and for any $\alpha, \beta \in \mathbb{N}_0^{2n}$, any $N \in \mathbb{N}$, there exists a k-independent constant $C_{\alpha,\beta,N,L} > 0$ such that the smooth kernel $A_k(x, y)$ of $\mathcal{A}_k$ satisfying

$$\left|\partial_x^\alpha \partial_x^\beta A_{k,s,t}(x, y)\right| \le C_{\alpha,\beta,N,L} k^{-N}, \text{for } k \gg 1, \tag{27}$$

locally uniformly on any compact subset $L \subset U \times V$, where s, t are local holomorphic trivialization of L over U, V, respectively, and $A_k(x, y) = A_{k,s,t}(x, y) s^k(x) \otimes (t^k)^*(y)$. Here, $(t^k)^*$ is the metric dual of t^k. If so, we denote $\mathcal{A}_k \equiv 0 \bmod O(k^{-\infty})$ or $A_k \equiv 0 \bmod O(k^{-\infty})$.

Notice that the condition of k-negligible is independent of the choice of local trivializations s, t and local coordinates x, y. Also, by Sobolev embedding, $\mathcal{A}_k$ is k-negligible if and only if $\mathcal{A}_k$ extends to an operator from $W^r_{\mathrm{comp}}(U, \bigwedge^{0,q} X \otimes L^k)$ to $W^{r+M}_{\mathrm{loc}}(V, \bigwedge^{0,q} X \otimes L^k)$ with operator norm $O(k^{-N})$, for any $r \in \mathbb{R}$, $M, N \in \mathbb{N}$.

3.2. Approximate Bergman kernel and semi-classical L^2-estimates

Let $x \in X(0)$, let $U \subset X(0)$ be a local trivialization open set and $x \in U$. We choose a local coordinate (U, z) centered at x and a local holomorphic trivialization s of L on U so that Fact 3 holds. That is,

$$\begin{aligned} &\phi(z) = \phi_0(z) + O(|z|^3), \quad \phi_0(z) = \sum_{i=1}^n \lambda_{i,x} |z^i|^2, \quad \lambda_{i.x} > 0, \\ &\omega(z) = \omega_0(z) + O(|z|), \quad \omega_0(z) = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i, \quad z \in U. \end{aligned} \tag{28}$$

Identifying U as a bounded domain in $\mathbb{C}^n$, we extend ϕ and ω by $\widehat{\phi}$ and $\widehat{\omega}$ to whole $\mathbb{C}^n$ by

$$\widehat{\phi} = \phi_0 + \underbrace{\theta_k(\phi - \phi_0)}_{\phi_1}, \quad \widehat{\omega} = \omega_0 + \underbrace{\theta_k(\omega - \omega_0)}_{\omega_1}, \text{ and } \quad \theta_k(z) = \theta(k^{1/2-\epsilon} z), \tag{29}$$

where $\theta \in C_c^\infty(\mathbb{C}^n)$ is a cut-off function such that $\theta = 1$ on $B_{1/2}(0)$ and $\operatorname{supp}\theta \subset B_1(0)$. Thus, $\phi_1 \in C_c^\infty(U, \mathbb{R})$, for any small $\epsilon > 0$. Since $U \subset X(0)$, we know that ϕ is strictly plruisubharmonic, i.e., there exists $C > 0$ such that $\frac{\partial^2\phi}{\partial z^j \partial \bar{z}^l}(z)\xi^j\overline{\xi}^l \geq C|\xi|^2$, for any $\xi \in \mathbb{C}^n \setminus \{0\}$, any $z \in U$. Thus, for sufficiently large k, $\widehat{\phi}$ is strictly plurisubharmonic on $\mathbb{C}^n$. Let $\widehat{\omega}_n := \widehat{\omega}^n/n! = \lambda(z)dm(z)$. Then $\lambda(z) = 1$ outside $B_{k^{\epsilon-1/2}}(0)$. In other words,

$$\begin{aligned}
&\widehat{\omega} = \omega, \quad \widehat{\phi} = \phi \text{ on } V_k := B_{\frac{1}{2}k^{\epsilon-1/2}}(0) \\
&\widehat{\omega} = \omega_0, \quad \widehat{\phi} = \phi_0 \text{ on } |z| > k^{\epsilon-1/2}.
\end{aligned}$$

We then consider L^2-space $L^2_{0,q}(\mathbb{C}^n, \widehat{\omega})$ which is the completion of $A_c^{0,q}(\mathbb{C}^n)$ with respect to the L^2-norm given by

$$(f|g)_{\widehat{\omega}} := \int_{\mathbb{C}^n} \langle f|g\rangle_{\widehat{\omega}}(z)\lambda(z)dm(z), \quad f, g \in A_c^{0,q}(\mathbb{C}^n).$$

Notation. In the remaining of this subsection, unless otherwise stated, we will denote $(\cdot|\cdot)_{\widehat{\omega}}$ simply by $(\cdot|\cdot)$ for the sake of brevity.

We now define the *deformed Cauchy–Riemann operator* $\bar{\partial}_{k\widehat{\phi}} := \bar{\partial} + k\epsilon(\bar{\partial}\widehat{\phi}) : A^{0,q}(\mathbb{C}^n) \to A^{0,q+1}(\mathbb{C}^n)$ and its formal adjoint $\bar{\partial}^*_{k\widehat{\phi}} = \bar{\partial}^* + k\iota(\bar{\partial}\widehat{\phi})$ with respect to $(\cdot|\cdot)_{\widehat{\omega}}$. Hence, as before, the **deformed Kodaira Laplacian** is then defined by

$$\triangle^{(q)}_{\widehat{\omega},\widehat{\phi}} = \bar{\partial}_{k\widehat{\phi}}\bar{\partial}^*_{k\widehat{\phi}} + \bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}. \tag{30}$$

Note that the analogous formula (18) and (19) still hold:

$$\begin{aligned}
\bar{\partial}_{k\widehat{\phi}} &= \sum_{j=1}^n \Big(\epsilon(e^j) \otimes (Z_j + kZ_j(\widehat{\phi})) + \epsilon(\bar{\partial}e^j)\iota(e^j)\Big) \\
\bar{\partial}^*_{k\widehat{\phi}} &= \sum_{j=1}^n \Big(\iota(e^j) \otimes (Z_j^* + k\overline{Z_j(\widehat{\phi})}) + \epsilon(e^j)\iota(\bar{\partial}e^j)\Big)
\end{aligned}. \tag{31}$$

Also, $\bar{\partial}_{\widehat{\omega},k\widehat{\phi}} = \bar{\partial}_{k,s}$, $\bar{\partial}^*_{\widehat{\omega},k\widehat{\phi}} = \bar{\partial}^*_{k,s}$, and $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}} = \Box^{(q)}_{k,s}$ on V_k. For $z \in \mathbb{C}^n \setminus U$, $\widehat{\omega} = \omega_0$, and thus $b^i_j(z) = \delta^i_j = c^i_j(z)$.

From Proposition 4, we see that for $q \geq 1$, $\alpha \in A_c^{0,q}(\mathbb{C}^n)$,

$$\begin{aligned}\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha = &\sum_{j=1}^{n} 1 \otimes (Z_j^* + k\overline{Z_j(\widehat{\phi})})(Z_j + kZ_j(\widehat{\phi}))\alpha \\ &+ k\sum_{j,l=1}^{n} \epsilon(e^j)\iota(e^l) \otimes (Z_j\overline{Z_l} - Z_l^* Z_j)(\widehat{\phi})\alpha \\ &+ O(1)(Z + kZ(\widehat{\phi}))\alpha + O(1)(Z^* + \overline{kZ(\widehat{\phi})})\alpha + O(1)\alpha.\end{aligned}$$

Therefore, we get

$$\begin{aligned}(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha|\alpha) = &\left(\sum_{j=1}^{n} 1 \otimes (Z_j^* + k\overline{Z_j(\widehat{\phi})})(Z_j + kZ_j(\widehat{\phi}))\alpha \middle| \alpha\right) \\ &+ k\left(\sum_{j,l=1}^{n} \epsilon(e^j)\iota(e^l) \otimes (Z_j\overline{Z_l} - Z_l^* Z_j)(\widehat{\phi})\alpha \middle| \alpha\right) \\ &+ \left(O(1)(Z + kZ(\widehat{\phi}))\alpha + O(1)\alpha \middle| \alpha\right) + \left(\alpha|O(1)(Z+kZ(\widehat{\phi}))\alpha\right) \\ \geq &\|(Z+kZ(\widehat{\phi}))\alpha\|^2 + k\left(\sum_{j,l=1}^{n} \epsilon(e^j)\iota(e^l) \otimes (Z_j\overline{Z_l} - Z_l^* Z_j)(\widehat{\phi})\alpha \middle| \alpha\right) \\ &- \left|\left(O(1)(Z + kZ(\widehat{\phi}))\alpha|\alpha\right)\right| - \left|\left(\alpha|O(1)(Z + k(\widehat{\phi}))\alpha\right)\right| - \left|(O(1)\alpha|\alpha)\right|,\end{aligned}$$

where $Z + kZ(\widehat{\phi}) = \sum_{j=1}^{n} Z_j + kZ_j(\widehat{\phi})$. By Cauchy–Schwartz inequality,

$$\begin{aligned}&\left|\left(O(1)(Z + kZ(\widehat{\phi}))\alpha|\alpha\right)\right| \leq \frac{1}{2}\left(\epsilon\left\|O(1)(Z + kZ(\widehat{\phi}))\alpha\right\|^2 + \frac{1}{\epsilon}\|\alpha\|^2\right), \\ &\left|(O(1)\alpha|\alpha)\right| \leq \frac{1}{2}\left(\|O(1)\alpha\|^2 + \|\alpha\|^2\right).\end{aligned}$$

By the construction, these $O(1)$ terms are supported in a compact set and uniformly bounded in k. Hence, there exists a constant $C' > 0$ independent of k so that

$$\|O(1)\alpha\|^2 \leq C'\|\alpha\|^2, \quad \|O(1)(Z + kZ(\widehat{\phi}))\alpha\|^2 \leq C'\|(Z + kZ(\widehat{\phi}))\alpha\|^2$$

and we choose $\epsilon > 0$ so that $\epsilon C' < 1$. Also, since $\widehat{\phi}$ is strictly plurisubharmonic, there exists a k-independent constant $C_0 > 0$ so that

$$\left(\sum_{j,l=1}^{n} \epsilon(e^j)\iota(e^l) \otimes (Z_j\overline{Z_l} - Z_l^* Z_j)(\widehat{\phi})\alpha \middle| \alpha\right) \geq C_0\|\alpha\|^2.$$

Combining these estimates, we obtain

$$\begin{aligned}(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha|\alpha) \geq &\left(1 - \frac{\epsilon}{C'}\right)\|(Z + kZ(\widehat{\phi}))\alpha\|^2 \\ &+ \left(C_0 k - \frac{1}{\epsilon} - \frac{C'}{2} - 1\right)\|\alpha\|^2, \quad \forall \alpha \in A_c^{0,q}(\mathbb{C}^n).\end{aligned}$$

Therefore, for sufficiently large k, there exists a constant C independent of k so that

$$(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha|\alpha) \geq Ck\|\alpha\|^2, \quad \forall \alpha \in A_c^{0,q}(\mathbb{C}^n), \quad q \geq 1. \tag{32}$$

We next consider Gaffney extension of $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ and show that (32) holds for any $\alpha \in \operatorname{Dom}\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$, for $q \geq 1$. Gaffney extension for deformed Kodaira Laplacian on $(0,q)$-forms is given by

$$\begin{aligned}\operatorname{Dom}\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}} &:= \{\alpha \in \operatorname{Dom}(\widehat{S}_q)\cap\operatorname{Dom}(\widehat{S}^\dagger_{q-1}) : S_q\alpha \in \operatorname{Dom}(\widehat{S}^\dagger_q), \widehat{S}^\dagger_{q-1}\alpha \in \operatorname{Dom}\widehat{S}_{q-1}\} \\ \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha &:= \widehat{S}^\dagger_q\widehat{S}_q\alpha + \widehat{S}_{q-1}\widehat{S}^\dagger_{q-1}\alpha, \quad \forall \alpha \in \operatorname{Dom}\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}.\end{aligned}$$

where $\widehat{S}_q$ is the maximal extension (cf. [32, Lemma 3.1.1.]) of $\bar{\partial}_{k\widehat{\phi}} : A^{0,q}(\mathbb{C}^n) \to A^{0,q+1}(\mathbb{C}^n)$ and $\widehat{S}^\dagger_{q-1}$ is the adjoint of $\widehat{S}_{q-1}$. It is known that Gaffney extension $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ is a self-adjoint, non-negative operator (cf. [32, Proposition 3.1.2]). Moreover, since $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ has the same principal symbol as usual Laplacian on $\mathbb{C}^n$, we know that it is elliptic. By elliptic regularity, we know that the L^2-projection $\mathcal{P}^{(q)}_{\widehat{\omega},k\widehat{\phi}} : L^2_{0,q}(\mathbb{C}^n, \widehat{\omega}_n) \to \ker\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ is a smoothing operator and thus has smoothing Schwartz kernel $P^{(q)}_{\widehat{\omega},k\widehat{\phi}}$. We call $P^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ the *approximate Bergman kernel for* $(0,q)$*-forms.* Now, we show

Lemma 5 (Approximation Lemma)**.** *Let* $\alpha \in \operatorname{Dom}\widehat{S}_q \cap \operatorname{Dom}\widehat{S}^\dagger_{q-1} \subset L^2_{0,q}(\mathbb{C}^n, \widehat{\omega}_n)$*. Then there exists a sequence* $\{\alpha_j\}_{j=1}^\infty \subset A_c^{0,q}(\mathbb{C}^n)$ *such that*

$$\|\alpha_j - \alpha\|, \quad \|\bar{\partial}_{k\widehat{\phi}}\alpha_j - \widehat{S}_q\alpha\|, \quad \|\bar{\partial}^*_{k\widehat{\phi}}\alpha_j - \widehat{S}^\dagger_{q-1}\alpha\| \to 0, \quad \textit{as } j \to \infty.$$

Proof. First of all, we show that if $\chi \in C_c^\infty(\mathbb{C}^n, \mathbb{R})$ is a test function, then for any $\alpha \in \mathrm{Dom}(\widehat{S}_q) \cap \mathrm{Dom}(\widehat{S}^\dagger_{q-1})$, $\chi\alpha \in \mathrm{Dom}(\widehat{S}_q) \cap \mathrm{Dom}(\widehat{S}^\dagger_{q-1})$. For $\alpha \in \mathrm{Dom}(\widehat{S}_q)$, we first claim that the Leibniz rule $\bar{\partial}_{k\widehat{\phi}}(\chi\alpha) = \chi\bar{\partial}_{k\widehat{\phi}}\alpha + \bar{\partial}_{k\widehat{\phi}}\chi \wedge \alpha$ holds in the distribution sense. To see this, observe that for any $\beta \in A_c^{0,q+1}(\mathbb{C}^n)$, $\gamma \in A_c^{0,q}(\mathbb{C}^n)$,

$$\begin{aligned}(\gamma|\bar{\partial}^*_{k\widehat{\phi}}(\chi\beta)) =& (\bar{\partial}_{k\widehat{\phi}}\gamma|\chi\beta) = (\chi\bar{\partial}_{k\widehat{\phi}}\gamma|\beta)\\ =& (\bar{\partial}_{k\widehat{\phi}}(\chi\gamma) - \bar{\partial}_{k\widehat{\phi}}\chi \wedge \gamma|\beta) = (\gamma|\chi\bar{\partial}^*_{k\widehat{\phi}}\beta) - (\gamma|\iota_{\bar{\partial}_{k\widehat{\phi}}\chi}\beta),\end{aligned}$$

where $\iota(\bar{\partial}_{k\widehat{\phi}}\chi)$ is the adjoint of $\epsilon(\bar{\partial}_{k\widehat{\phi}}\chi)$ with respect to $(\cdot|\cdot)$. Hence, $\bar{\partial}^*_{k\widehat{\phi}}(\chi\beta) = \chi\bar{\partial}^*_{k\widehat{\phi}}\beta - \iota_{\bar{\partial}_{k\widehat{\phi}}\chi}\beta$. From this, we deduce that

$$\begin{aligned}\bar{\partial}_{k\widehat{\phi}}(\chi\alpha)(\beta) :=& \chi\alpha(\bar{\partial}^*_{k\widehat{\phi}}\beta) = \alpha(\chi\bar{\partial}^*_{k\widehat{\phi}}\beta)\\ =& \alpha(\bar{\partial}^*_{k\widehat{\phi}}(\chi\beta)) + \alpha(\iota_{\bar{\partial}_{k\widehat{\phi}}\chi}\beta) = \chi\bar{\partial}_{k\widehat{\phi}}\alpha(\beta) + \bar{\partial}_{k\widehat{\phi}}\chi \wedge \alpha(\beta).\end{aligned}$$

