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MULTIPLICITY OF SOLUTIONS FOR SOME $p(x)$-BIHARMONIC PROBLEM

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Abstract

This paper deals with the study of some class of non-homogeneous problems involving the $p(x)$-biharmonic operator. Using direct variational methods, the existence of nontrivial solution is obtained. The multiplicity of solutions is obtained by combining Ekeland's variational principle with the Mountain pass theorem. Finally, the Fountain theorem is applied to prove the existence of infinetely many solutions for the given problem.

1. Introduction

In this article, we consider the following $p(x)$-biharmonic problem with Styklov boundary conditions and variable exponents

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda b(x)|u|^{\gamma(x)-2}u & \text{in } \Omega, \\ |\Delta u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $a \in L^\infty(\Omega)$ is such that $\operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0$. The function $b : \Omega \to \mathbb{R}$ is continuous and satisfying

$$b_0 \leq b(x) \leq b_1,$$

for some positive constants b_0 and b_1.
$\frac{\partial}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$. $p, \gamma : \overline{\Omega} \longrightarrow \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$

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and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Here, $\Delta^2_{p(x)}$ is the $p(x)$-biharmonic operator which is defined by

$$\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u).$$

The interest of studying problems involving variable exponents is due to their importance in several fields, also they can model various phenomena such as the electrorheological fluids [32, 34, 38], elastic mechanics [5], image processing [18].

Before giving our main results let us recall literature concerning related Styklov elliptic problems. In [29], Kandilakis and Magiropoulos considered the following problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = \mu\rho(x)|u|^{r-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda, \mu > 0$, $a : \Omega \to \mathbb{R}$ and $b, \rho : \partial\Omega \to \mathbb{R}$ are essentially bounded functions. Using variational method, the authors proved the existence of positive solution provided that λ is less than the first eigenvalue of the associated eigenvalue problem.

Allaoui in [2], considered the following problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda \left(a(x)|u|^{q(x)-2}u + b(x)|u|^{r(x)-2}u\right) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Under appropriate conditions and using variational method combined with the Mountain pass theorem, the author prove that if λ is small enough, then, problem (1.3) admits a nontrivial solution.

Fourth-order problems have various and important applications in different areas of applied mathematics and physics such as thin film theory, micro-electro-mechanical systems, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells (see [20, 25, 32]). In addition, problems of type (1.1), can describe the static from change of beam or the sport of rigid body. Due to their importance, many researchers focused on the study of such problems see for example [1, 7, 14] and references therein. The interplay between the fourth-order equation and the variable exponent equation goes to the $p(x)$-biharmonic problems. Note that the $p(x)$-biharmonic operator possesses more complicated nonlinearities than the p-biharmonic one, for example, it is inhomogeneous.

In recent years, many authors goes their attention to the $p(x)$-biharmonic problems. This trend is quite fresh, starting probably in 2009, with the papers [6, 21].

Problems of the form (1.1) have been extensively studied. For example, in the celebrate paper of Ambrosetti et al. [3], the authors use dual variational methods to prove existence theorems for critical points of a continuously differentiable functional. In the works of Bonder and D Rossi [11], the authors use variational and topological arguments to prove some existence results. Using variational methods and monotonicity arguments combined with the theory of the generalized Lebesgue Sobolev spaces, Chammem et al. [13] proved existence of positive solutions. By using variational principle and the mountain pass theorem Chen and Yao [15], established the existence of infinetely many solutions. Also, we refere the reader to [1, 9, 10, 27, 35] and references therein.

In this paper, motivated by the above mentioned works, we will use in Section 3, a direct variational method in order to prove the existence of a nontrivial solution for problem (1.1). Also, in Section 4, we will combine Ekeland's variational principle with the Mountain pass theorem in order to prove the multiplicity of solutions for such problem. Finally, In Section 5, we will apply the Fountain theorem to prove that problem (1.1) admits infinetely many solutions. Note that the main results of this paper improve and generalize the previous ones introduced in the literature. Now, we will introduce in Section 2, some notations and important results on the generalized Lebegue and Sobolev spaces which will be used in the rest of this paper.

