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PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS FOR NEUTRAL NONLINEAR COUPLED VOLTERRA INTEGRO-DIFFERENTIAL SYSTEMS WITH TWO VARIABLE DELAYS

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Abstract

In this paper, we study the existence of periodic and asymptotically periodic solutions of neutral nonlinear coupled Volterra integro-differential systems. We furnish sufficient conditions for the existence of such solutions. Krasnoselskii's fixed point theorem is used in this analysis.

1. Introduction

The study of the existence of periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Some contributions on the existence of periodic solutions for differential equations have been made (see [1]-[4], [19], [20]). On the other hand,

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the concept of asymptotic periodicity is more general than periodicity and from an applied perspective, asymptotically periodic systems describe world more realistically and accurately than periodic ones, we can see [8], [9], [12], [26], [27], for more details.

In 1928 Volterra [25] noted that many physical problems were being modeled by integral and integro-differential equations. Today we see that such models have application in several branches of applied science, such as control theory, mathematical biology, viscoelasticity, nuclear reactors, many other areas, and for this reason this type of equation has received much attention in recent years, (see for example [5], [7], [10], [11], [14]-[21], [23]). Motivated by the papers [6], [13], [22] and the references therein and by using Krasnoselskii's fixed point theorem, in this paper, we study the existence of periodic and asymptotically periodic solutions of the following system of coupled neutral nonlinear Volterra integro-differential equations with two delays

$$\begin{cases} x'(t) = h_1(t)x(t) + G_1(t, x(t), y(t), x(t-\tau_1(t)), y(t-\tau_2(t))) \\ \qquad +c_1(t)x'(t-\tau_1(t)) + \int_{-\infty}^{t} a_1(t,s) f_1(x(s), y(s))\, ds, \\ y'(t) = h_2(t)y(t) + G_2(t, x(t), y(t), x(t-\tau_1(t)), y(t-\tau_2(t))) \\ \qquad +c_2(t)y'(t-\tau_2(t)) + \int_{-\infty}^{t} a_2(t,s) f_2(x(s), y(s))\, ds, \end{cases} \tag{1.1}$$

where the functions h_i, c_i and a_i, $i = 1, 2$ are assumed to be continuous in their arguments throughout the paper. The functions $G_i(t, x, y, z, w)$, $i = 1, 2$ is continuous, periodic in t and Lipschitz continuous in x, y, z and w, $f_i(x, y)$, $i = 1, 2$ is continuous and Lipschitz continuous in x and y, and for some positive constants N_j and R_j, $j = \overline{1, 4}$ we have

$$|G_1(t, y_1, y_2, y_3, y_4) - G_1(t, x_1, x_2, x_3, x_4)| \le \sum_{j=1}^{4} N_j |y_j - x_j|,$$

$$|G_2(t, y_1, y_2, y_3, y_4) - G_2(t, x_1, x_2, x_3, x_4)| \le \sum_{j=1}^{4} R_j |y_j - x_j|,$$

and for some positive constants d_j and q_j, $j = 1, 2$ we have

$$|f_1(y_1, y_2) - f_1(x_1, x_2)| \le \sum_{j=1}^{2} d_j |y_j - x_j|,$$

$$|f_2(y_1,y_2)-f_2(x_1,x_2)|\leq\sum_{j=1}^{2}q_j|y_j-x_j|,$$

we also assume that $G_1(t,0,0,0,0)=G_2(t,0,0,0,0)=f_1(0,0)=f_2(0,0)=0$.