Hence, $\bar{\partial}_{k\widehat{\phi}}(\chi\alpha) = \chi\bar{\partial}_{k\widehat{\phi}}\alpha + \bar{\partial}_{k\widehat{\phi}}\chi \wedge \alpha$ holds indeed in the distribution sense. Therefore, since $\alpha \in \mathrm{Dom}(\widehat{S}_q)$ and $\chi \in C_c^\infty(\mathbb{C}^n)$, $\bar{\partial}_{k\widehat{\phi}}\alpha \in L^2_{0,q}(\mathbb{C}^n)$ and $\bar{\partial}_{k\widehat{\phi}}\chi = \bar{\partial}\chi + k\chi\bar{\partial}\phi \in A_c^{0,q}(\mathbb{C}^n)$, we see that $\bar{\partial}_{k\widehat{\phi}}(\chi\beta) \in L^2_{0,q+1}(\mathbb{C}^n)$. Next, for $\alpha \in \mathrm{Dom}(\widehat{S}^\dagger_{q-1})$, we need to show that there exists constant $C > 0$ such that

$$\left|(\bar{\partial}_{k\widehat{\phi}}u|\chi\alpha)\right| \leq C\|u\|, \quad \forall u \in \mathrm{Dom}(\widehat{S}_{q-1}).$$

However, $\left|(\bar{\partial}_{k\widehat{\phi}}u|\chi\alpha)\right| \leq \sup_{z\in\mathbb{C}^n}|\chi(z)|\left|(\bar{\partial}_{k\widehat{\phi}}u|\alpha)\right| \leq C'\|u\|$. This shows that $\chi\alpha \in \mathrm{Dom}(\widehat{S}^\dagger_{q-1})$.

Now, we can choose χ to belong to some partition of unity with compact supports and decompose $\alpha = \sum_{j=1}^n \chi_j\alpha$. It suffices to approximate each $\chi_j\alpha$ and thus we may assume that α supports in some compact set K. Then we apply the standard regularization technique by convoluting the coefficients of α by the mollifiers $\rho_j(z) := j^{2n}\rho(jz)$, where ρ is the standard modifier on $\mathbb{C}^n$. The result then follows from the classical Lemma of Friedrichs (cf. [14] Chapter VII, Lemma 3.3). $\square$

From above approximation Lemma and (32), we deduce that for $q \geq 1$,

$$\|\widehat{S}_q\alpha\|^2 + \|\widehat{S}^\dagger_{q-1}\alpha\|^2 \geq Ck\|\alpha\|^2, \quad \forall\alpha \in \mathrm{Dom}\,\widehat{S}_q \cap \mathrm{Dom}\,\widehat{S}^\dagger_{q-1}.$$

For $\alpha \in \operatorname{Dom} \triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}$, we also have $(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha|\alpha) = (\widehat{S}^{\dagger}_q \widehat{S}_q\alpha|\alpha)+(\widehat{S}_{q-1}\widehat{S}^{\dagger}_{q-1}\alpha|\alpha) = \|\widehat{S}_q\alpha\|^2 + \|\widehat{S}^{\dagger}_{q-1}\alpha\|^2$. Thus, this implies that (32) extends to all elements in $\operatorname{Dom} \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$, i.e.,

$$\left(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha\middle|\alpha\right) \geq Ck\|\alpha\|^2, \quad \forall \alpha \in \operatorname{Dom} \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}, \quad q \geq 1. \tag{33}$$

From this, we can prove

Corollary 6. *The deformed Kodaira Laplacian* $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}} : \operatorname{Dom} \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}} \to L^2_{0,q}(\mathbb{C}^n, \widehat{\omega})$ *is bijective and has a bounded inverse.*

Proof. First, it is clear from (33) that $\ker \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}} = 0$. For surjectivity, given any $\beta \in L^2_{0,q}(\mathbb{C}^n, \widehat{\omega})$, we consider the a linear functional on $\operatorname{im}(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}})$ given by $\ell_\beta(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha) = (\alpha|\beta)$, for any $\alpha \in \operatorname{Dom} \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$.

Injectivity of $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ implies that ℓ_β is well-defined. Also, (33) implies $\|\ell\| \leq \frac{\|\beta\|}{Ck}$. By Hahn–Banach Theorem, ℓ_β extends to a bounded linear functional on $\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$ with the same norm. By Riesz representation Theorem, there exists $\gamma \in L^2_{0,1}(\mathbb{C}^n, \widehat{\omega})$ with $\|\gamma\| \leq \frac{\|\beta\|}{Ck}$ such that $(\alpha|\beta) = \ell_\beta(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\alpha) = (\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}\alpha|\gamma)$, for any $\alpha \in \operatorname{Dom} \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}$. In other words, $\gamma \in \operatorname{Dom}(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}})^\dagger = \operatorname{Dom} \triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}$ and $\beta = \triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}}\gamma$. This shows the first assertion. The second assertion follows from (33) that $\|(\triangle^{(q)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\| \leq \frac{1}{Ck}$. □

Now, we turn to the L^2-existence Theorem for $\bar{\partial}_{k\widehat{\phi}}$ on $\mathbb{C}^n$.

Theorem 7. *If* $\alpha \in L^2_{0,1}(\mathbb{C}^n, \widehat{\omega})$ *with* $\bar{\partial}_{k\widehat{\phi}}\alpha = 0$ *in the distribution sense, then* $u = \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha$ *solves* $\bar{\partial}_{k\widehat{\phi}}u = \alpha$ *and we have the* L^2*-estimate*

$$\|u\| \leq \frac{1}{\sqrt{Ck}}\|\alpha\|.$$

Proof. Since $\beta := (\triangle^{-1}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha \in \operatorname{Dom}(\triangle_{\widehat{\omega},k\widehat{\phi}})$, thus $u = \bar{\partial}^*_{k\widehat{\phi}}\beta \in \operatorname{Dom} \bar{\partial}_{k\widehat{\phi}} \subset L^2(\mathbb{C}^n)$. The expression of u is legitimate. Then we have

$$\bar{\partial}_{k\widehat{\phi}}u = \triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}\beta - \bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta = \alpha - \bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta.$$

Now, we claim that $\bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta = 0$. To see this, we first compute that

$$\bar{\partial}_{k\widehat{\phi}}\bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta = \bar{\partial}_{k\widehat{\phi}}\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}\beta = \bar{\partial}_{k\widehat{\phi}}\alpha = 0.$$

Hence, $\bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta \in \ker \bar{\partial}_{k\widehat{\phi}}$. Also, $\bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta \in \ker \bar{\partial}^*_{k\widehat{\phi}}$ clearly. Therefore, $\bar{\partial}^*_{k\widehat{\phi}}\bar{\partial}_{k\widehat{\phi}}\beta \in \ker \triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}}$. From (33), we know that $\ker \triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}} = 0$ and thus $\bar{\partial}_{k\widehat{\phi}}u = \alpha$. As for the last statement,

$$\begin{aligned}\|u\|^2 =& (\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha|\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha)\\ =& \Big((\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha|\bar{\partial}_{k\widehat{\phi}}\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha\Big) = ((\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\alpha|\alpha) \le \frac{1}{Ck}\|\alpha\|^2. \quad \square\end{aligned}$$

From above Theorem, we deduce the following "Hodge decomposition":

Theorem 8. *Let* $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} := \mathcal{P}^{(0)}_{\widehat{\omega},k\widehat{\phi}} : L^2(\mathbb{C}^n,\widehat{\omega}) \to \ker \triangle^{(0)}_{\widehat{\omega},k\widehat{\phi}} = \ker(\bar{\partial}_{k\widehat{\phi}})$ *be the orthogonal projection. Then it is given by*

$$\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} = I - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}} \quad \text{on } C^\infty_c(\mathbb{C}^n). \tag{34}$$

Proof. First of all, for $u \in C^\infty_c(\mathbb{C}^n)$, we apply Theorem 7 to $v := \bar{\partial}_{k\widehat{\phi}}u$ and thus $u_0 = \bar{\partial}^*_{k\widehat{\phi}}(\triangle_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u$ solves $\bar{\partial}_{k\widehat{\phi}}u_0 = \bar{\partial}_{k\widehat{\phi}}u$. Therefore, $u - u_0 \in \ker \bar{\partial}_{k\widehat{\phi}}$. Also, for any $u \in \ker(\bar{\partial}_{k\widehat{\phi}})$, $(I - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}})u = u = \mathcal{P}_{\widehat{\omega},k\widehat{\phi}}u$. This shows that

$$(I - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}})^2u = u - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u.$$

Finally, we compute

$$\begin{aligned}(u-u_0|u_0) =& (u|u_0) - (u_0|u_0)\\ =& (\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u|u) - (\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u|\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u)\\ =& ((\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u|\bar{\partial}_{k\widehat{\phi}}u) - ((\triangle^{1}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u|\bar{\partial}_{k\widehat{\phi}}u) = 0,\end{aligned}$$

where $\bar{\partial}_{k\widehat{\phi}}\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u = \bar{\partial}_{k\widehat{\phi}}u$ as in the proof of Theorem 7. Therefore, $u - u_0 \perp u_0$ and hence we conclude that $u - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}u = \mathcal{P}_{\widehat{\omega},k\widehat{\phi}}u$. $\square$

Remark 1. Since $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}$ is a bounded operator on $L^2(\mathbb{C}^n,\widehat{\omega})$, we actually know that the (34) holds on $L^2(\mathbb{C}^n, \widehat{\omega})$ by density argument.

3.3. Symbolic calculus and asymptotic sum

To establish the asymptotic expansion for Bergman kernel, we develop the related symbol space and its asymptotic sum in this section. We first define a space of functions which is rapidly decreasing off-diagonal (cf. [28, section 3.1]).

Definition 4. The space $\widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$ consists of functions $a(x,y) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying for any $(\alpha,\beta) \in \mathbb{N}_0^{2d}$, there exists $l = l(\alpha,\beta,a) \in \mathbb{N}$ such that for any $N > 0$, there exists a constant $C = C_{\alpha,\beta,N}(a) > 0$,

$$\left|\partial_x^\alpha \partial_y^\beta a(x,y)\right| \le C \frac{(1+|x|+|y|)^{l(\alpha,\beta)}}{(1+|x-y|)^N}, \quad \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{35}$$

Equivalently, (35) means that for any $\alpha, \beta \in \mathbb{N}_0^d$, there exists $l = l(\alpha,\beta) \in \mathbb{N}$ such that for any $N > 0$,

$$\sup_{x,y\in U} \frac{(1+|x-y|)^N \left|\partial_x^\alpha \partial_y^\beta a(x,y)\right|}{(1+|x|+|y|)^{l(\alpha,\beta)}} < \infty. \tag{36}$$

Thus, if there exists $N_0 > 0$ such that (36) holds for $N > N_0$, then for $N \le N_0$,

$$\sup_{x,y\in U} \frac{(1+|x-y|)^N \left|\partial_x^\alpha \partial_y^\beta a(x,y)\right|}{(1+|x|+|y|)^{l(\alpha,\beta)}} \le \sup_{x,y\in U} \frac{(1+|x-y|)^{N_0} \left|\partial_x^\alpha \partial_y^\beta a(x,y)\right|}{(1+|x|+|y|)^{l(\alpha,\beta)}} < \infty.$$

In other words, it suffices to show the condition (35) for sufficiently large N.

Remark 2. One observes that if $a \in \widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then for each fixed $x, y \in U$, $a(x,\cdot)$, $a(\cdot,y) \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions. Hence, for any $\alpha, \beta \in \mathbb{N}_0^d$, any $a \in \widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$, for fixed $x, y \in U$, $\partial_x^\alpha \partial_y^\beta a(x,\cdot)$ and $\partial_x^\alpha \partial_y^\beta a(\cdot,y)$ are integrable in x and y, respectively.

Now, for a smooth function $a(x,y,k)$ with parameter k, recall that in Definition 2, we have defined a kind a semi-classical symbol space. For $m \in \mathbb{R}$, a function $a(x,y,k) \in \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ if

(i) $a(x,y,k) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, for each $k \in \mathbb{N}$, and

(ii) for any$(\alpha,\beta) \in \mathbb{N}_0^{2d}$, there exists $l = l(\alpha,\beta,a) \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that for any $N > 0$, there exists a constant $C = C_{\alpha,\beta,N}(a) > 0$,

$$\left|\partial_x^\alpha \partial_y^\beta a(x,y,k)\right| \le C k^{m+\frac{|\alpha|+|\beta|}{2}} \frac{(1+|\sqrt{k}x|+|\sqrt{k}y|)^{l(\alpha,\beta)}}{(1+|\sqrt{k}(x-y)|)^N}, \tag{37}$$

for any $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, any $k \ge k_0$.

Similarly, one only needs to verify (37) for $N > N_0$, for some $N_0 > 0$, and (37) shows that for each fixed $x, y \in \mathbb{R}^d$, $k \in \mathbb{N}$, $a(x,\cdot,k), a(\cdot,y,k) \in \mathscr{S}(\mathbb{R}^d)$.

Clearly, if $a(x,y) \in \widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then $a(\sqrt{k}x, \sqrt{k}y) \in \widehat{S}^0(\mathbb{R}^d \times \mathbb{R}^d)$ and for any $m \in \mathbb{R}$, $k^m a(\sqrt{k}x, \sqrt{k}y) \in \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$. Also, it is clear that $\widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d) \subset \widehat{S}^{m'}(\mathbb{R}^d \times \mathbb{R}^d)$ if $m < m'$.

We define *the space of symbols of rapidly decreasing in* k by $\widehat{S}^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d) := \bigcap_{m\in\mathbb{R}} \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$. To define the asymptotic expansion in our case, we need to establish the notion of asymptotic sum.