2. Preliminaries and Basic Properties

In this section, we recal some definitions and basic propertiesof the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{1,p(x)}(\Omega)$ and

$W_0^{1,p(x)}(\Omega)$. We refer the interested readers to the references [23, 28, 33, 37]. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. We consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), p(x) > 1, \forall\, x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$, we define,

$$p^- = \inf_{\overline{\Omega}} p(x), \;\; p^+ = \sup_{\overline{\Omega}} p(x).$$

We can denote $C_+(\partial\Omega)$ and p^-, p^+ for any $p(x) \in C(\partial\Omega)$. We define,

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R},\; \text{measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\},$$

and

$$L^{p(x)}(\partial\Omega) = \Big\{u : \partial\Omega \to \mathbb{R}, \text{measurable } : \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty\Big\},$$

with norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ defined respectively by

$$|u|_{L^{p(x)}(\Omega)} = \inf\Big\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \leq 1\Big\},$$

and

$$|u|_{L^{p(x)}(\partial\Omega)} = \inf\Big\{\mu > 0 : \int_{\partial\Omega} |\frac{u(x)}{\mu}|^{p(x)} d\sigma \leq 1\Big\},$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

Proposition 2.1 (see [28, 37])**.**

(1) *The space* $(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)})$ *is a separable, uniformly convex Banach space, and its conjugate space is* $L^{p'(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. *Moreover, the Hölder inequality holds, that is, for any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$, *we have*

$$\Big|\int_\Omega uv dx\Big| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)}.$$

(2) *If* $p_1, p_2 \in C_+(\overline{\Omega})$, *with* $p_1(x) \leq p_2(x)$, *for any* $x \in \overline{\Omega}$, *then,* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

Let us define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} + |\frac{\nabla u(x)}{\mu}|^{p(x)} dx \leq 1\}.$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$\|u\| = \inf\{\mu > 0 : \int_\Omega \left(a(x)|\frac{u(x)}{\mu}|^{p(x)} + b(x)|\frac{\nabla u(x)}{\mu}|^{p(x)} \right) dx \leq 1\}, \quad (2.1)$$

then, from the assumptions on the functions a and b, $\|u\|$ is an equivalent norm on $W^{1,p(x)}(\Omega)$.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.2 (see [33]).

(1) *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(2) *If $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x)$, for any $x \in \overline{\Omega}$, then, the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$, is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

(3) *If $q \in C_+(\partial\Omega)$ with $q(x) < p_*(x)$ for any $x \in \partial\Omega$, then, the trace embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\partial\Omega)$, is compact and continuous, where*

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Now, if we put

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx,$$
$$\Gamma(u) = \int_\Omega (|\Delta u|^{p(x)} + a(x)|u|^{p(x)}) dx,$$

and

$$\rho_\partial(u) = \int_{\partial\Omega} |u(x)|^{p(x)} dx.$$

Then, we have the following important results.

Proposition 2.3 (see [31]). *There exist positive constants ξ_1, ξ_2, such that*

(i) *if* $\Gamma(u) \geq 1$, *then,* $\xi_1 ||u||^{p^-} \leq \Gamma(u) \leq \xi_2 ||u||^{p^+}$,

(ii) *if* $\Gamma(u) \leq 1$, *then,* $\xi_1 ||u||^{p^+} \leq \Gamma(u) \leq \xi_2 ||u||^{p^-}$,

(iii) $\Gamma(u) \geq 1 (= 1, \leq 1) \Leftrightarrow ||u|| \geq 1 (= 1, \leq 1)$.

Proposition 2.4 (see [28, 36, 37]). *For all $u \in L^{p(x)}(\Omega)$, we have*

(1) $|u|_{L^{p(x)}(\Omega)} < 1$ $(resp = 1, > 1) \Leftrightarrow \rho(u) < 1$ $(resp = 1, > 1)$,

(2) $|u|_{L^{p(x)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega)}$,

(3) $|u|_{L^{p(x)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega)}$.

Proposition 2.5 (see [28, 36, 37]). *For all $u \in L^{p(x)}(\partial\Omega)$, we have*

(1) $|u|_{L^{p(x)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^+}_{L^{p(x)}(\partial\Omega)}$,

(2) $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^-}_{L^{p(x)}(\partial\Omega)}$.

Proposition 2.6 (see [28, 36, 37]). *If p and q are measurable functions, such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \leq p(x)q(x) \leq \infty$, for all $x \in \mathbb{R}^N$, then, for all $u \in L^{q(x)}(\mathbb{R}^N)$, with $u \neq 0$, we have*

(1) $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{q^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{q^-}_{p(x)q(x)}$,

(2) $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{q^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{q^+}_{p(x)q(x)}$.

Let

$$L(u) = \int_\Omega \frac{1}{p(x)} [|\Delta u|^{p(x)} + a(x)|u|^{p(x)}] dx.$$

Then we have the following result.

Proposition 2.7 (see [22, 7]). *The following statements hold true.*

1. *Then the functional $L : X \to \mathbb{R}$ is sequentially weakly lower semi continuous, $L \in C^1(X, \mathbb{R})$.*

2. *The mapping $L' : X \to X'$ is a strictly monotone, bounded homeomorphism and is of type S_+ , namely, $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} L'(u_n)(u_n - u) \leq 0$ implies that $u_n \to u$, where $\to$ and $\rightharpoonup$ denote the strong and weak convergence respectively.*

Throughout the rest of the paper, the letters $c_i, i = 1, 2, \ldots,$ denote positive constants which may change from line to line.