We assume that there exists a positive real number T, such that

$$\begin{aligned}h_i(t+T)=&h_i(t),\ c_i(t+T)=c_i(t),\\ a_i(t+T,s+T)=&a_i(t,s),\ \tau_i(t+T)=\tau_i(t),\quad i=1,2,\end{aligned}\tag{1.2}$$

with c_i continuously differentiable, τ_i twice continuously differentiable and $\tau_i(t)\geq\tau_i^*>0$ for $i=1,2$. To have a well behaved mapping we must assume that

$$\tau_i'(t)\neq 1,\ \int_0^T h_i(s)\,ds\neq 0,\ i=1,2.\tag{1.3}$$

Define $P_T=\{(\varphi,\psi):(\varphi,\psi)(t+T)=(\varphi,\psi)(t)\}$, where both φ and ψ are real valued continuous functions on $\mathbb{R}$. Then P_T is a Banach space when endowed with the maximum norm

$$\|(x,y)\|=\max\left\{\max_{t\in[0,T]}|x(t)|,\max_{t\in[0,T]}|y(t)|\right\}.$$

Definition 1.1. A function x is called asymptotically T-periodic if there exist two functions x_1 and x_2 such that x_1 is T-periodic, $\lim_{t\to\infty}x_2(t)=0$ and $x(t)=x_1(t)+x_2(t)$ for all t.

Lemma 1.2. *Assume* (1.2) *and* (1.3). *If $x,y\in P_T$, then x and y is a solution of* (1.1) *if and only if*

$$\begin{aligned}x(t)=&\frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)}-\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}r_1(u)\,x(u-\tau_1(u))\,du\\ &+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du\\ &+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\int_{-\infty}^{u}a_1(u,s)\,f_1(x(s),y(s))\,dsdu,\end{aligned}\tag{1.4}$$

and

$$y(t) = \frac{c_2(t)\, y(t-\tau_2(t))}{1-\tau_2'(t)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} r_2(u)\, y(u-\tau_2(u))\, du$$
$$+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} G_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))du$$
$$+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \int_{-\infty}^{u} a_2(u,s)\, f_2(x(s), y(s))\, dsdu, \tag{1.5}$$

where

$$r_1(u) = \frac{(c_1'(u) - c_1(u)\, h_1(u))(1-\tau_1'(u)) + \tau_1''(u)\, c_1(u)}{(1-\tau_1'(u))^2}, \tag{1.6}$$

and

$$r_2(u) = \frac{(c_2'(u) - c_2(u)\, h_2(u))(1-\tau_2'(u)) + \tau_2''(u)\, c_2(u)}{(1-\tau_2'(u))^2}. \tag{1.7}$$

Proof. Let $x, y \in P_T$ be a solution of (1.1). Next we multiply both sides of the first equation in (1.1) by $e^{-\int_0^t h_1(s)ds}$ and then integrate from t to $t+T$, to obtain

$$\int_t^{t+T} \left[x(u)e^{-\int_0^u h_1(s)ds}\right]' du$$
$$= \int_t^{t+T} e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du$$
$$+ \int_t^{t+T} e^{-\int_0^u h_1(s)ds} c_1(u)\, x'(u-\tau_1(u))\, du$$
$$+ \int_t^{t+T} e^{-\int_0^u h_1(s)ds} \int_{-\infty}^{u} a_1(u,s)\, f_1(x(s), y(s))\, dsdu.$$

Consequently, we have

$$x(t+T)e^{-\int_0^{t+T} h_1(s)ds} - x(t)e^{-\int_0^t h_1(s)ds}$$
$$= \int_t^{t+T} e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du$$
$$+ \int_t^{t+T} e^{-\int_0^u h_1(s)ds} c_1(u)\, x'(u-\tau_1(u))\, du$$
$$+ \int_t^{t+T} e^{-\int_0^u h_1(s)ds} \int_{-\infty}^{u} a_1(u,s)\, f_1(x(s), y(s))\, dsdu.$$

Multiply both sides with $e^{\int_0^{t+T} h_1(s)ds}$ and using the fact that $x(t+T) = x(t)$ and $e^{\int_t^{t+T} h_1(s)ds} = e^{\int_0^{T} h_1(s)ds}$, we obtain

$$\begin{aligned} x(t) = &\int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} G_1\left(u, x\left(u\right), y\left(u\right), x\left(u-\tau_1\left(u\right)\right), y\left(u-\tau_2\left(u\right)\right)\right) du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} \int_{-\infty}^{u} a_1\left(u,s\right) f_1\left(x\left(s\right), y\left(s\right)\right) dsdu. \end{aligned} \tag{1.8}$$