Theorem 9. *Given any sequence* $\{m_j\}_{j=0}^\infty$ *with* $m_j \searrow -\infty$ *and* $a_j(x,y,k) \in \widehat{S}^{m_j}(\mathbb{R}^d \times \mathbb{R}^d)$, *there exists* $a(x,y,k) \in \widehat{S}^{m_0}(\mathbb{R}^d \times \mathbb{R}^d)$ *such that for any* $q \in \mathbb{N}_0$, *there exists* $k_0(q) \in \mathbb{N}$ *such that if* $k \ge k_0$,

$$a(x,y,k) - \sum_{j=0}^{q} a_j(x,y,k) \in \widehat{S}^{m_{q+1}}(\mathbb{R}^d \times \mathbb{R}^d). \tag{38}$$

Moreover, such a *is uniquely modulo* $\widehat{S}^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. For any positive sequence $\{\mu_j\}_{j=0}^\infty$ with $\lambda_j \nearrow \infty$, we define

$$\tau_{j,k} = \mathbf{1}_{[0,1]}(\mu_j/k).$$

For any $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| + |\beta| \le j$, there exists $l_j \in \mathbb{N}$ such that for any $N > 0$, there exists a constant $C = C_{N,j} > 0$ satisfying

$$|\partial_x^\alpha \partial_y^\beta a_j(x,y,k)| \le C_{N,j} k^{m_j+\frac{j}{2}} \frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l_j}}{(1+|\sqrt{k}(x-y)|)^N}, \quad \forall x, y \in \mathbb{R}^d.$$

Let ϵ_j be a positive sequence tending to 0 to be chosen later and

$$a(x,y,k) := \sum_{j=0}^{\infty} A_j(x,y,k), \quad A_j(x,y,k) := a_j(x,y,k)\chi(k^{\frac{1}{2}-\epsilon_j}x, k^{\frac{1}{2}-\epsilon_j}y)\tau_{j,k}, \tag{39}$$

where $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $0 \le \chi \le 1$, $\chi(x,y) = 1$ for $|x|, |y| \le 1$, and $\chi(x,y) = 0$ for $|x|, |y| \ge 2$

First of all, $\tau_{j,k} \neq 0$ if and only if $\mu_j < k$. Since $\mu_j \nearrow \infty$, for each fixed k, there exists only finitely many j with $\mu_j < k$. Hence, for each fixed $k \in \mathbb{N}$, (39) is a finite sum and hence $a(x,y,k) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Now, for sufficiently large k so that $\tau_{j,k} = 1$, we have

$$A_j(x,y,k) - a_j(x,y,k) = \sum_{j=0}^{q} (\chi(k^{\frac{1}{2}-\epsilon_j}x, k^{\frac{1}{2}-\epsilon_j}y) - 1)a_j(x,y,k).$$

We claim that

Claim. For any $a \in \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ and $\epsilon > 0$, $a(x,y,k)(\chi(k^{\frac{1}{2}-\epsilon}x, k^{\frac{1}{2}-\epsilon}y) - 1) \in \widehat{S}^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, i.e., given any $r \in \mathbb{N}$, $a(\chi(k^{\frac{1}{2}-\epsilon}x, k^{\frac{1}{2}-\epsilon}y) - 1) \in \widehat{S}^{m-r}(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof of Claim. The key is the following. For any $M \in \mathbb{N}$, any $\alpha, \beta \in \mathbb{N}_0^d$, there exists $C = C_{M,\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta \chi(x,y) - 1| \le C_{M,\alpha,\beta}(|x|^M + |y|^M), \quad \forall x, y \in \mathbb{C}^n. \tag{40}$$

For the proof of (40), for $|x|, |y| < 1$, $1 - \chi(x,y) = 0$ and for $|x|, |y| > 2$, $1 - \chi(x,y) = 1$, the estimate holds obviously. For $1 \le |x|, |y| \le 2$, we can expand

$$\chi(x,y) - 1 = \sum_{0<|\alpha|+|\beta|\le M} \frac{\partial^{|\alpha|+|\beta|}\chi}{\partial x^\alpha \partial y^\beta}(0)x^\alpha y^\beta + \sum_{|\alpha|+|\beta|=M} R_{\alpha\beta}(x,y)x^\alpha y^\beta,$$

where $R_{\alpha\beta} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Therefore, $|\partial_x^\alpha \partial_y^\beta \chi(x,y) - 1| \le C_{\alpha,\beta,M}(|x|^M + |y|^M)$. Now, for any $\alpha, \beta \in \mathbb{N}_0^d$, any $N > 0$,

$$\begin{aligned}
&|\partial_x^\alpha \partial_y^\beta a(1 - \chi(k^{1/2-\epsilon}x, k^{1/2-\epsilon}y))| \\
&\quad \le \sum_{\alpha'+\alpha''=\alpha, \beta'+\beta''=\beta} C'_{\alpha,\beta} k^{m+\frac{|\alpha'|+|\beta'|}{2}} \frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l(\alpha',\beta')}}{(1+\sqrt{k}|x-y|)^N}
\end{aligned}$$

$$\begin{aligned}&\times C''_{\alpha'',\beta'',M}(|k^{\frac{1}{2}-\epsilon}x|^{M+|\alpha''|+|\beta''|}+|k^{\frac{1}{2}-\epsilon}y|^{M+|\alpha''|+|\beta''|})\\ \leq & C_3k^{m-\epsilon M+|\alpha|+|\beta|}\frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l_1(\alpha,\beta)+M}}{(1+\sqrt{k}|x-y|)^N},\end{aligned}$$

where $l_1(\alpha,\beta)=\max_{\alpha'+\alpha''=\alpha,\beta'+\beta''=\beta}(l(\alpha',\beta')+|\alpha''|+|\beta''|+M)$. Now, we choose M so that $\epsilon M>r$ and thus $a(1-\chi(k^{\frac{1}{2}-\epsilon}x,k^{\frac{1}{2}-\epsilon}y))\in\widehat{S}^{m-\epsilon M}(\mathbb{R}^d\times\mathbb{R}^d)\subset\widehat{S}^{m-r}(\mathbb{R}^d\times\mathbb{R}^d)$. □

As a result, $A_j(x,y,k)-a_j(x,y,k)\in\widehat{S}^{-\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ for sufficiently large k. On the other hand, for any $j\in\mathbb{N}$, any $\alpha,\beta\in\mathbb{N}_0^d$ with $|\alpha|+|\beta|\leq j$, there exists $l_j\in\mathbb{N}$ such that

$$|\partial_x^\alpha\partial_y^\beta a_j|\leq C_jk^{m_j-\frac{j}{2}}\frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l_j}}{(1+\sqrt{k}|x-y|)^j}.$$

Now, we apply above estimates to
$A_j(x,y,k)=a_j(x,y,k)\chi(k^{\frac{1}{2}-\epsilon_j}x,k^{\frac{1}{2}-\epsilon_j}y)\tau_{j,k}$ and

$$\begin{aligned}&|\partial_x^\alpha\partial_y^\beta A_j(x,y,k)|\\ &\quad\leq C_jk^{m_j-\frac{j}{2}-\frac{\epsilon_j j}{2}}\frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l_j}}{(1+\sqrt{k}|x-y|)^j}\sup_{|\alpha|+|\beta|\leq j}|(\partial_x^\alpha\partial_y^\beta\chi)(k^{\frac{1}{2}-\epsilon_j}x,k^{\frac{1}{2}-\epsilon_j}y)|.\end{aligned}$$

Since χ supports in $|x|,|y|\leq 2$, we see that $\sqrt{k}|x|,\sqrt{k}|y|\leq 2k^{\epsilon_j}$. Thus,

$$|\partial_x^\alpha\partial_y^\beta A_j(x,y,k)|\leq C'_jk^{m_j-\frac{j}{2}+\epsilon_jl_j}(1+\sqrt{k}|x-y|)^{-j}.$$

We take ϵ_j so that $k^{\epsilon_jl_j-1}\leq\frac{1}{2^jC'_j}$ for sufficiently large k and hence

$$C'_jk^{m_j-\frac{j}{2}+\epsilon_jl_j}\leq 2^{-j}k^{m_j-\frac{j}{2}+1}.$$

Given $\alpha,\beta\in\mathbb{N}_0^d$, $q\in\mathbb{N}$, we take $N\geq\max\{|\alpha|+|\beta|,q+1\}$ and $m_N+1\leq m_{q+1}$.

$$\begin{aligned}\Big|\partial_x^\alpha\partial_y^\beta(\sum_{j=N}^\infty A_j)\Big|&\leq\sum_{j=N}^\infty\frac{C'_jk^{m_j-\frac{j}{2}+\epsilon_jl_j}}{(1+\sqrt{k}|x-y|)^j}\\ &\leq\sum_{j=N}^\infty\frac{k^{m_{q+1}-\frac{N}{2}}}{2^j(1+\sqrt{k}|x-y|)^N}\leq\frac{k^{m_{q+1}-\frac{|\alpha|+|\beta|}{2}}}{(1+\sqrt{k}|x-y|)^N}.\end{aligned}$$

Hence,

$$\Big|\partial_x^\alpha\partial_y^\beta(a-\sum_{j=0}^{q}a_j)\Big| \le \Big|\partial_x^\alpha\partial_y^\beta(\sum_{j=N}^{\infty}A_j)\Big|+\Big|\partial_x^\alpha\partial_y^\beta\sum_{j=0}^{q}(a_j-A_j)\Big|+\Big|\partial_x^\alpha\partial_y^\beta\sum_{j=q+1}^{N-1}A_j\Big|.$$

Since $\chi \in \widehat{S}(\mathbb{R}^d\times\mathbb{R}^d)$, $A_j \in \widehat{S}^{m_j}(\mathbb{R}^d\times\mathbb{R}^d)$ (cf. Lemma 15). Also, previous argument and above claim show that $\sum_{j=N}^{\infty}A_j \in \widehat{S}^{m_{q+1}}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\sum_{j=0}^{q}(a_j-A_j)\in\widehat{S}^{-\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, we see that

$$|\partial_x^\alpha\partial_y^\beta(a-\sum_{j=0}^{q}a_j)| \le C_{\alpha,\beta,N}k^{m_{q+1}-\frac{|\alpha|+|\beta|}{2}}\frac{(1+|\sqrt{k}x|+|\sqrt{k}y|)^{l(\alpha,\beta)}}{(1+\sqrt{k}|x-y|)^N}.$$

In other words, $a-\sum_{j=0}^{q}a_j \in \widehat{S}^{m_{q+1}}$. □

If a and $\{a_j\}$ satisfy the conclusion of the Theorem 9, we then write $a\sim\sum_{j=0}^{\infty}a_j(x,y,k)$ and call a the *asymptotic sum* for $\{a_j\}_{j=1}^{\infty}$. Moreover, we define:

Definition 5. The space $\widehat{S}^m_{\mathrm{cl}}(\mathbb{R}^d\times\mathbb{R}^d)$ of *classical symbol* of order m consists of function $a(x,y,k)\in\widehat{S}^m(\mathbb{R}^d\times\mathbb{R}^d)$ such that there exists a sequence $a_j\in\widehat{S}(\mathbb{R}^d\times\mathbb{R}^d)$ for $j\in\mathbb{N}_0$ satisfying

$$a(x,y,k)\sim\sum_{j=0}^{\infty}k^{m-\frac{j}{2}}a_j(\sqrt{k}x,\sqrt{k}y) \tag{41}$$

Next, we define the *quantization* on symbol space $\widehat{S}^m(\mathbb{R}^d\times\mathbb{R}^d)$.

Definition 6. Given $a\in\widehat{S}^m(\mathbb{R}^d\times\mathbb{R}^d)$, we define a k-dependent continuous linear operator $Op_k(a)\in\widehat{L}^m(\mathbb{R}^d)$ by

$$Op_k(a)(u)(x)=\int_{\mathbb{R}^d}a(x,y,k)u(y)dm(y). \tag{42}$$

A k-dependent continuous linear operator $A_k: C_c^\infty(\mathbb{R}^n)\to D'(\mathbb{R}^d)$ is in the class $\widehat{L}^m(\mathbb{R}^d)$ if $A=Op_k(a)$ for some $a(x,y,k)\in\widehat{S}^m(\mathbb{R}^d\times\mathbb{R}^d)$.

In particular, $A_k\in\widehat{L}^m(\mathbb{R}^d)$ implies the Schwartz kernel $K_{A_k}(x,y)=a(x,y,k)\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. Thus, $A_k\in\widehat{L}^m(\mathbb{R}^d)$ is a smoothing operator for any $k\in\mathbb{N}$. Moreover, if $a\in\widehat{S}^{-\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, then $Op_k(a)$ is k-negligible in the sense of Definition 3 obviously. Hence, we may extend the Definition 6 by $A_k \in \widehat{L}^m(\mathbb{R}^d)$ if there exists $a \in \widehat{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$ such that $A_k - Op_k(a) = Op_k(a_1)$ with $a_1 \in \widehat{S}^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$.

We also define the subclass $\widehat{L}^m_{\mathrm{cl}}(\mathbb{R}^d) \subset \widehat{L}^m(\mathbb{R}^d)$ by $A_k \in \widehat{L}^m_{\mathrm{cl}}(\mathbb{R}^d)$ if $A_k - Op_k(a) = Op_k(a_1)$, for some $a \in \widehat{S}^m_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$ and $a_1 \in \widehat{S}^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$. For $A_k = Op_k(a) \in \widehat{L}^m_{\mathrm{cl}}(\mathbb{R}^d)$, we then define the *principal symbol* $\sigma(A_k)$ by the leading term $a_0(x,y) \in \widehat{S}(\mathbb{R}^d \times \mathbb{R}^d)$ in the asymptotic sum (41).

Theorem 10. *If* $A_k = Op_k(a) \in \widehat{L}^m(\mathbb{R}^d)$, $B_k = Op_k(b) \in \widehat{L}^{m'}(\mathbb{R}^d)$, *then*

(i) *The formal adjoint* $A_k^* \in \widehat{L}^m(\mathbb{R}^d)$ *with* $A_k^* = Op_k(a^*)$, *where* $a^*(x,y,k) := \overline{a(y,x,k)}$.

(ii) $A_k \circ B_k \in \widehat{L}^{m+m'-\frac{d}{2}}(\mathbb{R}^d \times \mathbb{R}^d)$ *with* $A_k \circ B_k = Op_k(a\#b)$, *where*

$$(a\#b)(x,y,k) := \int_{\mathbb{R}^d} a(x,t,k)b(t,y,k)dm(t), \tag{43}$$

i.e., if $a \in \widehat{S}^m_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$, $b \in \widehat{S}^{m'}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$, *then* $a\#b \in \widehat{S}^{m+m'-\frac{d}{2}}(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. For (i), for $u, v \in C_c^\infty(\mathbb{R}^d)$, the formal adjoint A_k^* is given by $(A_k(u)|v) = (u|A_k v)$. Hence, we compute A_k^* explicitly as

$$\begin{aligned}
(A_k u|v) &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} a(x,y,k)u(y)dm(y)\right)\overline{v(x)}dm(x)\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,y,k)u(y)\overline{v(x)}dm(y)dm(x)\\
&= \int_{\mathbb{R}^d} u(y)\overline{\int_{\mathbb{R}^d}\overline{a(x,y,k)}v(x)dm(x)}dm(y)\\
&= \int_{\mathbb{R}^d} u(y)\overline{A_k^*(v)(y)}dy.
\end{aligned}$$

This implies that $(A_k^* v)(x) = \int_{\mathbb{R}^d}\overline{a(y,x,k)}v(y)dm(y)$ and thus $A_k^* = Op_{a^*}$, where $a^*(x,y,k) := \overline{a(y,x,k)}$. Obviously, $a^* \in \widehat{S}^m_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d)$ if a is.