3. First Existence Result

In this section, we will a direct variational method to prove the existence of solutions for problem (1.1). Precisely, we use the following theorem.

Theorem 3.1 (see [17], Theorem 1.2)**.** *Assume that X is a reflexive Banach space of norm $\|.\|$ and the functional $\varphi : X \to \mathbb{R}$ satisfying*

- *φ is coercive.*
- *φ is (sequentially) weakly lower semicontinuous on X.*

Then φ is bounded from below on X and attains its infimum in X.

Throughout this paper, we shall work in the space $X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$.

Definition 3.1. *We say that u is a weak solution of problem* (1.1)*, if for any $v \in X$, we have*

$$\begin{aligned}
&\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_\Omega a(x)|u|^{p(x)-2} uv dx \\
&= \lambda \int_\Omega b(x)|u|^{\gamma(x)-2} uv dx + \int_\Omega f(x,u) v dx + \int_{\partial\Omega} g(x,u) v d\sigma,
\end{aligned}$$

where $d\sigma$ denotes the measure on the boundary $\partial\Omega$.

The Euler Lagrange functional associated to this problem is

$$\begin{aligned}
J_\lambda(u) = &\int_\Omega \frac{|\Delta u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)} dx - \lambda \int_\Omega b(x) \frac{|u|^{\gamma(x)}}{\gamma(x)} dx \\
&- \int_\Omega F(x,u) dx - \int_{\partial\Omega} G(x,u) d\sigma,
\end{aligned}$$

where $F(x,t) := \int_0^t f(x,s)ds$ and $G(x,t) := \int_0^t g(x,s)ds$.
By a standard computation we can see that the functional J_λ is well defined and belongs to $C^1(X,\mathbb{R})$. Moreover, for all $v \in X$, its Gâteaux derivative is given by

$$\begin{aligned}\langle J'_\lambda(u), v\rangle = &\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v + a(x)|u|^{p(x)-2}uv\, dx\\ &- \lambda \int_\Omega b(x)|u|^{\gamma(x)-2}uvdx - \int_\Omega f(x,u)vdx - \int_{\partial\Omega} g(x,u)vd\sigma.\end{aligned}$$

Hence $u \in X$ is a weak solution of the problem (1.1) if and only if u is a critical point of this problem.

In order to ensure the existence of solutions for problem (1.1), we assume the following hypotheses:

(f1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satysfiying

$$|f(x,t)| \le C_1|t|^{q(x)-1}, \quad \forall\ (t,x) \in (\mathbb{R} \times \Omega), \tag{3.1}$$

for some positive constant C_1, where $q \in C_+(\Omega)$ is such that $q^+ < p^-$.

(f2) f satisfies the Ambrosetti-Rabinowitz type condition which means that there exist $M_1 > 0$, $\theta_1 > p^+$ such that for all $x \in \Omega$, we have

$$0 < \theta_1 F(x,u) \le f(x,u)u, \quad |u| \ge M_1,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(g1) There exists $r \in C_+(\partial\Omega)$, such that the function $g \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$ satisfies

$$|g(x,t)| \le C_2|t|^{r(x)-1}, \quad \forall\ (t,x) \in (\mathbb{R} \times \partial\Omega), \tag{3.2}$$

where C_2 is a positive parameter and $r^+ < p^-$.

(g2) There exist $M_2 > 0$, $\theta_2 > p^+$ such that for all $x \in \partial\Omega$, we have

$$0 < \theta_2 G(x,u) \le g(x,u)u, \quad |u| \ge M_2,$$

where $G(x,t) = \int_0^t g(x,s)ds$.

Our first existence result of this paper is the following.

Theorem 3.2. *If hypotheses* $(\mathbf{f1}), (\mathbf{g1}), (\mathbf{f2}), (\mathbf{g2})$ *hold. If* $\max(q^+, r^+, \gamma^+) < p^-$, *then, problem* (1.1) *admits a weak solution.*

The proof of Theorem 3.2 is a consequence of the following lemma due to the theorem 3.1.