Letting

$$\begin{aligned} &\int_t^{t+T} e^{\int_u^{t+T} h_1(s)ds} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\ &\quad = \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds} c_1\left(u\right)}{1-\tau_1'\left(u\right)} \left(1-\tau_1'\left(u\right)\right) x'\left(u-\tau_1\left(u\right)\right) du. \end{aligned}$$

Performing an integration by parts, we get

$$\begin{aligned} &\int_t^{t+T} e^{\int_u^{t+T} h_1(s)ds} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\ &= \left[\frac{c_1(u)x(u-\tau_1(u))}{1-\tau_1'(u)} e^{\int_u^{t+T} h_1(s)ds}\right]_t^{t+T} - \int_t^{t+T} e^{\int_u^{t+T} h_1(s)ds} r_1(u)x(u-\tau_1(u))du \\ &= \frac{c_1(t)x(t-\tau_1(t))}{1-\tau_1'(t)}\left(1-e^{\int_0^{T} h_1(s)ds}\right) - \int_t^{t+T} e^{\int_u^{t+T} h_1(s)ds} r_1(u)x(u-\tau_1(u))du, \end{aligned} \tag{1.9}$$

where r_1 is given by (1.6). Substituting (1.9) into (1.8), we obtain

$$\begin{aligned} x(t) = &\frac{c_1\left(t\right) x\left(t-\tau_1\left(t\right)\right)}{1-\tau_1'\left(t\right)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} r_1\left(u\right) x\left(u-\tau_1\left(u\right)\right) du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^{T} h_1(s)ds}} \int_{-\infty}^{u} a_1\left(u,s\right) f_1\left(x\left(s\right), y\left(s\right)\right) dsdu. \end{aligned}$$

The proof of (1.5) is similar and hence we omit it. □

2. Periodic Solutions

Lemma 2.1 ([24]). *Let $\mathbb{M}$ be a bounded closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*

(i) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Let $\gamma_i(t) = \frac{c_i(t)}{1-\tau_i'(t)}$, $i = 1, 2$, we assume that $\sup_{t\in[0,T]} |\gamma_i(t)| = \mu_i < 1$, and let $\mu = \max\{\mu_1, \mu_2\}$. Let β_i, $i = 1, 2$ be positive constants such that $0 < \mu_i + \beta_i < 1$. Moreover, we assume the existence of positive constants M_i, K_i, α_i, L_i and θ_i, $i = 1, 2$ such that

$$|f_1(x, y)| \le M_1, \tag{2.1}$$

$$|f_2(x, y)| \le M_2, \tag{2.2}$$

$$|G_1(t, x, y, z, w)| \le K_1,\ |G_2(t, x, y, z, w)| \le K_2, \tag{2.3}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| \int_{-\infty}^u |a_1(u, s)|\, ds du \le \alpha_1, \tag{2.4}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} \right| \int_{-\infty}^u |a_2(u, s)|\, ds du \le \alpha_2, \tag{2.5}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |r_1(u)|\, du \le \beta_1, \tag{2.6}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} \right| |r_2(u)|\, du \le \beta_2, \tag{2.7}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| du \le L_1,\ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} \right| du \le L_2, \tag{2.8}$$

and

$$\int_{-\infty}^u |a_1(u, s)|\, ds \le \theta_1,\ \int_{-\infty}^u |a_2(u, s)|\, ds \le \theta_2. \tag{2.9}$$

Set

$$M=\max\left\{\frac{L_1K_1+\alpha_1M_1}{1-\mu_1-\beta_1},\frac{L_2K_2+\alpha_2M_2}{1-\mu_2-\beta_2}\right\}. \tag{2.10}$$