For (ii), for each $f \in C_c^\infty(\mathbb{R}^d)$, $(B_k f)(t) = \int_{\mathbb{R}^d} b(t,y,k)f(y)dy$ and thus

$$(A_k \circ B_k)(f)(x) = \int_{\mathbb{R}^d} a(x,t,k)\left(\int_{\mathbb{R}^d} b(t,y,k)f(y)dm(y)\right)dm(t)$$

Observe that for fixed $x \in \mathbb{R}^d$, $k \in \mathbb{N}$, $a(x,t,k)b(t,y,k)f(y)$ is integrable in t and y. Therefore, by Fubini-Tonelli Theorem, we then have

$$
\begin{aligned}
(A_k \circ B_k)(f)(x) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} a(x,t,k)b(t,y,k)dm(t) \right) f(y)dm(y) \\
&= \int_{\mathbb{R}^d} (a\#b)(x,y,k)f(y)dm(y).
\end{aligned}
$$

We then see that $A_k \circ B_k = Op_{a\#b}$. Now, we show that $a\#b \in \widehat{S}^{m+m'-\frac{d}{2}}(\mathbb{R}^d \times \mathbb{R}^d)$. To see this, for any $\alpha, \beta \in \mathbb{N}_0^d$,

$$
\begin{aligned}
\left| \partial_x^\alpha \partial_y^\beta (a\#b) \right| \leq & \int_U \left| \partial_x^\alpha a(x,t,k) \partial_y^\beta b(t,y,k) \right| dm(t) \\
\leq & C_{\alpha,\beta,N} k^{m+m'+\frac{|\alpha|+|\beta|}{2}} \\
& \times \int_{\mathbb{R}^d} \frac{(1+\sqrt{k}|x|+\sqrt{k}|t|)^{l(\alpha,0)}}{(1+|\sqrt{k}(x-t)|)^N} \frac{(1+|\sqrt{k}t|+|\sqrt{k}y|)^{l'(0,\beta)}}{(1+|\sqrt{k}(t-y)|)^N} dm(t)
\end{aligned}
$$

We make the change of variable $s = \sqrt{k}t$ and thus $dm(s) = k^{\frac{d}{2}} dm(t)$. This implies

$$
\begin{aligned}
& \left| \partial_x^\alpha \partial_y^\beta (a\#b) \right| \\
\leq & C k^{m+m'-\frac{d}{2}+\frac{|\alpha|+|\beta|}{2}} \int_{\mathbb{R}^d} \frac{(1+\sqrt{k}|x|+|s|)^{l(\alpha,0)}}{(1+|\sqrt{k}x-s|)^N} \frac{(1+|s|+|\sqrt{k}y|)^{l'(0,\beta)}}{(1+|s-\sqrt{k}y)|)^N} dm(s).
\end{aligned}
$$

Let $l(\alpha,\beta) = \max\{l(\alpha,0), l'(0,\beta)\}$. We observe that for any $M > 0$,

$$
\begin{aligned}
& (1+|\sqrt{k}x-s|)^M (1+|\sqrt{k}y-s|)^M \\
& \geq (1+|\sqrt{k}(x-y)|+|\sqrt{k}x-s||\sqrt{k}y-s|)^M \\
& = (1+|\sqrt{k}(x-y)|)^M \left(1+\frac{|\sqrt{k}x-s||\sqrt{k}y-s|}{1+\sqrt{k}|x-y|} \right)^M \geq (1+\sqrt{k}|x-y|)^M
\end{aligned}
$$

Hence, by taking $M = N/2$ and $u = s - \sqrt{k}x$, we can write

$$
\begin{aligned}
& \left| \partial_x^\alpha \partial_y^\beta (a\#b) \right| \\
\leq & C \frac{k^{m+m'-\frac{d}{2}+\frac{|\alpha|+|\beta|}{2}}}{(1+\sqrt{k}|x-y|)^{N/2}} \int_{\mathbb{R}^d} \frac{(1+\sqrt{k}|x|+|s|)^{l(\alpha,\beta)}(1+\sqrt{k}|y|+|s|)^{l(\alpha,\beta)}}{(1+|\sqrt{k}x-s|)^{N/2}(1+|\sqrt{k}y-s|)^{N/2}} ds
\end{aligned}
$$

$$\leq C\frac{k^{m+m'-\frac{d}{2}+\frac{|\alpha|+|\beta|}{2}}}{(1+\sqrt{k}|x-y|)^{N/2}}\int_{\mathbb{R}^d}\frac{(1+2\sqrt{k}|x|+|u|)^{l(\alpha,\beta)}(1+\sqrt{k}|x|+\sqrt{k}|y|+|u|)^{l(\alpha,\beta)}}{(1+|u|)^{N/2}}du.$$

Hence, there exists $N_0 = N_0(\alpha,\beta,d)$ such that if $N > N_0$, the integral converges. By expanding the numerator of the integrand, we can find $l'(\alpha,\beta)$ so that for any $N > N_0$,

$$\left|\partial_x^\alpha\partial_y^\beta(a\#b)\right| \leq C_{\alpha,\beta,N}k^{m+m'-\frac{d}{2}+\frac{|\alpha|+|\beta|}{2}}\frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l'(\alpha,\beta)}}{(1+\sqrt{k}|x-y|)^{N/2}}.$$

Thus, $a\#b \in \widehat{S}^{m+m'-\frac{d}{2}}(\mathbb{R}^d\times\mathbb{R}^d)$. □

3.4. Asymptotic expansion of approximate kernel

We now establish the asymptotic expansion of approximate Bergman kernel $P^{(0)}_{\widehat{\omega},k\widehat{\phi}}$ through the symbolic calculus presented in section 3.3.

We first consider the case $P^{(q)}_{\omega_0,k\phi_0}$, where $\phi_0 = \sum_{j=1}^n \lambda_{j,x}|z^j|^2$, $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{j=1}^n dz^j\wedge d\bar{z}^j$, which is the orthogonal projection $L^2_{0,q}(\mathbb{C}^n) := L^2_{0,q}(\mathbb{C}^n, \omega_0) \to \ker\triangle_{\omega_0,k\phi_0}$, where the analogous Laplacian $\triangle^{(q)}_{\omega_0,k\phi_0}$ is given by

$$\triangle^{(q)}_{\omega_0,k\phi_0} := \bar{\partial}^{*,\omega_0}_{k\phi_0}\bar{\partial}_{k\phi_0} + \bar{\partial}_{k\phi_0}\bar{\partial}^{*,\omega_0}_{k\phi_0},$$

$\bar{\partial}_{k\phi_0} := \bar{\partial} + k\epsilon(\bar{\partial}\phi_0)$, and $\bar{\partial}^{*,\omega_0}_{k\phi_0}$ is the formal adjoint with respect to the L^2-inner product

$$(\alpha|\beta)_0 := \sum\nolimits'_{|I|=J}\int_{\mathbb{C}^n}\alpha_I\overline{\beta_I}dm(z).$$

Let $\delta_k(z) = \frac{z}{\sqrt{k}}$ be the scaling map on $\mathbb{C}^n$ with inverse $\delta_k^{-1}(z) = \sqrt{k}z$. Then for $u \in C^\infty(\mathbb{C}^n)$,

$$\begin{aligned}\delta_k\bar{\partial}_{k\phi_0}\delta_k^{-1}u(z) &= \delta_k\left(\sum_{i=1}^n\left(\sqrt{k}\frac{\partial u}{\partial\bar{z}^i}(\sqrt{k}z) + k\frac{\partial\phi_0}{\partial\bar{z}^i}(z)u(\sqrt{k}z)\right)d\bar{z}^i\right)\\ &= \delta_k\left(\sum_{i=1}^n\left(\sqrt{k}\frac{\partial u}{\partial\bar{z}^i}(\sqrt{k}z) + k\lambda_{i,x}z^iu(\sqrt{k}z)\right)d\bar{z}^i\right)\end{aligned}$$

$$= \delta_k \left(\sqrt{k} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial \bar{z}^i}(\sqrt{k}z) + \sqrt{k}\lambda_{i,x} z^i u(\sqrt{k}z) \right) d\bar{z}^i \right)$$
$$= \sqrt{k}\delta_k \delta_k^{-1}(\bar{\partial}_{\phi_0} u)(z) = \sqrt{k}\bar{\partial}_{\phi_0} u(z).$$

Therefore, we can deduce that

$$\bar{\partial}_{\phi_0}\delta_k = \frac{1}{\sqrt{k}}\delta_k \bar{\partial}_{k\phi_0}, \quad \bar{\partial}^{*,\omega_0}_{\phi_0}\delta_k = \frac{1}{\sqrt{k}}\delta_k \bar{\partial}^{*,\omega_0}_{k\phi_0}. \tag{44}$$

Hence, from (44) we get

$$\triangle_{\omega_0,\phi_0}\delta_k = \frac{1}{k}\delta_k \triangle_{\omega_0,k\phi_0}. \tag{45}$$

Using (45), if $\{\sigma_j(z)\}_{j=1}^d$, where $d \in \mathbb{N}_0 \cup \{\infty\}$, is an orthonormal basis of $\ker \triangle_{\omega_0,k\phi_0}$ with respect to $(\cdot|\cdot)_{\omega_0}$, then $\delta_k\sigma_j$ satisfies $\triangle_{\omega_0,\phi_0}\delta_k\sigma_j = \frac{1}{k}\delta_k\triangle_{\omega_0,k\phi_0}\sigma_j = 0$. Their inner product is given by

$$(\delta_k\sigma_i|\delta_k\sigma_j)_{\omega_0} = \int_{\mathbb{C}^n} \delta_k\sigma_i(w)\overline{\delta_k\sigma_j(w)}dm(w) = k^n \int_{\mathbb{C}^n} \sigma_i(w)\overline{\sigma_j(w)}dm(w) = k^n\delta_{ij}.$$

This shows that $\{k^{-\frac{n}{2}}\delta_k\sigma_j\}_{j=1}^d$ is an orthonormal basis for $\ker \triangle_{\omega_0,\phi_0}$. Since P_{ω_0,ϕ_0} is the kernel of orthogonal projection onto $\ker \triangle_{\omega_0,k\phi_0}$, we know

$$P_{\omega_0,\phi_0}(z,w) = k^{-n}\sum_{j=1}^{d} \sigma_j\left(\frac{z}{\sqrt{k}}\right)\overline{\sigma_j\left(\frac{z}{\sqrt{k}}\right)} = k^{-n}P_{\omega_0,k\phi_0}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right). \tag{46}$$

For P_{ω_0,ϕ_0}, we can compute it explicitly.

Proposition 11. *The approximate Bergman kernel $P^{(0)}_{\omega_0,\phi_0}(z,w)$ is given by*

$$P^{(0)}_{\omega_0,\phi_0}(z,w) = \frac{2^n\lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{\sum_{j=1}^n \lambda_{j,x}\left(2z^j\overline{w}^j - |z^j|^2 - |w^j|^2\right)}.$$

Proof. First, we consider the trivial line bundle $L = \mathbb{C}\times\mathbb{C}^n$ over $\mathbb{C}^n$ with weight $|1|^2_{h^L} = e^{-2\phi_0}$. Its L^2-section can be identified as the weighted L^2-space $L^2(\mathbb{C}^n, e^{-2\phi_0}dm)$, and the subspace of holomorphic sections is identified the subspace $\mathcal{F}$, known as *Bargmann–Fock space*, consisting of entire functions f satisfying

$$\|f\|^2_{\omega_0,\phi_0} = \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\sum_{j=1}^n \lambda_{j,x}|z^j|^2} dm(z) < \infty, \quad \bar\partial f = 0, \quad \forall f \in \mathcal{F}.$$

We denote $K_{\mathrm{BF}}(z,w)$ by the Schwartz kernel of the orthogonal projection $\Pi_{\mathrm{BF}} : L^2(\mathbb{C}^n, L) \to H^0(X,L)$, i.e.,

$$(\Pi_{\mathrm{BF}} f)(z) = f(z) = \int_{\mathbb{C}^n} K_{\mathrm{BF}}(z,w)(f(w))dm(w), \quad \forall f \in H^0(X,L).$$

Claim. $K_{\mathrm{BF}}(z,w) = \frac{2^n\lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{2\sum_{j=1}^n \lambda_{j,x}(z^j\overline{w}^j - |w^j|^2)}$.

Proof. For multi-index $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$, we let $z^\alpha := (z^1)^{\alpha_1}\ldots(z^n)^{\alpha_n}$. Clearly, $\bar\partial(z^\alpha) = 0$, for any $\alpha \in \mathbb{N}_0^n$. Hence, $z^\alpha \in \mathcal{O}(\mathbb{C}^n)$. For $\alpha, \beta \in \mathbb{N}_0^n$, using polar coordinate $z^j = r_j e^{-\sqrt{-1}\theta_j}$ and Fubini–Torelli Theorem,

$$\begin{aligned}(z^\alpha|z^\beta)_{\omega_0,\phi_0} &= \int_{\mathbb{C}^n} z^\alpha \bar z^\beta e^{-2\sum_{j=1}^n \lambda_{x,j}|z^j|^2} dm \\ &= \prod_{j=1}^n \left[\int_0^\infty \int_0^{2\pi} r_j^{\alpha_j+\beta_j+1} e^{\sqrt{-1}(\alpha_j-\beta_j)\theta_j} e^{-2\lambda_{x,j}r_j^2} d\theta_j dr_j\right].\end{aligned}$$

If $\alpha_j \neq \beta_j$, then $\int_0^{2\pi} e^{\sqrt{-1}(\alpha_j-\beta_j)\theta_j} d\theta_j = 0$. Hence, $(z^\alpha, z^\beta) = 0$ if $\alpha \neq \beta$. Now, observe that for $l \in \mathbb{N}$,

$$\int_0^\infty r^{2l+1} e^{-2\lambda r^2} dr = \frac{1}{2(2\lambda)^{l+1}} \int_0^\infty u^l e^{-u} du = \frac{\Gamma(l+1)}{2(2\lambda)^{l+1}} = \frac{l!}{2(2\lambda)^{l+1}}, \quad (47)$$

where $u = 2\lambda r^2$. Therefore, the square of norm of z^α is given by

$$\|z^\alpha\|^2_{\omega_0,\phi_0} = \prod_{j=1}^n \frac{2\pi\alpha_j!}{2(2\lambda_{j,x})^{\alpha_j+1}} = \frac{\pi^n \alpha!}{2^{|\alpha|+n}\lambda^{\alpha+1}},$$

where $\lambda^{\alpha+1} := \prod_{j=1}^n \lambda_{j,x}^{\alpha_j+1}$. As a result, $\left\{\Psi_\alpha := \sqrt{\frac{2^{|\alpha|+n}\lambda^{\alpha+1}}{\pi^n\alpha!}} z^\alpha\right\}_{\alpha\in\mathbb{N}_0^n}$ is an orthonormal basis for $\mathcal{F}$ and $K_{\mathrm{BF}}(z,w)$ is given by

$$K_{\mathrm{BF}}(z,w) = \sum_{\alpha\in\mathbb{N}_0^n} \Psi_\alpha(z)\overline{\Psi_\alpha(w)} e^{-2\phi_0(w)}.$$

We then compute that

$$
\begin{aligned}
K_{\mathrm{BF}}(z,w) &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{2^{|\alpha|+n}\lambda^{\alpha+1}}{\pi^n \alpha!} z^\alpha \overline{w}^\alpha e^{-2\sum_{j=1}^n \lambda_{j,x}|w^j|^2} \\
&= \sum_{d=1}^{\infty} \frac{2^{d+n}}{\pi^n d!} \sum_{|\alpha|=d} \frac{d!}{\alpha!} (\lambda z)^\alpha \overline{w}^\alpha e^{-2\sum_{j=1}^n \lambda_{j,x}|w^j|^2} \\
&= \frac{2^n \lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{2\sum_{j=1}^n \lambda_{j,x}(z^j\overline{w}^j - |w^j|^2)}. \qquad \square
\end{aligned}
$$

Now, observe that $L^2(\mathbb{C}^n, e^{-2\sum_{j=1}^n \lambda_{j,x}|z^j|^2}dm)$ and $L^2(\mathbb{C}^n)$ is isometric via $u \mapsto ue^{\phi_0(z)}$. As in section 3.1, we know that

$$
\bar{\partial}(ue^{\sum_{j=1}^n \lambda_{j,x}|z^j|^2}) = e^{\sum_{j=1}^n \lambda_{j,x}|z^j|^2} \bar{\partial}_{\phi_0} u, \quad \forall u \in C^\infty(\mathbb{C}^n).
$$

Therefore, $\mathcal{P}_{\omega_0,\phi_0}$ and Π_{BF} are related by $\mathcal{P}_{\omega_0,\phi_0} = e^{-\phi_0}\Pi_{\mathrm{BF}}e^{\phi_0}$, and their Schwartz kernels have the relation

$$
\begin{aligned}
P_{\omega_0,\phi_0}(z,w) =& e^{-\phi_0(z)} K_{\mathrm{BF}}(z,w) e^{\phi_0(w)} \qquad (48) \\
=& \frac{2^n \lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{\sum_{j=1}^n \lambda_{j,x}\left(2z^j\overline{w}^j - |z^j|^2 - |w^j|^2\right)}. \qquad \square
\end{aligned}
$$

Our goal is to obtain asymptotic of $P_{\widehat{\omega},k\widehat{\phi}}$. Recall that in (29), $\widehat{\phi} = \phi_0 + \phi_1$ with $\phi_1 \in C_c^\infty(\mathbb{C}^n)$. We consider $e^{-k(\widehat{\phi}-\phi_0)}u = e^{-k\phi_1}u$. Notice that $\phi_1 \in C_c^\infty(\mathbb{C}^n)$ implies that

$$
\int_{\mathbb{C}^n} |u|^2 e^{\pm 2k\phi_1} dm(z) < \infty, \quad \forall k \in \mathbb{N}, u \in L^2(\mathbb{C}^n).
$$

Hence, by similar argument as in Proposition 11, the map $u \mapsto ue^{-k\phi_1}$ defines an isometry on $L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ which maps $\ker \bar{\partial}_{k\phi_0}$ bijectively onto $\ker \bar{\partial}_{k\widehat{\phi}}$ with inverse map $v \mapsto e^{k\phi_1}v$. On the other hand, we consider the change of base metric from ω_0 to $\widehat{\omega}$. Observe that $\widehat{\omega} = \omega_0 + \omega_1$ with ω_1 supports in $B_{k^{\epsilon-\frac{1}{2}}}(0)$. This implies that the L^2-norm $\|\cdot\|_{\omega_0}$ and $\|\cdot\|_{\widehat{\omega}}$ are equivalent and thus $L^2(\mathbb{C}^n) = L^2(\mathbb{C}^n, \widehat{\omega})$. We may regard $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} : L^2(\mathbb{C}^n) \to \ker \bar{\partial}_{k\widehat{\phi}}$.