Lemma 3.1. *Under assumptions* $(\mathbf{f1}), (\mathbf{g1}), (\mathbf{f2}), (\mathbf{g2})$. *If* $\max(q^+, r^+, \gamma^+) < p^-$, *then* J_λ *is coercive and sequentially weakly lower semicontinuous.*

Proof. From $(\mathbf{f1})$ and $(\mathbf{g1})$, we obtain

$$|F(x,t)| \leq c_1 \frac{|t|^{q(x)}}{q(x)}, \quad \forall\, (x,t) \in (\Omega \times \mathbb{R}), \tag{3.3}$$

and

$$|G(x,t)| \leq c_2 \frac{|t|^{r(x)}}{r(x)}, \quad \forall\, (x,t) \in (\partial\Omega \times \mathbb{R}). \tag{3.4}$$

Let $u \in X$ large enough, such that $\|u\| > \max(M_1, M_2)$, where M_1 and M_2 are given respectively by assumptions $(\mathbf{f2})$ and $(\mathbf{g2})$. From the embeddings $X \hookrightarrow L^{q(x)}(\Omega)$ and $X \hookrightarrow L^{r(x)}(\partial\Omega)$, we get

$$\begin{aligned} J_\lambda(u) =& \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} + a(x)\frac{|u|^{p(x)}}{p(x)} dx - \lambda \int_\Omega b(x) \frac{|u|^{\gamma(x)}}{\gamma(x)} dx - \int_\Omega F(x,u) dx \\ & - \int_{\partial\Omega} G(x,u) d\sigma \\ \geq & \frac{1}{p^+}\Gamma(u) - \frac{\lambda b_1}{\gamma^-} \int_\Omega |u|^{\gamma(x)} dx - \frac{c_1}{q^-} \int_\Omega |u|^{q(x)} dx - \frac{c_2}{r^-} \int_{\partial\Omega} |u|^{r(x)} d\sigma \\ \geq & c_3 \|u\|^{p^-} - c_4 \|u\|^{\gamma^+} - c_5 \|u\|^{q^+} - c_6 \|u\|^{r^+}. \end{aligned} \tag{3.5}$$

Since $\max(q^+, r^+, \gamma^+) < p^-$, then, we infer that J_λ is coercive and bounded from bellow on X.

Now we aim to prove that J_λ is sequentially weakly lower semicontinuous. Let $(u_n) \subset X$ be such that $u_n \rightharpoonup u$. Then, from the compact embedding (see the Proposition 2.2), we obtain

$$u_n \to u \quad in \quad L^{p(\cdot)}(\Omega) \quad and \quad u_n \to u \quad in \quad L^1(\Omega) \tag{3.6}$$

$$u_n \to u \quad in \quad L^{\gamma(\cdot)}(\Omega), \tag{3.7}$$

and

$$u_n \to u \quad in \quad L^{p(\cdot)}(\partial\Omega) \quad and \quad u_n \to u \quad in \quad L^1(\partial\Omega). \tag{3.8}$$

Put

$$I_1 = \int_\Omega F(x,u)dx \quad \text{and} \quad I_2 = \int_\Omega G(x,u)d\sigma.$$

Then by the mean value theorem, there exists v and $\hat{v}$ such that

$$|I_1(u_n)-I_1(u)| \leq \int_\Omega |F(x,u_n)-F(x,u)|dx \leq \int_\Omega |u_n-u| \sup_{x\in\Omega} |f(x,v)|dx,$$

and

$$|I_2(u_n)-I_2(u)| \leq \int_{\partial\Omega} |G(x,u_n)-G(x,u)|d\sigma \leq \int_{\partial\Omega} |u_n-u| \sup_{x\in\partial\Omega} |g(x,\hat{v})|d\sigma.$$

Hence, from (**f1**) and (**g1**), we infer that I_1 and I_2 are weakly continuous. Similarly, the function $u \mapsto \int_\Omega b(x)\frac{|u|^{\gamma(x)}}{\gamma(x)}dx$, is weakly continuous. Moreover, from Proposition 2.7, we know that L is sequentially weakly lower semicontinuous. Finally, all the above informations imply that J_λ is sequentially weakly lower semicontinuous. $\square$

Proof of theorem 3.2 From the classical theorem investigated by Tonelli-Weierstrass based on Theorem 3.1, we deduce that we can find $\bar{u}$ in which J_λ attains its infimum. Moreover, we can easily seen that $\bar{u}$ is a nontrivial solution for problem (1.1).

4. Second Existence Result

In this section, mountain pass theorem is combined with Ekland's variational principle in order to prove the existence of at least two nontrivial solutions. Precisely, our second main result of this paper is the following.

Theorem 4.1. *Assume that the hypotheses* (**f1**), (**g1**), (**f2**), (**g2**) *hold. Then, the problem* (1.1) *admits at least two nontrivial solutions.*

In order to prove Theorem 4.1, we give the following definition.

Definition 4.1. *Let X be a Banach space and $\varphi \in C^1(X,\mathbb{R}), c \in \mathbb{R}$. We say that φ satisfies the (PS) condition at level c if any sequence $u_n \subset X$, such that*

$$\varphi(u_n) \to c, \text{ and } \varphi'(u_n) \to 0, \quad \text{in } X^*, \text{ as } n \to \infty,$$

contains a convergent subsequence.

Our main tool to prove Theorem 4.1, is the following result.