We define subset $\Omega_{x,y}$ of P_T as follows

$$\Omega_{x,y}=\left\{(x,y):(x,y)\in P_T \text{ with } \|(x,y)\|\leq M\right\}.$$

Then $\Omega_{x,y}$ is a bounded, closed and convex subset of P_T. Now for $(x,y)\in\Omega_{x,y}$ we can define an operator $E:\Omega_{x,y}\to P_T$ by

$$E\left(x,y\right)\left(t\right)=\left(E_1\left(x,y\right)\left(t\right),E_2\left(x,y\right)\left(t\right)\right),$$

where

$$\begin{aligned}
&E_1(x,y)(t)\\
&=\frac{c_1(t)x(t-\tau_1(t))}{1-\tau_1'(t)}-\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}r_1(u)x(u-\tau_1(u))du\\
&\quad+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du\\
&\quad+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\int_{-\infty}^u a_1(u,s)f_1(x(s),y(s))dsdu,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&E_2\left(x,y\right)\left(t\right)\\
&=\frac{c_2\left(t\right)y\left(t-\tau_2\left(t\right)\right)}{1-\tau_2'\left(t\right)}-\int_t^{t+T}\frac{e^{\int_u^{t+T}h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}}r_2\left(u\right)y\left(u-\tau_2\left(u\right)\right)du\\
&\quad+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}}G_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du\\
&\quad+\int_t^{t+T}\frac{e^{\int_u^{t+T}h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}}\int_{-\infty}^u a_2(u,s)f_2(x(s),y(s))dsdu.
\end{aligned} \tag{2.12}$$

To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is continuous and compact. Therefore, we state

(2.11) as

$$E_1(x,y)(t) = B_1(x,y)(t) + A_1(x,y)(t), \tag{2.13}$$

where $B_1, A_1 : \Omega_{x,y} \to P_T$ are given by

$$B_1(x,y)(t) = \frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)}, \tag{2.14}$$

and

$$\begin{aligned} A_1(x,y)(t) = &-\int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} r_1(u)x(u-\tau_1(u))du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s)f_1(x(s),y(s))dsdu. \end{aligned} \tag{2.15}$$

And we state (2.12) as

$$E_2(x,y)(t) = B_2(x,y)(t) + A_2(x,y)(t), \tag{2.16}$$

where $B_2, A_2 : \Omega_{x,y} \to P_T$ are given by

$$B_2(x,y)(t) = \frac{c_2(t)\,y(t-\tau_2(t))}{1-\tau_2'(t)}, \tag{2.17}$$

and

$$\begin{aligned} A_2(x,y)(t) = &-\int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} r_2(u)y(u-\tau_2(u))du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} G_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \int_{-\infty}^{u} a_2(u,s)f_2(x(s),y(s))dsdu. \end{aligned} \tag{2.18}$$

Now for $(x,y) \in \Omega_{x,y}$ we can define the operators $B, A : \Omega_{x,y} \to P_T$ by

$$B(x,y)(t) = (B_1(x,y)(t),\ B_2(x,y)(t)),$$

and

$$A(x,y)(t) = (A_1(x,y)(t),\ A_2(x,y)(t)).$$

Observe that, since the functions $G_i(t,x_1,x_2,x_3,x_4)$, $i=1,2$ is Lipschitz continuous in x_1, x_2, x_3, x_4 and $f_i(x_1,x_2)$, $i=1,2$ is Lipschitz continuous in x_1, x_2 we have