We then define an intermediate operator $\widehat{\mathcal{P}}_{\omega_0,k\phi_0} : L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)$ by

$$\widehat{\mathcal{P}}_{\omega_0,k\phi_0} = e^{-k\phi_1} \circ \mathcal{P}_{\omega_0,k\phi_0} \circ e^{k\phi_1}. \tag{49}$$

By uniqueness of Schwartz kernel, we see that its Schwartz kernel $\widehat{P}_{\omega_0,k\phi_0}(z,w)$ is given by

$$\widehat{P}_{\omega_0,k\phi_0}(z,w) = e^{-k\phi_1(z)} P_{\omega_0,k\phi_0}(z,w) e^{k\phi_1(w)}. \tag{50}$$

Now, we observe that

Lemma 12. $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} = \widehat{\mathcal{P}}_{\omega_0,k\phi_0} \circ \mathcal{P}_{\widehat{\omega},k\widehat{\phi}}$ *and* $\widehat{\mathcal{P}}_{\omega_0,k\phi_0} = \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \widehat{\mathcal{P}}_{\omega_0,k\phi_0}$.

Proof. First, it is easy to see that the map $u \mapsto e^{-k\phi_1} u$ sends $\ker \bar{\partial}_{k\phi_0}$ onto to $\ker \bar{\partial}_{k\widehat{\phi}}$ and $u \mapsto e^{k\phi_1} u$ sends $\ker \bar{\partial}_{k\widehat{\phi}}$ onto to $\ker \bar{\partial}_{k\phi_0}$. By (34) and above observation,

$$\begin{aligned}
\widehat{\mathcal{P}}_{\omega_0,k\phi_0} - \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \widehat{\mathcal{P}}_{\omega_0,k\phi_0} &= (I - \mathcal{P}_{\widehat{\omega},k\widehat{\phi}}) \circ \widehat{\mathcal{P}}_{\omega_0,k\phi_0} \\
&= \bar{\partial}^*_{k\widehat{\phi}} (\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1} \bar{\partial}_{k\widehat{\phi}} e^{-k\phi_1} \mathcal{P}_{\omega_0,k\phi_0} e^{k\phi_1} = 0; \\
\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} - \widehat{\mathcal{P}}_{\omega_0,k\phi_0} \circ \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} &= e^{-k\phi_1} (I - \mathcal{P}_{\omega_0,k\phi_0}) e^{k\phi_1} \circ \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \\
&= e^{-k\phi_1} \bar{\partial}^{*,\omega_0}_{k\phi_0} (\triangle^{(1)}_{\omega_0,k\phi_0})^{-1} \bar{\partial}_{k\phi_0} e^{k\phi_1} \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} = 0. \quad \square
\end{aligned}$$

Moreover, let $\widehat{\mathcal{P}}^{*,\widehat{\omega}}_{\omega_0,k\phi_0}$ be the formal adjoint of $\widehat{\mathcal{P}}_{\omega_0,k\phi_0}$ with respect to the norm $(\cdot|\cdot)_{\widehat{\omega}}$. By direct computation, we know that its Schwartz kernel is given by

$$\widehat{P}^{*,\widehat{\omega}}_{\omega_0,k\phi_0}(z,w) = \lambda^{-1}(z) e^{k\phi_1(z)} P_{k\phi_0}(z,w) e^{-k\phi_1(w)} \lambda(w), \tag{51}$$

where λ is the density of $\frac{\widehat{\omega}^n}{dm}$, i.e. $\widehat{\omega}_n = \lambda dm$. If we define $\mathcal{R} := \widehat{\mathcal{P}}^{*,\widehat{\omega}}_{\omega_0,k\phi_0} - \widehat{\mathcal{P}}_{\omega_0,k\phi_0}$ to measure the extent which $\widehat{\mathcal{P}}_{\omega_0,k\phi_0}$ is not formally self-adjoint with respect to $(\cdot|\cdot)_{\widehat{\omega}}$, then its Schwartz kernel is given by

$$R(z,w) = P_{\omega_0,k\phi_0}(z,w) \left(\lambda^{-1}(z) \lambda(w) e^{k\phi_1(z) - k\phi_1(w)} - e^{k\phi_1(w) - k\phi_1(z)} \right). \tag{52}$$

Now, if we take adjoint in the first formula in Lemma 12, we get

$$\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} = \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \widehat{\mathcal{P}}^{*,\widehat{\omega}}_{\omega_0,k\phi_0} = \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ (\widehat{\mathcal{P}}_{\omega_0,k\phi_0} + \mathcal{R}) = \widehat{\mathcal{P}}_{\omega_0,k\phi_0} + \mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \mathcal{R},$$

where we use the second formula in Lemma 12 in the last line. We then get

$$\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(I-\mathcal{R})=\widehat{\mathcal{P}}_{\omega_0,k\phi_0}. \tag{53}$$

Now, for any $M\in\mathbb{N}$, if we multiply $(I+\mathcal{R}+\mathcal{R}^2+\cdots+\mathcal{R}^{M-1})$ from the right on the both sides of (53), then we obtain

$$\widehat{\mathcal{P}}_{\omega_0,k\phi_0}+\widehat{\mathcal{P}}_{\omega_0,k\phi_0}\circ\mathcal{R}+\cdots+\widehat{\mathcal{P}}_{\omega_0,k\phi_0}\circ\mathcal{R}^{M-1}+\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}\circ\mathcal{R}^M=\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}. \tag{54}$$

(54) is the key observation for establishing asymptotic expansion for $P_{\widehat{\omega},k\widehat{\phi}}$ near $(0,0)$. We will now employ the symbolic calculus developed in previous section to (54) to achieve this. First of all, from Proposition 11 and

$$2z^j\overline{w}^j-|z^j|^2-|w^j|^2=-|z^j-w^j|^2+2\sqrt{-1}\mathrm{Im}z^j\overline{w}^j,\quad\forall z,w\in\mathbb{C}^n,$$

we know that

$$P_{\omega_0,\phi_0}(z,w)=\frac{2^n\lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n}e^{-\sum_{j=1}^n\lambda_{j,x}|z^j-w^j|^2+2\sqrt{-1}\mathrm{Im}z^j\overline{w}^j}\in\widehat{S}(\mathbb{C}^n\times\mathbb{C}^n).$$

Therefore, $P_{\omega_0,k\phi_0}(z,w)=k^nP_{\omega_0,\phi_0}(\sqrt{k}z,\sqrt{k}w)\in\widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$. Now, we show

Lemma 13. *For $\epsilon\in[0,1/6)$, we have*

$$\widehat{P}_{\omega_0,k\phi_0}(z,w)\in\widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n),\quad R(z,w)\in\widehat{S}^{n-\frac{1}{2}}_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n).$$

Proof. By our choice of ϕ_1 and ω_1 as in (29), we know that for $|z|,|w|>k^{\epsilon-1/2}$, $\widehat{P}_{\omega_0,k\phi_0}(z,w)=P_{\omega_0,k\phi_0}(z,w)\in\widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$. For $|z|,|w|<k^{\epsilon-1/2}$, since $\phi_1(z)=O(|z|^3)$, $|k\phi_1(z)|\le Ck|z|^3\le Ck^{-1/2}|\sqrt{k}z|^3$. Since $|z|<k^{\epsilon-1/2}$, we see that $|k\phi_1(z)|\le Ck^{3\epsilon-1/2}$. This shows that

$$\left|e^{k\phi_1(x)-k\phi_1(y)}-1\right|\le Ce^{k^{3\epsilon-\frac{1}{2}}}\sup_{|x|,|y|<k^{-\frac{1}{2}+\epsilon}}(k|x|^3+k|y|^3)\le Ck^{3\epsilon-1/2}e^{k^{3\epsilon-\frac{1}{2}}}, \tag{55}$$

where x,y are the underlying real coordinates for z and w. Hence, if $\epsilon\in[0,\frac{1}{6})$, then $3\epsilon-\frac{1}{2}<0$. Therefore, $\widehat{P}_{\omega_0,k\phi_0}\in\widehat{S}^n(\mathbb{C}^n\times\mathbb{C}^n)$. Furthermore,

$$\left|e^{k\phi_1(x)-k\phi_1(y)}-\sum_{l=1}^N\frac{(k\phi_1(x)-k\phi_1(y))^l}{l!}\right|$$

$$\begin{aligned}&\lesssim \int_0^1 |k\phi_1(x)-k\phi_1(y)|^N e^{k\phi_1(tx)-k\phi_1(ty)}dt\\ &\lesssim k^{(N+1)(3\epsilon-1/2)}e^{k^{3\epsilon-1/2}}.\end{aligned} \tag{56}$$

This shows that $\widehat{P}_{\omega_0,k\phi_0}\in \widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$. Now, if we expand $\lambda(x)\lambda^{-1}(y)$ in Taylor expansion:

$$\begin{aligned}\lambda^{-1}(x)\lambda(y)=&1+\sum_{j=1}^{N-1}\sum_{|\alpha|+|\beta|=j}\frac{\partial_x^\alpha\partial_y^\beta(\lambda^{-1}(x)\lambda(y))(0,0)}{(\alpha+\beta)!}x^\alpha y^\beta\\ &+N\int_0^1(1-t)^{N-1}\sum_{|\alpha|+|\beta|=N}\frac{\partial_x^\alpha\partial_y^\beta(\lambda^{-1}(x)\lambda(y))(tx,ty)x^\alpha y^\beta}{(\alpha+\beta)!}dt.\end{aligned}$$

This shows that

$$\begin{aligned}\frac{\lambda(y)}{\lambda(x)}-1-\sum_{j=1}^{N-1}k^{-\frac{j}{2}}\sum_{|\alpha|+|\beta|=j}\frac{\partial_x^\alpha\partial_y^\beta(\lambda^{-1}(x)\lambda(y))(0,0)}{(\alpha+\beta)!}(\sqrt{k}x)^\alpha(\sqrt{k}y)^\beta\\ =O(k^{-\frac{N}{2}}).\end{aligned} \tag{57}$$

Hence, we get

$$\begin{aligned}|R(z,w)|\le&|P_{\omega_0,k\phi_0}(z,w)|\Big(\Big|\lambda^{-1}(z)\lambda(w)-1\Big|\Big|e^{k\phi_1(z)-k\phi_1(w)}\Big|\\ &+\Big|e^{k\phi_1(z)-k\phi_1(w)}-e^{k\phi_1(w)-k\phi_1(z)}\Big|\Big)\\ \le&|P_{\omega_0,k\phi_0}(z,w)|\Big(\Big|\lambda^{-1}(z)\lambda(w)-1\Big|(1+Ck^{3\epsilon-1/2}e^{k^{3\epsilon-\frac{1}{2}}})\\ &+2Ck^{3\epsilon-1/2}e^{k^{3\epsilon-\frac{1}{2}}}\Big).\end{aligned}$$

The derivative estimate of R follows similarly as above. This shows that $R\in\widehat{S}^{n-\frac{1}{2}}(\mathbb{C}^n\times\mathbb{C}^n)$ if $\epsilon\in[0,\frac{1}{6})$. Moreover, (56) and (57) shows that $R\in\widehat{S}^{n-\frac{1}{2}}_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$. □

From Theorem 10, we know that for any $j\in\mathbb{N}$ and $R_j:=R^{\#j}$,

$$\widehat{P}_{\omega_0,k\phi_0}\#R_j\in\widehat{S}^{n-j/2}_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n).$$

Before proving our main result for this section, we need to first show that the remainder kernel $P_{\widehat{\omega},k\widehat{\phi}}\#R^{\#j}$ for $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}\circ\mathcal{R}^j$ in (54) is well-defined.

Lemma 14. *Let $R_j := R^{\#j}$ be the Schwartz kernel of $\mathcal{R}^j$. Then $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \mathcal{R}^j$ is well-defined as a smoothing operator with smoothing kernel $P_{\widehat{\omega},k\widehat{\phi}}\#R_j$, for any $j \in \mathbb{N}$.*

Proof. For any $\alpha, \beta \in \mathbb{N}_0^{2n}$ any $x_0, y_0 \in \mathbb{C}^n$, by Cauchy–Schwartz inequality,

$$\begin{aligned}
&\left|\partial_x^\alpha \partial_y^\beta \int_{\mathbb{C}^n} P_{\widehat{\omega},k\widehat{\phi}}(x_0,u)R_j(u,y_0)dm(u)\right| \\
&\quad \leq \int_{\mathbb{C}^n} |\partial_x^\alpha P_{\widehat{\omega},k\widehat{\phi}}(x_0,u)||\partial_y^\beta R_j(u,y_0)|dm(u) \\
&\quad \leq \left(\int_{\mathbb{C}^n} |\partial_x^\alpha P_{\widehat{\omega},k\widehat{\phi}}(x_0,u)|^2 dm(u)\right)^{1/2} \left(\int_{\mathbb{C}^n} |\partial_y^\beta R_j(u,y_0)|^2 dm(u)\right)^{1/2}.
\end{aligned}$$

First of all, since $R_j \in \widehat{S}^{n-j/2}(\mathbb{C}^n \times \mathbb{C}^n)$, we know that $R_j(\cdot, y) \in \mathscr{S}(\mathbb{R}^{2n})$ for fixed y_0 and thus $(\partial_y^\beta R_j)(\cdot, y_0) \in L^2(\mathbb{C}^n)$, for any $\beta \in \mathbb{N}_0^{2n}$. On the other hand, since $\|\cdot\|_{\omega_0}$ and $\|\cdot\|_{\widehat{\omega}}$ are equivalent, there exists a constant $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{C}^n} |\partial_x^\alpha P_{\widehat{\omega},k\widehat{\phi}}(x_0,u)|^2 dm(u) &\leq C \int_{\mathbb{C}^n} |\partial_x^\alpha P_{\widehat{\omega},k\widehat{\phi}}(x_0,u)|^2 \widehat{\omega}_n(u) \\
&= C \partial_x^\alpha \partial_y^\alpha P_{\widehat{\omega},k\widehat{\phi}}(x_0,x_0) < \infty.
\end{aligned}$$

It is clear that $P_{\widehat{\omega},k\widehat{\phi}}\#R_j$ is the Schwartz kernel of $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \mathcal{R}^j$ and thus $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}} \circ \mathcal{R}^j$ is a smoothing operator, for $P_{\widehat{\omega},k\widehat{\phi}}\#R_j$ is smooth. □

Hence, the kernel version of (54) is well-defined:

$$\begin{aligned}
&\widehat{P}_{\omega_0,k\phi_0} + \widehat{P}_{\omega_0,k\phi_0}\#R + \cdots + \widehat{P}_{\omega_0,k\phi_0}\#R_{M-1} + P_{\widehat{\omega},k\widehat{\phi}}\#R_M \\
&\quad = P_{\widehat{\omega},k\widehat{\phi}}, \forall M \in \mathbb{N}. \qquad (58)
\end{aligned}$$

Also, we need the following simple observation. Let $\chi, \widetilde{\chi} \in C_c^\infty(\mathbb{C}^n)$ with

$$\operatorname{supp}\chi \subset B_1(0),\ \operatorname{supp}\widetilde{\chi} \subset B_2(0),\ \widetilde{\chi} = 1 \text{ on } \operatorname{supp}\chi,\ \chi = 1 \text{ on } B_{1/2}(0).$$

and set $\chi_k(z) := \chi(8k^{1/2-\epsilon}z)$ and $\widetilde{\chi}_k(z) := \widetilde{\chi}(8k^{1/2-\epsilon}z)$.