Theorem 4.2. (*Mountain pass theorem*). *Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $e \in X$ with $||e|| > r$ for some $r > 0$. Assume that*

$$\inf_{||u||=r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

If φ satisfies the (PS) condition at level c, then, c is a critical value of φ, where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \quad \text{and} \;\; \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

The proof of Theorem 4.1, is divided into several lemmas.

Lemma 4.2. *The functional J_λ satisfies the Palais Smale condition, that is, if a sequence $\{u_n\} \subset X$ is such that $J_\lambda(u_n)$ is bounded and $J'_\lambda(u_n)$ tends to zero as n tends to infinity, then, $\{u_n\}$ has a convergent subsequence.*

Proof. Since J_λ is coercive then u_n is bounded. So up to a subsequence, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. On the other hand, from the compact embeddings, we get

$$u_n \longrightarrow u \quad \text{strongly respectively in} \quad L^{q(x)}(\Omega), L^{\gamma(x)}(\Omega) \;\text{ and }\; L^{r(x)}(\partial\Omega).$$

Moreover, by the fact that $J'_\lambda(u_n)$ tends to zero, we obtain

$$\begin{aligned} 0 =& \langle J'_\lambda(u_n), u_n - u\rangle \\ =& \int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n \Delta(u_n - u) + a(x)|u_n|^{p(x)-2} u_n(u_n - u) dx \\ & - \lambda \int_\Omega b(x)|u_n|^{\gamma(x)-2} u_n(u_n - u) dx \\ & - \int_\Omega f(x, u_n)(u_n - u) dx - \int_{\partial\Omega} g(x, u_n)(u_n - u) d\sigma. \end{aligned}$$

From the Hölder inequality, the compact embedding and the fact that $b(x) \leq b_1$, we have

$$\int_\Omega b(x)|u_n|^{\gamma(x)-2} u_n(u_n - u) dx \to 0 \text{ as } n \to \infty.$$

Now, according to the Krasnoselki's theorem, the Nemytskii operators

$$\begin{array}{rccc} N_f: & L^{q(x)}(\Omega) & \rightarrow & L^{\frac{q(x)}{q(x)-1}}(\Omega) \\ & u & \longmapsto & f(\cdot,u) \end{array}$$

and

$$\begin{array}{rccc} N_g: & L^{r(x)}(\Omega) & \rightarrow & L^{\frac{r(x)}{r(x)-1}}(\partial\Omega) \\ & u & \longmapsto & g(\cdot,u) \end{array}$$

are continuous. Hence, $N_f(u_n) \to N_f(u)$ in $L^{\frac{q(x)}{q(x)-1}}(\Omega)$ and $N_g(u_n) \to N_g(u)$ in $L^{\frac{r(x)}{r(x)-1}}(\partial\Omega)$.

In addition, in view of the Holder's inequality and the continuous embeddings of X into $L^{q(x)}(\Omega)$ and $L^{r(x)}(\partial\Omega)$, we get

$$\begin{aligned} &\Big|\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)(x)dx\Big| \\ &\quad \le 2\|N_f(u_n)-N_f(u)\|_{\frac{q(x)}{q(x)-1}}\|u_n-u\|_{q(x)} \\ &\quad \le c_1\|N_f(u_n)-N_f(u)\|_{\frac{q(x)}{q(x)-1}}\|u_n-u\|. \end{aligned}$$

And

$$\begin{aligned} &\Big|\int_{\partial\Omega} (g(x,u_n)-g(x,u))(u_n-u)(x)dx\Big| \\ &\quad \le 2\|N_g(u_n)-N_g(u)\|_{\frac{r(x)}{r(x)-1}}\|u_n-u\|_{r(x)} \\ &\quad \le c_2\|N_f(u_n)-N_f(u)\|_{\frac{r(x)}{r(x)-1}}\|u_n-u\|. \end{aligned}$$

Thus, the fact that L' is of type S_+ (Proposition 2.7) completes the proof. □

Lemma 4.3. *Under the same hypothesis of Theorem* 4.1*, we have*

(i) *There exist two positive constants ρ and ζ, such that*

$$\textit{If } \ \|u\| = \rho \ \textit{ then } \ J_\lambda(u) \ge \zeta.$$

(ii) *There exists $e \in X$ with $\|e\| > \rho$ and $J_\lambda(e) < 0$.*

Proof. (i) Let $u \in X$ with $\|u\| > 1$. Then by using equation (3.5), we have

$$\begin{aligned} J_\lambda(u) &\ge c_1\|u\|^{p^-} - c_2\|u\|^{\gamma^+} - c_3\|u\|^{q^+} - c_4\|u\|^{r^+} \\ &\ge \|u\|^{p^-}\left(c_1 - c_2\|u\|^{\gamma^+-p^-} - c_3\|u\|^{q^+-p^-} - c_4\|u\|^{r^+-p^-}\right). \end{aligned}$$