$$\begin{aligned}
&\left|G_1\left(t,x_1,x_2,x_3,x_4\right)\right| \\
&\quad=\left|G_1\left(t,x_1,x_2,x_3,x_4\right)-G_1\left(t,0,0,0,0\right)+G_1\left(t,0,0,0,0\right)\right| \\
&\quad\leq\left|G_1\left(t,x_1,x_2,x_3,x_4\right)-G_1\left(t,0,0,0,0\right)\right|+\left|G_1\left(t,0,0,0,0\right)\right| \\
&\quad\leq\sum_{j=1}^{4}N_j\left|x_j\right|, \\
&\left|G_2\left(t,x_1,x_2,x_3,x_4\right)\right| \\
&\quad=\left|G_2\left(t,x_1,x_2,x_3,x_4\right)-G_2\left(t,0,0,0,0\right)+G_2\left(t,0,0,0,0\right)\right| \\
&\quad\leq\left|G_2\left(t,x_1,x_2,x_3,x_4\right)-G_2\left(t,0,0,0,0\right)\right|+\left|G_2\left(t,0,0,0,0\right)\right| \\
&\quad\leq\sum_{j=1}^{4}R_j\left|x_j\right|,
\end{aligned}$$

$$\begin{aligned}
\left|f_1\left(x_1,x_2\right)\right| &=\left|f_1\left(x_1,x_2\right)-f_1\left(0,0\right)+f_1\left(0,0\right)\right| \\
&\leq\left|f_1\left(x_1,x_2\right)-f_1\left(0,0\right)\right|+\left|f_1\left(0,0\right)\right| \\
&\leq\sum_{j=1}^{2}d_j\left|x_j\right|,
\end{aligned}$$

and

$$\begin{aligned}
\left|f_2\left(x_1,x_2\right)\right| &=\left|f_2\left(x_1,x_2\right)-f_2\left(0,0\right)+f_2\left(0,0\right)\right| \\
&\leq\left|f_2\left(x_1,x_2\right)-f_2\left(0,0\right)\right|+\left|f_2\left(0,0\right)\right| \\
&\leq\sum_{j=1}^{2}q_j\left|x_j\right|.
\end{aligned}$$

Theorem 2.2. *Suppose* (1.2), (1.3) *and* (2.1)-(2.9) *hold. Suppose that*

$$\beta_1+L_1\sum_{j=1}^{4}N_j+\alpha_1\sum_{j=1}^{2}d_j\leq 1,\ \ \textit{and}\ \beta_2+L_2\sum_{j=1}^{4}R_j+\alpha_2\sum_{j=1}^{2}q_j\leq 1.$$

Then (1.1) *has a T-periodic solution.*

Proof. In order to prove that (1.1) has a T-periodic solution, we shall make sure that A and B satisfy the conditions of Lemma 2.1. For all $(x,y)\in\Omega_{x,y}$, we have $(x,y)(t+T) = (x,y)(t)$ and $\|(x,y)\| \leq M$. Now let us discuss $B(x,y) + A(x,y)$. We have

$$\begin{aligned} B_1(x,y)(t+T) =& \frac{c_1(t+T)\,x(t+T-\tau_1(t+T))}{1-\tau_1'(t+T)} \\ =& \frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)} = B_1(x,y)(t), \end{aligned}$$

and

$$\begin{aligned} & A_1(x,y)(t+T) \\ =& -\int_{t+T}^{t+2T} \frac{e^{\int_u^{t+2T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} r_1(u)\,x(u-\tau_1(u))\,du \\ & + \int_{t+T}^{t+2T} \frac{e^{\int_u^{t+2T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ & + \int_{t+T}^{t+2T} \frac{e^{\int_u^{t+2T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s)\,f_1(x(s),y(s))\,dsdu \\ =& -\int_{t}^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} r_1(u)\,x(u-\tau_1(u))\,du \\ & + \int_{t}^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ & + \int_{t}^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s)\,f_1(x(s),y(s))\,dsdu \\ =& A_1(x,y)(t). \end{aligned}$$

Then $E_1(x,y)(t+T) = E_1(x,y)(t)$. In a similar way we can easily show that $E_2(x,y)(t+T) = E_2(x,y)(t)$. Therefore, $E(x,y)(t+T) = E(x,y)(t)$.