Lemma 15. *For any $a \in \widehat{S}^m(\mathbb{C}^n \times \mathbb{C}^n)$, $\widetilde{\chi}_k(x)a(x,y,k)\chi_k(y) \in \widehat{S}^m(\mathbb{C}^n \times \mathbb{C}^n)$.*

Proof. For any $\alpha, \beta \in \mathbb{N}_0^n$, we estimate

$$|\partial_x^\alpha \partial_y^\beta a(x,y,k)\widetilde{\chi}_k(x)\chi_k(y)| \leq C_{\alpha,\beta} \sum_{\alpha' \leq \alpha, \beta' \leq \beta} |\partial_x^{\alpha'} \widetilde{\chi}_k||\partial_y^{\beta'} \chi_k||\partial_x^{\alpha-\alpha'} \partial_y^{\beta-\beta'} a|.$$

Since $a \in \widehat{S}^m(\mathbb{C}^n \times \mathbb{C}^n)$, for each $\alpha-\alpha'$, $\beta-\beta'$, there exists $l(\alpha-\alpha', \beta-\beta') \in \mathbb{N}$ such that for any $N \in \mathbb{N}$, for any $x, y \in \mathbb{C}^n$, we have

$$\begin{aligned}&|\partial_x^{\alpha-\alpha'} \partial_y^{\beta-\beta'} a(x,y,k)| \\ &\quad \leq C_{\alpha-\alpha',\beta-\beta',N} k^{m+\frac{|\alpha-\alpha'|+|\beta-\beta'|}{2}} \frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l(\alpha-\alpha',\beta-\beta')}}{(1+\sqrt{k}|x-y|)^N}.\end{aligned}$$

On the other hand, we have $|\partial_x^{\alpha'} \widetilde{\chi}_k| \leq C_{\alpha'} k^{|\alpha'|(1/2-\epsilon)}$ and $|\partial_y^{\beta} \chi_k| \leq C_{\beta'} k^{|\beta'|(1/2-\epsilon)}$. Hence, we conclude that

$$\begin{aligned}&|\partial_x^\alpha \partial_y^\beta a(x,y,k)\widetilde{\chi}_k(x)\chi_k(y)| \\ &\quad \leq C_{\alpha,\beta,N} k^{m+\frac{|\alpha|+|\beta|}{2}} \frac{(1+\sqrt{k}|x|+\sqrt{k}|y|)^{l(\alpha,\beta)}}{(1+\sqrt{k}|x-y|)^N}, \quad \forall x,y \in \mathbb{C}^n,\end{aligned}$$

where $l(\alpha, \beta) := \max\{l(\alpha-\alpha', \beta-\beta') : \alpha' \leq \alpha, \beta' \leq \beta\}$. □

We are ready to establish the asymptotic expansion of $P_{\widehat{\omega},k\widehat{\phi}}$ near $(0,0)$.

Theorem 16. *For $\epsilon \in [0, 1/6)$, we have*

$$\widetilde{\chi}_k(x) P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y) \in \widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n \times \mathbb{C}^n),$$

where χ_k, $\widetilde{\chi}_k$ as above.

Proof. We first show that $\widetilde{\chi}_k P_{\widehat{\omega},k\widehat{\phi}} \chi_k \in \widehat{S}^n(\mathbb{C}^n \times \mathbb{C}^n)$. For z, w in a compact set K of $0 \in \mathbb{C}^n$, by standard scaling technique (cf. [27, Theorem 4.3]), one can prove that for any $\alpha \in \mathbb{N}_0^{2n}$, there exists a constant $C = C_K > 0$ such that any $u \in \ker \bar{\partial}_{k\widehat{\phi}}$,

$$|(\partial_x^\alpha u)(z)| \leq C_{\alpha,K} k^{\frac{n+|\alpha|}{2}} \|u\|_{\widehat{\omega}}, \quad \forall z \in K.$$

Let $\{\Psi_j\}_{j=1}^{d_k}$ be an orthonormal basis of $\ker \triangle_{\widehat{\omega},k\widehat{\phi}}$ with respect to $(\cdot|\cdot)_{\widehat{\omega}}$. Fix $\alpha \in \mathbb{N}_0^{2n}$ and $x_0 \in K$, we may assume that $\sum_{j=1}^{d_k} |\partial_x^\alpha \Psi_j(x_0)|^2 \neq 0$. We then set

$$u(z) := \frac{\sum_{j=1}^{d_k} \overline{(\partial_x^\alpha \Psi_j)(x_0)} \Psi_j(z)}{\left(\sum_{j=1}^{d_k} |\partial_x^\alpha \Psi_j(x_0)|^2\right)^{\frac{1}{2}}}.$$

Since $P_{\widehat{\omega},k\widehat{\phi}}(z,w) = \sum_{j=1}^{d_k} \Psi_j(z)\overline{\Psi_j(w)}$ is smooth, the sum $\sum_{j=1}^{d_k} |\partial_x^\alpha \Psi_j(x_0)|^2$ converges, and thus $u \in \ker \triangle_{\omega_0,k\widehat{\phi}}$ and $\|u\|_0^2 = 1$. By above argument, there exists a constant $C_{\alpha,K}$ so that

$$\left(\sum_{j=1}^{d_k} |\partial_x^\alpha \Psi_j(x_0)|^2\right)^{\frac{1}{2}} = |(\partial_x^\alpha u)(x_0)| \leq C_{\alpha,K} k^{\frac{n+|\alpha|}{2}}.$$

Since $|\partial_x^\alpha \partial_y^\beta (P_{\widehat{\omega},k\widehat{\phi}})(x_0,x_0)|$ is dominated by

$$\left(\sum_j |\partial_x^\alpha \Psi_j(x_0)|^2\right)^{1/2} \left(\sum_j |\partial_y^\beta \Psi_j(x_0)|^2\right)^{1/2},$$

$$|\partial_x^\alpha \partial_y^\beta (P_{\widehat{\omega},k\widehat{\phi}})(x_0,x_0)| \leq C_{\alpha,\beta,K} k^{n+\frac{|\alpha|+|\beta}{2}}, \tag{59}$$

and the same estimates holds for any $z \in K$ with the same constant $C_{\alpha,K}$.

Now, for off-diagonal estimates, we notice that $|z| < \frac{1}{4}k^{-1/2+\epsilon}$, $|w| < \frac{1}{8}k^{-1/2+\epsilon}$. Therefore, $|z-w| < \frac{5}{8}k^{-1/2+\epsilon}$. For any $M \in \mathbb{N}$, we now multiply $\widetilde{\chi}_k(x)\chi_k(y)$ on the both sides of (58):

$$\begin{aligned}&\widetilde{\chi}_k(x) P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y)\\ &\quad= \widetilde{\chi}_k(x)(P_{\widehat{\omega},k\widehat{\phi}} \# R_M)(x,y)\chi_k(y) + \sum_{j=0}^{M-1} \widetilde{\chi}_k(x)\widehat{P}_{\omega_0,k\phi_0} \# R_j(x,y)\chi_k(y).\end{aligned}$$

By Lemma 15, we know that $\widetilde{\chi}_k \widehat{P}_{\omega_0,k\phi_0} \# R_j \chi_k \in \widehat{S}^{n-j/2}(\mathbb{C}^n \times \mathbb{C}^n)$. Hence, given any $N \in \mathbb{N}$, to estimate $(1+\sqrt{k}|z-w|)^N |\widetilde{\chi}_k P_{\widehat{\omega},k\widehat{\phi}} \chi_k|$, it remains to estimate $(1+\sqrt{k}|z-w|)^N |\widetilde{\chi}_k P_{\widehat{\omega},k\widehat{\phi}} \# R_M \chi_k|$. We observe that

$$\begin{aligned}&(1+\sqrt{k}|z-w|)^N \widetilde{\chi}_k(z) \left| \int_{\mathbb{C}^n} P_{\widehat{\omega},k\widehat{\phi}}(z,u) R_M(u,w) dm(u) \right| \chi_k(w)\\ &\leq \left(1+\frac{5}{8}k^\epsilon\right)^N C_L k^{n-M/2} \int_{\mathbb{C}^n} |P_{\widehat{\omega},k\widehat{\phi}}(z,u)| \frac{(1+\sqrt{k}|u|+\sqrt{k}|w|)^l}{(1+\sqrt{k}|u-w|)^L} dm(u)\end{aligned}$$

$$\leq\Big(1+\frac{5}{8}k^{\epsilon}\Big)^{N}C_Lk^{n-M/2}|P_{\widehat{\omega},k\widehat{\phi}}(z,z)|^{1/2}\Big(\int_{\mathbb{C}^n}\frac{(1+\sqrt{k}|u|+\sqrt{k}|w|)^{2l}}{(1+\sqrt{k}|u-w|)^{2L}}dm(u)\Big)^{1/2}.$$

By above on-diagonal estimates, we know that $|P_{\widehat{\omega},k\widehat{\phi}}(z,z)|^{1/2}\lesssim k^{n/2}$. Now, as in the proof of Theorem 10, we make the change of variable $t=\sqrt{k}u-\sqrt{k}w$:

$$\int_{\mathbb{C}^n}\frac{(1+\sqrt{k}|u|+\sqrt{k}|w|)^{2l}}{(1+\sqrt{k}|u-w|)^{2L}}dm(u)\leq k^{-n}\int_{\mathbb{C}^n}\frac{(1+|t|+2\sqrt{k}|w|)^{2l}}{(1+|t|)^{2L}}dm(t).$$

By choosing $L=l+2n$, we get

$$\begin{aligned}&(1+\sqrt{k}|z-w|)^{N}\widetilde{\chi}_k(z)\Big|(P_{\widehat{\omega},k\widehat{\phi}}\#R_M)(z,w)\Big|\chi_k(w)\\ &\quad\leq C_{n,l}(1+\frac{5}{8}k^{\epsilon})^{N}k^{n-M/2}(1+\sqrt{k}|w|)^{l}.\end{aligned}$$

If we choose $M>2\epsilon N$, then $k^{-M/2}(1+\frac{5}{8}k^{\epsilon})^{N}\leq 2^{N}$. Similar estimate works for $(1+\sqrt{k}|z-w|)^{N}|\partial_x^{\alpha}\partial_y^{\beta}(\widetilde{\chi}_k(P_{\widehat{\omega},k\widehat{\phi}}\#R_M)\chi_k)|$ with the same M but now l may depends on $\alpha,\beta\in\mathbb{N}_0^{2n}$. Hence, we conclude that $\widetilde{\chi}_kP_{\widehat{\omega},k\widehat{\phi}}\chi_k\in\widehat{S}^{n}(\mathbb{C}^n\times\mathbb{C}^n)$.

Finally, we show that $\widetilde{\chi}_kP_{\widehat{\omega},k\widehat{\phi}}\chi_k\in\widehat{S}^{n}_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$. To see this, for any $\alpha,\beta\in\mathbb{N}_0^{2n}$, any $M,N\in\mathbb{N}$, above argument shows that one can find $M'>M$ so that

$$\begin{aligned}&(1+\sqrt{k}|z-w|)^{N}\Big|\partial_x^{\alpha}\partial_y^{\beta}\widetilde{\chi}_k(x)(P_{\widehat{\omega},k\widehat{\phi}}\#R_{M'})(x,y)\chi_k(y)\Big|\\ \leq&C_{\alpha,\beta,N}k^{n-\frac{M}{2}}(1+\sqrt{k}|z|+\sqrt{k}|w|)^{l(\alpha,\beta)}.\end{aligned}$$

Therefore, by Lemma 15 and above estimate,

$$\begin{aligned}&\widetilde{\chi}_k(x)P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y)-\sum_{j=0}^{M}\widetilde{\chi}_k(x)\widehat{P}_{\omega_0,k\phi_0}\#R_j(x,y)\chi_k(y)\\ &=\chi_k(x)(P_{\widehat{\omega},k\widehat{\phi}}\#R_{M'})(x,y)\chi_k(y)+\sum_{j=M+1}^{M'-1}\widetilde{\chi}_k(x)\widehat{P}_{\omega_0,k\phi_0}\#R_j(x,y)\chi_k(y)\end{aligned}$$

is in $\widehat{S}^{n-M/2}(\mathbb{C}^n \times \mathbb{C}^n)$. In view of Theorem 9, we conclude that

$$\widetilde{\chi}_k(x)P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y) \sim \sum_{j=0}^{\infty} \widetilde{\chi}_k(x)\widehat{P}_{\omega_0,k\phi_0}\#R_j(x,y)\chi_k(y)$$

and thus $\widetilde{\chi}_k(x)P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y) \in \widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n \times \mathbb{C}^n)$. □

3.5. Localization of global Bergman kernel

In this section, we complete the proof of Theorem 1 by localizing global Bergman kernel to the approximate Bergman kernel whose asymptotic expansion is already established in section 3.4.