Put

$$h(t) = c_1 - c_2 t^{\gamma^+ - p^-} - c_3 t^{q^+ - p^-} - c_4 t^{r^+ - p^-}.$$

Since $\lim_{t\to\infty} h(t) = c_1 > 0$, then we can choose $\rho > 1$ large enough such that $h(\rho) > 0$. Let

$$\zeta = \rho^{p^-} \left(c_1 - c_2 \rho^{\gamma^+ - p^-} - c_3 \rho^{q^+ - p^-} - c_4 \rho^{r^+ - p^-} \right).$$

Then, it is easy to see that $\zeta > 0$. Moreover if $\|u\| = \rho$, then $J_\lambda \geq \zeta > 0$.

(ii) Let $e_0 \in C^\infty(X)$ be such that $e_0 \geq 0$ and $e_0 \neq 0$. Then, using the Ambrosetti Rabinowitz conditions $(\mathbf{f2}), (\mathbf{g2})$ and the compact embeddings, we get

$$J_\lambda(te_0) \leq \frac{t^{p^+}}{p^-} \|e_0\|^{p^+} - \frac{\lambda b_0}{\gamma^+} t^{\gamma^+} \|e_0\|^{\gamma^+} - c_1 t^{\theta_1} \int_\Omega |e_0|^{\theta_1} dx - c_2 t^{\theta_2} \int_\Omega |e_0|^{\theta_2} dx.$$

Since $\gamma^+ < p^-$ and $\min(\theta_1, \theta_2) > p^+$, then, we have $\lim_{t\to\infty} J_\lambda(te_0) = -\infty$. So, we can find t large enough such that if $e = te_0$, then $\|e\| > \rho$ and $J_\lambda(e) < 0$. □

Proof of theorem 4.1 By combining Lemma 4.2 and Lemma 4.3 with Theorem 4.2, we deduce that J_λ admits a critical point $u_1 \in X \setminus \{0, e\}$ which can be characterized by

$$J_\lambda(u_1) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)), \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Moreover, we have

$$J_\lambda(u_1) \geq \xi > 0. \tag{4.1}$$

On the other hand, from Lemma 4.3, we obtain

$$\inf_{u\in\partial B(0,R)} J_\lambda(u) > 0.$$

Moreover, for $u \in B(0, R)$, we have

$$J_\lambda(u) \geq c_1 \|u\|^{p^-} - c_2 \|u\|^{\gamma^+} - c_3 \|u\|^{q^+} - c_4 \|u\|^{r^+}.$$

So, by Lemma 4.3, we get

$$-\infty < \underline{c} = \inf_{u \in \overline{B(0,R)}} (J_\lambda(u)) < 0.$$

Let $\epsilon > 0$ be such that

$$0 < \epsilon < \inf_{u \in \partial B(0,R)} (J_\lambda(u)) - \inf_{u \in B(0,R)} (J_\lambda(u)).$$

Hence, Ekeland's variational principle applied to the functional $J_\lambda : \overline{B(0,R)} \to \mathbb{R}$, implies the existence of $u_\epsilon \in \overline{B(0,R)}$, such that

$$\begin{cases} \underline{c} \leq J_\lambda(u_\epsilon) \leq \underline{c} + \epsilon \\ J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon ||u - u_\epsilon||, \quad u \neq u_\epsilon. \end{cases} \tag{4.2}$$

Furthermore, we have

$$J_\lambda(u_\epsilon) \leq \inf_{u \in \overline{B(0,R)}} (J_\lambda(u)) + \epsilon \leq \inf_{B(0,R)} (J_\lambda(u)) + \epsilon < \inf_{\partial B(0,R)} (J_\lambda(u)). \tag{4.3}$$

Then, we deduce that $u_\epsilon \in B(0,R)$.

Now, we define $\Lambda_\lambda : \overline{B(0,R)} \to \mathbb{R}$ by

$$\Lambda_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|.$$

It is clear that u_ϵ is a minimum of Λ_λ. Therefore, for $t \in (0,1)$ small enough and for any $v \in B(0,1)$, we have

$$\frac{\Lambda_\lambda(u_\epsilon + tv) - \Lambda_\lambda(u_\epsilon)}{t} \geq 0;$$

which means that

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} + \epsilon \|v\| \geq 0.$$

By letting t tends to zero, we obtain

$$J'_\lambda(u_\epsilon)(v) + \epsilon \|v\| \geq 0.$$

This implies that

$$\|J'_\lambda(u_\epsilon)\| \leq \epsilon.$$

Combining the results mentioned above, we deduce the existence of a sequence $\{w_n\} \subset B(0,R)$, such that

$$J_\lambda(w_n) \to \underline{c} < 0, \quad \text{and} \quad J'_\lambda(w_n) \to 0.$$