For any $(x,y) \in \Omega_{x,y}$, we will show that $|E(x,y)(t)| \leq M$. In view of the above estimates, we have

$$|B_1(x,y)(t)| \leq \left|\frac{c_1(t)}{1-\tau_1'(t)}\right| |x(t-\tau_1(t))| \leq \mu_1 M,$$

and

$$|A_1(x,y)(t)| \leq \int_t^{t+T} \left|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right| |r_1(u)|\,|x(u-\tau_1(u))|\,du$$

$$+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right||G_1(u,x(u),y(u),x(u-\tau_1(u)),$$
$$y(u-\tau_2(u)))|du$$
$$+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right|\int_{-\infty}^{u}|a_1(u,s)||f_1(x(s),y(s))|dsdu$$
$$\leq \beta_1 M+L_1K_1+\alpha_1 M_1.$$

As a consequence of (2.10)

$$\frac{L_1K_1+\alpha_1M_1}{1-\mu_1-\beta_1}\leq M,$$

so,

$$L_1K_1+\alpha_1M_1\leq(1-\mu_1-\beta_1)M.$$

This implies that

$$|E_1(x,y)(t)|\leq \mu_1M+\beta_1M+L_1K_1+\alpha_1M_1$$
$$\leq \mu_1M+\beta_1M+(1-\mu_1-\beta_1)M=M.$$

In a similar way we can easily show that

$$|E_2(x,y)(t)|\leq M.$$

Thus, E maps $\Omega_{x,y}$ into itself, i.e. $E(\Omega_{x,y})\subseteq\Omega_{x,y}$. We will now show that A is continuous. Let $\{(x_n,y_n)\}$ be a sequence in $\Omega_{x,y}$ such that

$$\lim_{n\to\infty}\|(x_n,y_n)-(x,y)\|=0.$$

Since $\Omega_{x,y}$ is closed, we have $(x,y)\in\Omega_{x,y}$. Then by the definition of A we have

$$\|A(x_n,y_n)-A(x,y)\|$$
$$=\max\left\{\max_{t\in[0,T]}|A_1(x_n,y_n)(t)-A_1(x,y)(t)|,\max_{t\in[0,T]}|A_2(x_n,y_n)(t)-A_2(x,y)(t)|\right\},$$

in which

$$|A_1(x_n,y_n)(t)-A_1(x,y)(t)|$$

$$\leq \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |r_1(u)| \, |x_n(u - \tau_1(u)) - x(u - \tau_1(u))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |G_1(u, x_n(u), y_n(u), x_n(u-\tau_1(u)), y_n(u-\tau_2(u)))$$
$$-G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u)))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| \int_{-\infty}^u |a_1(u,s)| \, |f_1(x_n(s), y_n(s)) - f_1(x(s), y(s))| \, dsdu,$$

the continuity of G_1 and f_1 along with the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \max_{t\in[0,T]} |A_1(x_n, y_n)(t) - A_1(x, y)(t)| = 0.$$

By a similar argument we can easily argue that

$$\lim_{n\to\infty} \max_{t\in[0,T]} |A_2(x_n, y_n)(t) - A_2(x, y)(t)| = 0.$$

Thus

$$\lim_{n\to\infty} \|A(x_n, y_n) - A(x, y)\| = 0.$$

This result proves that A is continuous.

We now have to show that A is compact. For $n \in \mathbb{N}$, let $(x_n, y_n) \in \Omega_{x,y}$, we have

$$|A_1(x_n, y_n)(t)|$$
$$\leq \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |r_1(u)| \, |x_n(u - \tau_1(u))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |G_1(u, x_n(u), y_n(u), x_n(u-\tau_1(u)), y_n(u-\tau_2(u)))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| \int_{-\infty}^u |a_1(u,s)| \, |f_1(x_n(s), y_n(s))| \, dsdu$$
$$\leq \left(\beta_1 + L_1 \sum_{j=1}^4 N_j + \alpha_1 \sum_{j=1}^2 d_j \right) M \leq M.$$