Our goal is to establish the relation between $\Pi_{k,s}$ and $\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}$. To achieve this, we need to modify approximate Bergman kernel to a kernel defined on U. First, we consider a sequence of bump functions $\{\psi_i\}_{i=1}^{\infty} \subset C_c^{\infty}(U,[0,1])$ such that for any compact set $K \subset U$, $K \cap \operatorname{supp}\psi_i \neq \emptyset$, for only finitely many i, and $\sum_{i=1}^{\infty}\psi_i = 1$ on U, and we define

$$\eta(z,w) := \sum_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset} \psi_i(z)\psi_j(w) \tag{60}$$

Lemma 17. *η is smooth and $\eta \equiv 1$ on a neighborhood Ω of the diagonal $\Delta_U \subset U \times U$. Furthermore, the projection $\operatorname{supp}\eta \to U$ on both z and w directions are proper maps.*

Proof. Clearly, it suffices to prove that η is smooth on a neighborhood of any $(x_0,y_0) \in U \times U$. For any $(x_0,y_0) \in U \times U$, any neighborhoods $W, W' \Subset U$ of x_0 and y_0, respectively. By construction of $\{\psi_i\}$, we know that there exist only finitely many $i,j \in \mathbb{N}$ such that $\operatorname{supp}\psi_i \cap W \neq \emptyset$ and $\operatorname{supp}\psi_j \cap W' \neq \emptyset$. Therefore, the sum in (60) is a finite sum on $W \times W'$ and thus $\eta \in C^{\infty}(W \times W')$. For the second assertion, observe that

$$\begin{aligned} 1-\eta(z,w) &= \sum_{i=1}^{\infty}\psi_i(z) - \chi(z,w) = \sum_{i=1}^{\infty}\psi_i(z)\left(\sum_{j=1}^{\infty}\psi_j(w)\right) - \chi(z,w) \\ &= \sum_{i,j=1}^{\infty}\psi_i(z)\psi_j(w) - \sum_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset}\psi_i(z)\psi_j(w) \end{aligned}$$

$$= \sum_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset} \psi_i(z)\psi_j(w).$$

If $1-\eta(z,w)\neq 0$, then $\psi_i(z)\neq 0$ and $\psi_j(w)\neq 0$, for some pair (i,j) with $\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j=\emptyset$. We know that for such pair (i,j), $z\in\operatorname{supp}\psi_i$ and $w\in\operatorname{supp}\psi_j$ and thus $(z,w)\in\operatorname{supp}\psi_i\times\operatorname{supp}\psi_j$. This shows that

$$\operatorname{supp}(1-\eta) = \bigcup_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset} \operatorname{supp}\psi_i \times \operatorname{supp}\psi_j,$$

and thus the intersection of $\operatorname{supp}(1-\eta)$ with the diagonal of $U\times U$ is empty. Furthermore, for each $z_0\in V$, if $z_0\in\operatorname{supp}\psi_{i_0}$, then there exists only finitely many $j_1,\ldots,j_N$ such that $\operatorname{supp}\psi_{i_0}\cap\operatorname{supp}\psi_{j_l}\neq\emptyset$, for $l=1,\ldots,N$. Thus, we can pick a neighborhood W of z_0 such that $W\cap\bigcup_{l=1}^N\operatorname{supp}\psi_{j_l}=\emptyset$. This shows that $(W\times W)\cap\operatorname{supp}(1-\eta)=\emptyset$. As a result, $\eta\equiv 1$ on an open neighborhood Ω of the diagonal.

Finally, for each compact set $K\subset U$, the pre-image of it under the first projection is then given by $(K\times U)\cap\operatorname{supp}\eta$. Since there exists only finitely many index $i_1,\ldots,i_N$ such that $\operatorname{supp}\psi_{i_l}\cap K\neq\emptyset$, for $l=1,\ldots,N$, and for each $l=1,\ldots,N$, there exists only finitely many $j_{l,m}$, for $m=1,\ldots,M$, such that $\operatorname{supp}\psi_{i_l}\cap\operatorname{supp}\psi_{j_{l,m}}\neq\emptyset$. This shows that

$$\begin{aligned}\operatorname{supp}\eta\cap(K\times U) &= (K\times U)\cap \bigcup_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset} \operatorname{supp}\psi_i \times \operatorname{supp}\psi_j \\ &= \bigcup_{l=1}^{N}\operatorname{supp}\psi_{i_l} \times \bigcup_{m=1}^{M}\operatorname{supp}\psi_{j_{l,m}},\end{aligned}$$

and thus $\operatorname{supp}\eta\cap(K\times U)$ is compact. The proof for second projection is the same. □

We define *localized approximate projection* $\widehat{\Pi}_k : L^2_{\text{comp}}(U,\omega)\to L^2(U,\omega)$ by

$$(\widehat{\Pi}_k u)(z) = \int_U P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)u(w)\omega_n(w) \tag{61}$$

whose Schwartz kernel is given by $\widehat{K}_k(z,w)=P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)$. By Lemma 17, we know that the projections $(z,w)\in\operatorname{supp}\eta\to w\in U$ is proper, and thus $\eta(z,w)u(w)$ also has compact support in z. This shows that

$\widehat{\Pi}_k : L^2_{\rm comp}(U,\omega) \to L^2_{\rm comp}(U,\omega)$. On the other hand, by the properness of $(z,w) \in \operatorname{supp}\eta \to w \in U$, we know that for $u \in L^2(U,\omega)$, any $\tau \in C^\infty_c(U)$, $\tau(z)(\widehat{\Pi}_k u)(z) \in L^2(U,\omega)$. This implies that $\widehat{\Pi}_k : L^2(U,\omega) \to L^2_{\rm loc}(U,\omega)$.

On the other hand, we also define *localized approximate projection* $\widetilde{\Pi}_k$: $L^2(U,\omega) \to L^2_{\rm comp}(U,\omega)$ *concentrated near origin* by

$$(\widetilde{\Pi}_k u)(z) := \widetilde{\chi}_k(z)(\widehat{\Pi}_k(\chi_k u))(z), \tag{62}$$

where $\widetilde{\chi}_k(z) = \widetilde{\chi}(8k^{1/2-\epsilon}z)$, $\chi_k(z) = \chi(8k^{1/2-\epsilon}z)$ and

$$\operatorname{supp}\chi_k \subset B_1(0), \quad \operatorname{supp}\widetilde{\chi} \subset B_2(0), \quad \widetilde{\chi} = 1 \text{ on } \operatorname{supp}\chi,\ \chi = 1 \text{ on } B_{1/2}(0).$$

By construction, we know that $\operatorname{supp}\widetilde{\chi}_k, \operatorname{supp}\chi_k \subset V_k$. We denote $\widetilde{K}_k(z,w)$ by the Schwartz kernel of $\widetilde{\Pi}_k$. Therefore, we have

$$\widetilde{K}_k(z,w) = \widetilde{\chi}_k(z)\widehat{K}_k(z,w)\chi_k(w) = \widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w).$$

We first prove a crucial result which is important in our later arguments.

Theorem 18. *For $\epsilon \in (0,1/6)$, $(1-\widetilde{\chi}_k)\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}\chi_k$ is a k-negligible operator in the sense of Definition 3.*

In view of Definition 3, it suffices to prove that for any $l, N \in \mathbb{N}$, any compact set $K \subset \mathbb{C}^n \times \mathbb{C}^n$, $\|(1-\widetilde{\chi}_k(x))P_{\widehat{\omega},k\widehat{\phi}}(x,y)\chi_k(y)\|_{C^l(K)} \le C_{N,l}k^{-N}$, for some constant $C_{N,l} > 0$ independent of k. We first prove a lemma.

Lemma 19. *For any $m \in \mathbb{R}$, $a \in \widehat{S}^m(\mathbb{C}^n \times \mathbb{C}^n)$,*

$$(1-\widetilde{\chi}_k(z))P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) \equiv 0 \bmod O(k^{-\infty}).$$

Proof. Notice that $(1-\widetilde{\chi}_k)a\chi_k$ supports in $(\operatorname{supp}\chi_k)^c \times \operatorname{supp}\chi_k$. Since $\delta := d(\operatorname{supp}(1-\widetilde{\chi}), \operatorname{supp}\chi) > 0$, we only need to consider

$$|w| < \frac{1}{8}k^{-1/2+\epsilon}, \quad |z| > \frac{1}{8}k^{-1/2+\epsilon}, \quad \text{and } |z-w| \ge \frac{1}{8}k^{-\frac{1}{2}+\epsilon}\delta.$$

Now, given any $\alpha, \beta \in \mathbb{N}_0^{2n}$, any $L \in \mathbb{N}$, since $a \in \widehat{S}^m(\mathbb{C}^n \times \mathbb{C}^n)$, we have the following esimtate

$$\left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k)a\chi_k\right| \le C_{\alpha,\beta,L}k^{m+\frac{|\alpha|+|\beta|}{2}-\epsilon(|\alpha|+|\beta|)}\frac{(1+\sqrt{k}|z|+\sqrt{k}|w|)^{l(\alpha,\beta)}}{(1+\sqrt{k}|z-w|)^L}.$$

Now, given any compact set $K \subset \mathbb{C}^n \times \mathbb{C}^n$ with $(\operatorname{supp} \chi_k)^c \times \operatorname{supp} \chi_k$, we may assume $\frac{1}{8}k^{-1/2+\epsilon} < |z| \leq R$, for some $R := R_K > 0$ depending on K. Hence, we get

$$\begin{aligned}\left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k) a \chi_k\right| &\leq C_{\alpha,\beta,L} k^{m+\frac{|\alpha|+|\beta|}{2}} \frac{(1+|k|^\epsilon + \sqrt{k}R)^{l(\alpha,\beta)}}{(1+8k^\epsilon\delta)^L} \\ &\leq C_{\alpha,\beta,N} k^{m+\frac{|\alpha|+|\beta|+l(\alpha,\beta)}{2}} k^{-L\epsilon}(8\delta)^{-L} R^{l(\alpha,\beta)} \\ &\leq C_{\alpha,\beta,L,K}(8\delta)^{-L} k^{m+\frac{|\alpha|+|\beta|+l(\alpha,\beta)}{2}-\epsilon L}.\end{aligned}$$

For any $N \in \mathbb{N}$, we choose $L > \frac{m+\frac{|\alpha|+|\beta|+l(\alpha,\beta)}{2}+N}{2\epsilon}$. Therefore, we see that

$$\left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k) a \chi_k\right| \leq C_{\alpha,\beta,m,N,\epsilon,K} k^{-N}. \qquad \square$$

***Proof of Theorem* 18.** Now, if we multiply (58) by $(1-\widetilde{\chi}_k)(z)\chi_k(w)$, then by above Lemma,

$$(1-\widetilde{\chi}_k)(z)\widehat{P}_{\omega_0,k\phi_0} \# R_j \chi_k \equiv 0 \mod O(k^{-\infty}),$$

for any $j \in \mathbb{N}$. Similar to Lemma 14, we estimate

$$\begin{aligned}&|(1-\widetilde{\chi}_k(z))|\left|\int_{\mathbb{C}^n} P_{\widehat{\omega},k\widehat{\phi}}(z,u) R_M(u,w) dm(u)\right| \chi_k(w) \\ \leq& |1-\widetilde{\chi}_k(z)| P_{\widehat{\omega},k\widehat{\phi}}(z,z)|^{1/2} C_L k^{n-M/2} \Big(\int_{\mathbb{C}^n} \frac{(1+|t|+2\sqrt{k}|w|)^{2l}}{(1+|t|)^{2L}}\Big)^{1/2} |\chi_k(w)| \\ \leq& C_{L,K} k^{n-\frac{M}{2}+\frac{l}{2}} \leq C_{n,l,K} k^{-N},\end{aligned}$$

where we choose $L = L_0(n,l)$ so that the integral converges and $M > 2n + l + 2N$, for any $N \in \mathbb{N}$. The derivatives estimates proceeds in similar fashion but l may depends on the degree of differentiation. In conclusion, for any $N \in \mathbb{N}$, any $\alpha, \beta \in \mathbb{N}_0^{2n}$, there exists $M > 2n + l(\alpha,\beta) + 2N \in \mathbb{N}$ and $C_{n,\alpha,\beta,K} > 0$ independent of k such that

$$\sup_K \left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k) P_{\widehat{\omega},k\widehat{\phi}} \# R_M \chi_k\right| \leq C_{n,\alpha,\beta,K} k^{-N}.$$

Hence, given any $\alpha, \beta \in \mathbb{N}_0^{2n}$, any $N \in \mathbb{N}$, we choose $M > 2n + l(\alpha,\beta,2N)$

so that

$$\begin{aligned}
&\sup_K \left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k(z))P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w)\right| \\
&\le \sum_{j=0}^{M-1} \sup_K \left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k)(z)(\widehat{P}_{\omega_0,k\phi_0}\#R_j)(z,w)\chi_k(w)\right| \\
&\quad + \sup_K \left|\partial_x^\alpha \partial_y^\beta (1-\widetilde{\chi}_k)(P_{\widehat{\omega},k\widehat{\phi}}\#R_M)(z,w)\chi_k(w)\right| \\
&\le C_{\alpha,\beta,K,N} k^{-N}.
\end{aligned}$$

□

Remark 3. Notice that the condition on Lemma 19 can be relaxed. We actually proved that $\chi_k P_{\widehat{\omega},k\widehat{\phi}}\tau_k \equiv 0 \bmod O(k^{-\infty})$, for $\chi_k \in C_c^\infty(\mathbb{C}^n,[0,1])$, $\tau_k \in C^\infty(\mathbb{C}^n,[0,1])$ with $d(\operatorname{supp}\chi_k, \operatorname{supp}\tau_k) > \frac{1}{8}k^{-1/2+\epsilon}\delta$, for some $\delta > 0$ independent of k. Particularly, we can exchange the role of $1-\widetilde{\chi}_k$ and χ_k. Also, by Theorem 8, we see that

$$(1-\widetilde{\chi}_k)\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}\chi_k = -(1-\widetilde{\chi}_k)\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}}\chi_k \equiv 0 \bmod O(k^{-\infty}). \quad (63)$$

Theorem 20. *$\Box^{(0)}_{k,s}\widetilde{\Pi}_k$ is k-negligible, i.e., $\Box^{(0)}_{k,s}\widetilde{\Pi}_k \equiv 0 \bmod O(k^{-\infty})$ on U.*

Proof. It suffices to prove $\bar{\partial}_{k,s}\widetilde{\Pi}_k \equiv 0 \bmod O(k^{-\infty})$ on $L^2(U,\omega)$. In view of Definition 3, it remains to prove that

$$\bar{\partial}_{k,s}\left(\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w)\right) \equiv 0 \bmod O(k^{-\infty}), \qquad \text{on } U.$$

To see this, we write

$$\begin{aligned}
&\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w) \\
&=\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) + (1-\eta(z,w))\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w)
\end{aligned}$$

For the latter term, from the proof of Lemma 17, we see that

$$\begin{aligned}
&(1-\eta(z,w))\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) \\
&= \sum_{\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset} \psi_i(z)\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w)\psi_j(w).
\end{aligned}$$

Now, notice that the proof of Theorem 18 works for $\psi_i(z)\widetilde{\chi}_k(z)$ and

$\psi_j(w)\chi_k(w)$ (cf. Remark 3), and thus we see that

$$(1-\eta(z,w))\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) \equiv 0 \operatorname{mod} O(k^{-\infty}).$$

For the first term,

$$\begin{aligned}&\bar{\partial}_{k,s}(\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w))\\ &= (\bar{\partial}+k\bar{\partial}\phi)(\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w))\\ &= (\bar{\partial}\widetilde{\chi}_k)(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) + \widetilde{\chi}_k(z)\bar{\partial}_{k,s}(P_{\widehat{\omega},k\widehat{\phi}}(z,w))\chi_k(w).\end{aligned}$$

Since $\chi_k(w)$ supports in V_k and $\omega=\widehat{\omega}$, $\widehat{\phi}=\phi$ on V_k, we have $\bar{\partial}_{k,s}(P_{\widehat{\omega},k\widehat{\phi}}(z,w))=0$. On the other hand, Theorem 18 (cf. again Remark 3) shows that

$$(\bar{\partial}\widetilde{\chi}_k)(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) \equiv 0 \operatorname{mod} O(k^{-\infty}). \qquad \square$$

Since $\Pi_{k,s}: L^2_{\text{comp}}(U,\omega) \to L^2(U,\omega)$ and $\widetilde{\Pi}_k : L^2(U,\omega) \to L^2_{\text{comp}}(U,\omega)$, the composition $\Pi_k \circ \widetilde{\Pi}_{k,s} : L^2(U) \to L^2(U)$ makes sense. Recall that if the local spectral gap condition (3) holds on an open set $U \subset X(0)$. By local unitary identification in section 3.1, we have

$$\|(I-\Pi_{k,s})u\|_\omega \le \frac{1}{Ck^d}\|\Box^{(0)}_{k,s}u\|_\omega, \quad u \in C^\infty_c(U). \tag{64}$$

Hence, we can prove

Theorem 21. *If the local spectral gap condition* (3) *holds on an open set* $U \subset X(0)$, *then the operator* $\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k$ *is* k*-negligible on* U.