Since $\{w_n\} \subset B(0,R)$, then, $\{w_n\}$, is bounded in X. So, up to a subsequence, there exists $u_2 \in X$, such that, $\{w_n\}$ converges weakly to $u_2 \in X$. Hence, if we proceed as in the proof of the Proposition 4.2, then we can prove that $w_n \to u_2$ strongly in X.
Since $J_\lambda \in C^1(X,\mathbb{R})$, then

$$J'_\lambda(w_n) \to J_\lambda(u_2) \text{ as } n \to \infty$$

Hence, we conclude that

$$J'_\lambda(u_2) = 0, \ \|w\| < R, \text{ and } J_\lambda(u_2) < 0. \tag{4.4}$$

This implies that u_2 is a nontrivial solution for problem (1.1).

Finally, from (4.1), we have $J_\lambda(u_2) < 0 < J_\lambda(u_1)$. So problem (1.1) has at least two nontrivial solutions.

5. Third Existence Result

In this section, we will state and prove the existence of infinetely many solutions for problem (1.1). So, we assume that all hypothesis of Theorem 3.2 are satisfied. Moreover, we assume the following supplementary conditions:

(f3) $f(x,-t) = -f(x,t), \quad \forall\, t \in \mathbb{R}, x \in \Omega.$

(g3) $g(x,-t) = -g(x,t), \quad \forall\, t \in \mathbb{R}, x \in \partial\Omega.$

Theorem 5.1. *Assume that the hypotheses of Theorem* 3.2 *are satisfied. If in addition* (**f3**), (**g3**) *hold and* $0 < \lambda < \frac{\gamma^-}{2p^+ b_1}$, *then problem* (1.1) *admits infinitely weak solutions.*

The proof of Theorem 5.1, mainly rests on an application of the Fountain theorem. It is well known that X is a reflexive and separable Banach space. Therefore, there exist $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

$$X = \overline{span\{e_n, n \in \mathbb{N}\}}, \quad X^* = \overline{span\{e_n^*, n \in \mathbb{N}\}}, \quad \langle e_n, e_j^* \rangle = \delta_{i,j},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

For $k \in \mathbb{N}$, we put

$$X_k = \mathbb{R}e_k = span\{e_k\}, \quad Y_k = \overset{k}{\underset{i=1}{\oplus}} X_i, \quad Z_k = \overset{\infty}{\underset{i=k}{\oplus}} X_i.$$

Theorem 5.2 (Fountain Theorem, see [12]). *Suppose that an even functional $J \in C^(X, \mathbb{R})$ satisfies the Palais smail condition, and that there is $k_0 > 0$ such that for every $k \geq k_0$ there exist $\rho_k > r_k > 0$ so that the following properties hold:*

(i) $a_k = \max\limits_{u \in Y_k, \|u\| = \rho_k} J(u) \leq 0.$

(ii) $b_k = \inf\limits_{u \in Z_k, \|u\| = r_k} J(u) \to \infty$, *as* $k \to \infty$.

Then, J has a sequence of critical points $\{u_k\}$ such that $J(u_k) \to \infty$.

Put

$$\eta_k = \sup_{u \in Z_k, \|u\|=1} |u|_{q(x)} \quad \text{and} \quad \mu_k = \sup_{u \in Z_k, \|u\|=1} |u|_{r(x)}.$$

Then, we have the following important result.

Lemma 5.1. *The following statements hold true:*

1. *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $\lim\limits_{k \to +\infty} \eta_k = 0$.*

2. *If $r \in C_+(\overline{\Omega})$ and $r(x) < p_*(x)$ for all $x \in \overline{\partial\Omega}$, then $\lim\limits_{k \to +\infty} \mu_k = 0$.*

Proof. 1. It is clear that the sequence $\{\eta_k\}$ is nonnegative and decreasing. So, there exists $\eta \geq 0$ such that η_k converges to η . Let $u_k \in Z_k$ be such that

$$\|u_k\| = 1 \quad \text{and} \quad 0 \leq \eta_k - |u_k|_{q(x)} < \frac{1}{k}. \tag{5.1}$$

Since η_k is bounded, we see that (u_k) is also bounded. So up to a subsequence, there exists $u \in X$, such that $u_k \rightharpoonup u$ weakly in X. Moreover, for all $i \in \mathbb{N}$, we have

$$\langle e_i^*, u\rangle = \lim_{k\to+\infty} \langle e_i^*, u_k\rangle = 0.$$

Thus, $u = 0$ and $u_k \rightharpoonup 0$ weakly in X. Since X is compactly embedded in $L^{q(x)}(\Omega)$, then $u_k \to 0$ strongly in $L^{q(x)}(\Omega)$. Hence, from (5.1), we get $\lim\limits_{k\to 0} \eta_k = 0$.