In a similar way we can easily show that

$$|A_2(x_n,y_n)(t)| \leq \left(\beta_2 + L_2\sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j\right) M \leq M.$$

Thus

$$\|A(x_n,y_n)\| \leq M.$$

If we calculate $(A(x_n,y_n))'(t)$, then

$$\begin{aligned}
&(A_1(x_n,y_n))'(t)\\
&\leq G_1(t,x_n(t),y_n(t),x_n(t-\tau_1(t)),y_n(t-\tau_2(t))) - r_1(t)x_n(t-\tau_1(t))\\
&\quad + \int_{-\infty}^{t} a_1(t,s) f_1(x_n(s),y_n(s))ds + h_1(t)\\
&\quad \times \Bigg[\int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} G_1(u,x_n(u),y_n(u),x_n(u-\tau_1(u)),y_n(u-\tau_2(u)))du\\
&\qquad - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} r_1(u)x_n(u-\tau_1(u))du\\
&\qquad + \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s) f_1(x(s),y(s))dsdu\Bigg]\\
&= G_1(t,x_n(t),y_n(t),x_n(t-\tau_1(t)),y_n(t-\tau_2(t))) - r_1(t)x_n(t-\tau_1(t))\\
&\quad + \int_{-\infty}^{t} a_1(t,s) f_1(x_n(s),y_n(s))ds + h_1(t)A_1(x_n,y_n)(t).
\end{aligned}$$

Hence, for some positive constant D_1, we obtain

$$\begin{aligned}
&\left|(A_1(x_n,y_n))'(t)\right|\\
&\quad = |G_1(t,x_n(t),y_n(t),x_n(t-\tau_1(t)),y_n(t-\tau_2(t)))| + |r_1(t)|\,|x_n(t-\tau_1(t))|\\
&\qquad + \int_{-\infty}^{t} |a_1(t,s)|\,|f_1(x_n(s),y_n(s))|\,ds + |h_1(t)|\,|A_1(x_n,y_n)(t)|\\
&\quad \leq \left[\sum_{j=1}^{4} N_j + \theta_1 \sum_{j=1}^{2} d_j + \theta_3 + \theta_4\right] M \leq D_1,
\end{aligned}$$

where $\sup_{t\in[0,T]}|r_1(t)| = \theta_3$, $\sup_{t\in[0,T]}|h_1(t)| = \theta_4$. In a similar way we can show for some positive constant D_2 that

$$\left|\left(A_2\left(x_n, y_n\right)\right)'(t)\right| \leq \left[\sum_{j=1}^{4} R_j + \theta_2 \sum_{j=1}^{2} q_j + \theta_5 + \theta_6\right] M \leq D_2,$$

where $\sup_{t\in[0,T]} \left|r_2(t)\right| = \theta_5$, $\sup_{t\in[0,T]} \left|h_2(t)\right| = \theta_6$. Thus

$$\left\|\left(A\left(x_n, y_n\right)\right)'\right\| \leq D,$$

where $D = \max\left(D_1, D_2\right)$. Thus, the sequence $\left(A\left(x_n, y_n\right)\right)$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that there exists a subsequence $\left(A\left(x_{n_k}, y_{n_k}\right)\right)$ of $\left(A\left(x_n, y_n\right)\right)$ converges uniformly to a continuous T-periodic function $\left(x^*, y^*\right)$. Thus, A is compact.