Proof. For any $u \in C^\infty_c(U)$, we have the following estimate for L^2-norm.

$$\|(\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k)u\|_\omega = \|(\Pi_{k,s}-I)\widetilde{\Pi}_k u\|_\omega \le C^{-1}k^{-d}\|\Box^{(0)}_{k,s}\widetilde{\Pi}_k u\|_\omega.$$

Using Theorem 20, $\Box^{(0)}_{k,s}\widetilde{\Pi}_k \equiv 0 \operatorname{mod} O(k^{-\infty})$. Thus, for any $N>0$, there exists a k-independent constant $C := C_{M,N} > 0$ so that $\|\Box^{(0)}_{k,s}\widetilde{\Pi}_k u\|_\omega. \le Ck^{-N}\|u\|_{W^{-M},\omega}$, for any $M>0$. Also, notice that $\Box^{(0)}_{k,s}(\widetilde{\Pi}_k - \Pi_{k,s}\widetilde{\Pi}_k) = \Box^{(0)}_{k,s}\widetilde{\Pi}_k \equiv 0 \operatorname{mod} O(k^{-\infty})$. By elliptic estimate, for any $u \in C^\infty_c(U)$, we know that there exists a constant $C>0$ independent of k and $l>0$ so that for any $m \in \mathbb{N}$, any $N \in \mathbb{N}$,

$$\begin{aligned}
&\|(\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k)u\|_{W^{2m},\omega} \\
&\quad \leq Ck^{lm}(\|(\Box^{(0)}_{k,s})^m(\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k)u\|_\omega + \|(\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k)u\|_\omega) \\
&\quad \lesssim k^{lm}(\|(\Box^{(0)}_{k,s})^m\widetilde{\Pi}_k u\|_\omega + k^{-d}\|\Box^{(0)}_{k,s}\widetilde{\Pi}_k u\|_\omega) \lesssim k^{-N}\|u\|_{W^{-2m},\omega},
\end{aligned}$$

for coefficient of $\Box^{(0)}_{k,s}$ has at most polynomial growth in k. By density argument, above estimate holds for any $u \in L^2(U,\omega)$ and thus

$$\Pi_{k,s}\widetilde{\Pi}_k - \widetilde{\Pi}_k : W^{-2m}(U,\omega) \to W^{2m}(U,\omega)$$

has operator norm $O(k^{-N})$, for any $m \in \mathbb{N}$ and $N \in \mathbb{N}$. We conclude that $\Pi_{k,s}\widetilde{\Pi}_k \equiv \widetilde{\Pi}_k \bmod O(k^{-\infty})$. □

On the other hand, we prove

Theorem 22. *The operator $\chi_k\widehat{\Pi}_k\widetilde{\chi}_k\Pi_{k,s} - \chi_k\Pi_{k,s}$ is k-negligible on U.*

An asymptotic upper bound of $\Pi_{k,s}$ is needed for Theorem 22.

Lemma 23. *For any $\alpha,\beta \in \mathbb{N}_0^{2n}$, there exists a constant $C_{\alpha,\beta,U} > 0$ so that*

$$\left|\partial_x^\alpha\partial_y^\beta(K_{k,s})(x',y')\right| \leq C_{\alpha,\beta,U}k^{n+\frac{|\alpha|+|\beta}{2}}, \quad \forall x',y' \in U.$$

Proof. This proceeds similar to the proof of (59) in the first part of the proof of Theorem 16. □

Proof of Theorem 22. Since $\chi_k, \widetilde{\chi}_k$ supports in V_k and $\widehat{\omega} = \omega$, $\widehat{\phi} = \phi$ on V_k, by Thoerem 8, for any $u \in L^2_{\text{comp}}(U,\omega)$, we can write

$$\begin{aligned}
(\chi_k\widehat{\Pi}_k\widetilde{\chi}_k\Pi_{k,s}u)(z) &= \sum_{\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j\neq\emptyset} \chi_k(z)\psi_i(z)(\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(\psi_j\widetilde{\chi}_k\Pi_{k,s}u))(z) \\
&= -\sum_{\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j=\emptyset} \chi_k(z)\psi_i(z)(\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(\psi_j\widetilde{\chi}_k\Pi_{k,s}u))(z) + \chi_k(z)\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(\widetilde{\chi}_k\Pi_{k,s}u)(z) \\
&= -\sum_{\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j=\emptyset} \chi_k(z)\psi_i(z)(\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(\psi_j\widetilde{\chi}_k\Pi_{k,s}u))(z) \\
&\qquad + \chi_k(z)(I - \bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\bar{\partial}_{k\widehat{\phi}})(\widetilde{\chi}_k\Pi_{k,s}u)(z) \\
&= \chi_k(z)(\Pi_{k,s}u)(z) - \sum_{\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j=\emptyset} \chi_k(z)\psi_i(z)(\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}(\psi_j\widetilde{\chi}_k\Pi_{k,s}u))(z)
\end{aligned}$$

$$-\chi_k(z)\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}(\bar{\partial}\widetilde{\chi}_k)\Pi_{k,s}u)(z)-\chi_k(z)\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\widetilde{\chi}_k(\bar{\partial}_{k\widehat{\phi}}\Pi_{k,s}u)(z).$$

Applying Theorem 18 and Remark 3 to the last two terms, we have

$$\sum_{\operatorname{supp}\psi_i\cap\operatorname{supp}\psi_j=\emptyset}\psi_i\mathcal{P}_{\widehat{\omega},k\widehat{\phi}}\psi_j\equiv 0 \bmod O(k^{-\infty}),$$
$$\chi_k\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}(\bar{\partial}\widetilde{\chi}_k)\equiv 0 \bmod O(k^{-\infty}).$$

Also, $\widehat{\phi}=\phi$ on V_k, $\chi_k(z)\bar{\partial}^*_{k\widehat{\phi}}(\triangle^{(1)}_{\widehat{\omega},k\widehat{\phi}})^{-1}\widetilde{\chi}_k(\bar{\partial}_{k\phi}\Pi_{k,s}u)(z)=0$.

Combining with Lemma 23 which shows that derivatives of $K_{k,s}$ is at most polynomials in k, we see that

$$\chi_k(z)\Pi_{k,s}\equiv\chi_k\widehat{\Pi}_k\widetilde{\chi}_k\Pi_{k,s} \bmod O(k^{-\infty}). \qquad \square$$

***Proof of Theorem* 1.** If we take adjoint in Theorem 22, we get

$$\Pi_{k,s}\chi_k\equiv\Pi_{k,s}\widetilde{\chi_k}\widehat{\Pi}_k^{*,\omega}\chi_k \bmod O(k^{-\infty}),$$

where $\widehat{\Pi}_k^{*,\omega}$ means the adjoint with respect to ω. As in the proof of Theorem 21 , since $\chi_k,\widetilde{\chi}_k$ supports in V_k and $\widehat{\omega}=\omega$, $\widehat{\phi}=\phi$ on V_k, we see that

$$\widetilde{\chi_k}\widehat{\Pi}_k^{*,\omega}\chi_k=\widetilde{\chi}_k\widehat{\Pi}_k^{*,\widehat{\omega}}\chi_k=\widetilde{\chi_k}\widehat{\Pi}_k\chi_k.$$

The last identity follows from $P_{\widehat{\omega},k\widehat{\phi}}$ is self-adjoint (with respect to $\widehat{\omega}$), $\eta(z,w)=\overline{\eta(w,z)}=\eta(w,z)$, and $\widehat{K}_k(z,w)=P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)$. By Theorem 21 and the assumption that U satisfies local spectral gap, we conclude that

$$\Pi_{k,s}\chi_k\equiv\Pi_{k,s}\widetilde{\Pi}_k\equiv\widetilde{\Pi}_k \bmod O(k^{-\infty}).$$

In terms of kernels, this shows that

$$K_{k,s}(z,w)\chi_k(w)\equiv\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w) \bmod O(k^{-\infty}).$$

Since $\eta(z,w)$ is k-independent, by multiplying η to (58), we see that

$$\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w)\in\widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n\times\mathbb{C}^n)$$

and

$$\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w) \sim \sum_{j=0}^{\infty} \widetilde{\chi}_k(z)(\widehat{P}_{\omega_0,k\phi_0}\#R^{\#j})(z,w)\eta(z,w)\chi_k(w).$$

Finally, by multiplying ρ on above, we have

$$\rho(z)\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w) \equiv \rho(z)K_{k,s}(z,w)\chi_k(w) \bmod O(k^{-\infty}).$$

We see that

$$\rho(z)K_{k,s}(z,w)\chi_k(w) \in \widehat{S}^n_{\mathrm{cl}}(\mathbb{C}^n \times \mathbb{C}^n)$$

as it is supported in $U \times V_k$. □

We also deduce Theorem 2.

Proof of Theorem 2. From the proof of Theorem 16, we know that

$$\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w) \sim \sum_{j=0}^{\infty} \widetilde{\chi}_k(z)(\widehat{P}_{\omega_0,k\phi_0}\#R^{\#j})(z,w)\chi_k(w)$$

and by Theorem 18, $\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\eta(z,w)\chi_k(w) \equiv P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w)$. Hence,

$$\rho(z)K_{k,s}(z,w)\chi_k(w) \sim \rho(z)\widetilde{\chi}_k(z)P_{\widehat{\omega},k\widehat{\phi}}(z,w)\chi_k(w).$$

The first coefficient in the asymptotic sum is given by

$$\widehat{P}_{\omega_0,k\phi_0}(z,w) = \frac{2^n k^n \lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{k\sum_{j=1}^{n}\lambda_{j,x}(2z^j\overline{w}^j - |z^j|^2 - |w^j|^2) - k(\phi_1(z)-\phi_1(w))}.$$

By (55), we know that $1 - e^{-k(\phi_1(z)-\phi_1(w))}$ is of lower degree in k, we get

$$a_0(z,w) = \frac{2^n \lambda_{1,x}\cdots\lambda_{n,x}}{\pi^n} e^{\sum_{j=1}^{n}\lambda_{j,x}(2z^j\overline{w}^j - |z^j|^2 - |w^j|^2)}. \quad \square$$

References

1. Robert Berman, Bo Berndtsson, and Johannes Sjöstrand, A direct approach to Bergman kernel asymptotics for positive line bundles, *Ark. Mat.*, **46** (2008), No.2, 197-217.

2. L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szego. In *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, Astérisque, No. 34-35, pages 123-164. Société Mathématique de France, Paris, 1976.

3. Stefan Bergmann, Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen. *Math. Ann.*, **86** (1922), No.3-4, 238-271.

4. Martin Bordemann, Eckhard Meinrenken, and Martin Schlichenmaier, Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits, *Comm. Math. Phys.*, **165**(1994), No.2, 281-296.

5. Thierry Bouche, Convergence de la métrique de Fubini-Study d'un fibré linéaire positif, *Ann. Inst. Fourier (Grenoble)*, **40** (1990), No.1, 117-130.

6. Thierry Bouche, Two vanishing theorems for holomorphic vector bundles of mixed sign, *Math. Z.*, **218** (1995), No.4, 519-526.

7. Robert Berman and Johannes Sjöstrand, Asymptotics for Bergman-Hodge kernels for high powers of complex line bundles, *Ann. Fac. Sci. Toulouse Math. (6)*, **16** (2007), No.4, 719-771.

8. David Catlin, The Bergman kernel and a theorem of Tian. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 1-23. Birkhäuser Boston, Boston, MA, 1999.

9. Xiuxiong Chen, Simon Donaldson, and Song Sun, Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities, *J. Amer. Math. Soc.*, **28** (2015), No.1, 183-197.

10. Xiuxiong Chen, Simon Donaldson, and Song Sun, Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof, *J. Amer. Math. Soc.*, **28** (2015), No.1, 235-278.

11. So-Chin Chen and Mei-Chi Shaw, *Partial differential equations in several complex variables*, volume 19 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

12. E. B. Davies, *Spectral theory and differential operators*, volume 42 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

13. Jean-Pierre Demailly, Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète, *Ann. Sci. École Norm. Sup. (4)*, **15** (1982), No.3, 457-511.

14. Jean-Pierre Demailly, Complex analytic and algebraic geometry, `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`, 2012.

15. Michael R. Douglas and Semyon Klevtsov, Bergman kernel from path integral, *Comm. Math. Phys.*, **293** (2010), No.1, 205-230.

16. Xianzhe Dai, Kefeng Liu, and Xiaonan Ma, On the asymptotic expansion of Bergman kernel, *J. Differential Geom.*, **72** (2006), No.1, 1-41.

17. S. K. Donaldson, Scalar curvature and projective embeddings. I, *J. Differential Geom.*, **59**(2001), No.3, 479-522.

18. Harold Donnelly, Spectral theory for tensor products of Hermitian holomorphic line bundles, *Math. Z.*, **245** (2003), No.1, 31-35.

19. Simon Donaldson and Song Sun, Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, *Acta Math.*, **213** (2014), No.1, 63-106.

20. G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex.* Annals of Mathematics Studies, No. 75. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972.

21. Matthew P. Gaffney, Hilbert space methods in the theory of harmonic integrals, *Trans. Amer. Math. Soc.*, **78** (1955), 426-444.

22. Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

23. Lars Hörmander, L^2 estimates and existence theorems for the $\bar{\partial}$ operator, *Acta Math.*, **113** (1965), 89-152.

24. Lars Hörmander, *An introduction to complex analysis in several variables*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

25. Lars Hörmander, *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

26. Hamid Hezari, Casey Kelleher, Shoo Seto, and Hang Xu, Asymptotic expansion of the Bergman kernel via perturbation of the Bargmann-Fock model, *J. Geom. Anal.*, 26 (2016), No.4, 2602-2638.

27. Chin-Yu Hsiao and George Marinescu, Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles, *Comm. Anal. Geom.*, **22** (2014), No.1, 1-108.

28. Chin-Yu Hsiao and Nikhil Savale, Bergman-szegö kernel asymptotics in weakly pseudoconvex finite type cases, arXiv:2009.07159, Sep 2020.

29. Chin-Yu Hsiao, Bergman kernel asymptotics and a pure analytic proof of the Kodaira embedding theorem, In *Complex analysis and geometry*, volume 144 of *Springer Proc. Math. Stat.*, pages 161-173. Springer, Tokyo, 2015.

30. George Marinescu, The Laplace operator on high tensor powers of line bundles, Preprint, `http://www.mi.uni-koeln.de/~gmarines/PREPRINTS/habil.pdf`, 2005.

31. Xiaonan Ma and George Marinescu, The first coefficients of the asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator, *Internat. J. Math.*, **17** (2006), No.6, 737-759.

32. Xiaonan Ma and George Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 2007.

33. Xiaonan Ma and George Marinescu, Generalized Bergman kernels on symplectic manifolds, *Adv. Math.*, **217** (2008), No.4, 1756-1815.

34. Xiaonan Ma and George Marinescu, Berezin-Toeplitz quantization on Kähler manifolds, *J. Reine Angew. Math.*, **662** (2012), 1-56.

35. Ophélie Rouby, Johannes Sjöstrand, and San Vũ Ngọc, Analytic Bergman operators in the semiclassical limit, *Duke Math. J.*, **169** (2020), No.16, 3033-3097.

36. Wei-Dong Ruan, Canonical coordinates and Bergman metrics, *Comm. Anal. Geom.*, **6** (1998), No.3, 589-631.

37. Martin Schlichenmaier, Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results. *Adv. Math. Phys.*, pages Art. ID 927280, 38, 2010.

38. Shoo Seto, *On the Asymptotic Expansion of the Bergman Kernel*, PhD thesis, UC Irvine, 2015.

39. Yum Tong Siu, A vanishing theorem for semipositive line bundles over non-Kähler manifolds, *J. Differential Geom.*, **19** (1984), No.2, 431-452.

40. Gang Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Differential Geom.*, **32** (1990), No.1, 99-130.

41. Wells, Jr., Raymond O, *Differential Analysis on Complex Manifolds*, third edition, Graduate Texts in Mathematics, 65, Springer, New York, 2008.

42. Kosaku Yosida, *Functional analysis*, Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

43. Steve Zelditch, Szego kernels and a theorem of Tian, *Internat. Math. Res. Notices*, **6** (1998), 317-331.