2. Since the embedding from X into $L^{r(x)}(\partial\Omega)$ is compact. Then, the result follows similarly as in the first step. □

Proof of Theorem 5.1. From Section 3, we have $J_\lambda \in C^1(X, \mathbb{R})$. Moreover, from Lemma 4.2, J_λ satisfies the Palais-Smale condition. On the other hand, from condition (**f3**), (**g3**) , we see that J_λ is an even functional. Now, we shall verify that J_λ satisfies the conditions of Theorem 5.2 item by item.

(i) Let $u \in Y_k$ such that $\|u\| > 1$. Then by using the conditions (**f2**), (**g2**), we have

$$\begin{aligned} J_\lambda(u) &\le \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda b_0}{\gamma^+}\int_\Omega |u|^\gamma dx - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma \\ &\le \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda b_0}{\gamma^+}\int_\Omega |u|^\gamma dx - c_1\int_\Omega |u|^{\theta_1}dx - c_2\int_{\partial\Omega}|u|^{\theta_2}d\sigma. \end{aligned}$$

Since the space Y_k has finite dimension, then all norms are equivalents. Hence, we obtain

$$\begin{aligned} J_\lambda(u) &\le \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda b_0}{\gamma^+}\int_\Omega |u|^\gamma dx - c_1\int_\Omega |u|^{\theta_1}dx - c_2\int_{\partial\Omega}|u|^{\theta_2}d\sigma \\ &\le \frac{1}{p^-}\|u\|^{p^+} - c_1\frac{\lambda b_0}{\gamma^+}\|u\|^{\gamma^+} - c_2\|u\|^{\theta_1} - c_3\|u\|^{\theta_2}. \end{aligned}$$

From the fact that
$min(\theta_1, \theta_2) > p^+$, it is easy to see that

$$J_\lambda(u) \to -\infty \quad \text{as } \|u\| \to \infty, u \in Y_k.$$

Hence, it follows that for some $\rho_k = \|u\| > 0$ large enough we have

$$a_k = \max_{u\in Y_k, \|u\|=\rho_k} J_\lambda(u) \leq 0.$$

That is condition (i) of Theorem 5.2 holds.
(ii) Let $u \in Z_k$ such that $\|u\| > 1$, it is easy to prove that

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda b_1}{\gamma^-}\|u\|^{\gamma^+} - c_1\int_\Omega |u|^{q(x)}dx - c_2\int_{\partial\Omega}|u|^{r(x)}d\sigma. \quad (5.2)$$

Without loss of generality, we can assume that
$min(|u|_{q(x)}, |u|_{r(x)}) > 1$. So we get

$$\int_\Omega |u|^{q(x)}dx \leq |u|^{q^+}_{q(x)} \leq (\eta_k\|u\|)^{q^+} \text{ and } \int_{\partial\Omega}|u|^{r(x)}d\sigma \leq |u|^{r^+}_{r(x)} \leq (\mu_k\|u\|)^{r^+}. \quad (5.3)$$

Now, by combining (5.2) with (5.3), we obtain

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda b_1}{\gamma^-}\|u\|^{\gamma^+} - c_1\eta_k^{q^+}\|u\|^{q^+} - c_2\mu_k^{r^+}\|u\|^{r^+}.$$

Let $t > 1$ and $v \in Z_k$ be such that $\|v\| = 1$. Since $\max(\gamma^+, q^+, r^+) < p^-$, then we have

$$\begin{aligned} J_\lambda(tv) &\geq \frac{1}{p^+}t^{p^-} - \frac{\lambda b_1}{\gamma^-}t^{\gamma^+} - c_1\eta_k^{q^+}t^{q^+} - c_2\mu_k^{r^+}t^{r^+} \\ &\geq t^{\max(\gamma^+,q^+,r^+)}\left(\frac{1}{p^+} - \left(c_1\eta_k^{q^+} + c_2\mu_k^{r^+}\right) - \frac{\lambda b_1}{\gamma^-}\right). \end{aligned}$$

For k large enough, choosing $c_1\eta_k^{q^+} + c_2\mu_k^{r^+} < \frac{1}{2p^+}$, we can deduce

$$J_\lambda(tv) \geq t^{\max(\gamma^+,q^+,r^+)}\left(\frac{1}{2p^+} - \frac{\lambda b_1}{\gamma^-}\right).$$

Put $t = r_k$ and $u = tv$, we see that $u \in Z_k$, $\|u\| = r_k$. Moreover, from the above inequality, if $0 < \lambda < \frac{\gamma^-}{2p^+b_1}$, then, $\inf_{u\in Z_k, \|u\|=r_k} J_\lambda(u) \to \infty$ as $k \to \infty$. That is condition (ii) of Theorem 5.2 holds. This completes the proof of Theorem 5.1.

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