For all $\left(x_1, y_1\right), \left(x_2, y_2\right) \in \Omega_{x,y}$

$$\begin{aligned}\left|B_1\left(x_1, y_1\right)(t) - B_1\left(x_2, y_2\right)(t)\right| &= \left|\frac{c_1(t)\, x_1\left(t-\tau_1(t)\right)}{1-\tau_1'(t)} - \frac{c_1(t)\, x_2\left(t-\tau_1(t)\right)}{1-\tau_1'(t)}\right| \\ &= \left|\frac{c_1(t)}{1-\tau_1'(t)}\right| \left|x_1\left(t-\tau_1(t)\right) - x_2\left(t-\tau_1(t)\right)\right| \\ &\leq \mu_1 \left|x_1\left(t-\tau_1(t)\right) - x_2\left(t-\tau_1(t)\right)\right|,\end{aligned}$$

hence B_1 is contraction because $\mu_1 < 1$. In a similar way we can easily show that

$$\left|B_2\left(x_1, y_1\right)(t) - B_2\left(x_2, y_2\right)(t)\right| \leq \mu_2 \left|y_1\left(t-\tau_2(t)\right) - y_2\left(t-\tau_2(t)\right)\right|,$$

hence B_2 is contraction because $\mu_2 < 1$. Then

$$\begin{aligned}&\left|B\left(x_1, y_1\right)(t) - B\left(x_2, y_2\right)(t)\right| \\ &\quad = \max\left\{\left|B_1\left(x_1, y_1\right)(t) - B_1\left(x_2, y_2\right)(t)\right|, \left|B_2\left(x_1, y_1\right)(t) - B_2\left(x_2, y_2\right)(t)\right|\right\},\end{aligned}$$

this implies that

$$\left\|B\left(x_1, y_1\right) - B\left(x_2, y_2\right)\right\| \leq \mu \left\|\left(x_1, y_1\right), \left(x_2, y_2\right)\right\|.$$

Hence B is contraction.

Thus, the conditions of Lemma 2.1 are satisfied and there is a $(x, y) \in \Omega_{x,y}$, such that $(x, y) = A(x, y) + B(x, y)$. □

In the next theorem we relax condition (2.2).

Theorem 2.3. *Suppose* (1.2), (1.3), (2.1) *and* (2.3)-(2.9) *hold. Suppose that*

$$\beta_1 + L_1 \sum_{j=1}^{4} N_j + \alpha_1 \sum_{j=1}^{2} d_j \leq 1, \text{ and } \beta_2 + L_2 \sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j \leq 1.$$

In addition, we assume the existence of continuous nondecreasing function W_2 *such that*

$$|f_2(x,y)| \leq f_2(|x|, y) \leq Q_2 W_2(|x|), \tag{2.19}$$

for some positive constant Q_2*, and for* $u > 0$ *we ask that*

$$\frac{W_2(u)}{u} \leq \frac{1 - \mu_2 - \beta_2 - \frac{L_2 K_2}{M}}{\alpha_2 Q_2}. \tag{2.20}$$

Then (1.1) *has a* T*-periodic solution.*

Proof. Set

$$M = \max\left\{ \frac{L_1 K_1 + \alpha_1 M_1}{1 - \mu_1 - \beta_1}, \frac{L_2 K_2 + \alpha_2 Q_2 W_2(M)}{1 - \mu_2 - \beta_2} \right\}. \tag{2.21}$$

Note that due to (2.20) we have

$$M \geq \frac{L_2 K_2 + \alpha_2 Q_2 W_2(M)}{1 - \mu_2 - \beta_2},$$

and hence (2.20) is well defined. For any $(x, y) \in \Omega_{x,y}$, we have by the proof of the previous theorem that

$$|E_1(x,y)(t)| \leq M.$$

Thus

$$|B_2(x,y)(t)| \leq \left| \frac{c_2(t)}{1 - \tau_2'(t)} \right| |y(t - \tau_2(t))| \leq \mu_2 M,$$

and

$|A_2(x,y)(t)|$

$$\begin{aligned}
&\leq \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| |r_2(u)| \, |x(u-\tau_1(u))| \, du \\
&\quad + \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| |G_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))| \, du \\
&\quad + \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| \int_{-\infty}^t |a_2(u,s)| \, f_2(|x(s)|, y(s)) \, dsdu \\
&\leq M \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| |r_2(u)| \, du + K_2 \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| du \\
&\quad + Q_2 W_2(M) \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_2(s)ds}}{1-e^{\int_0^T h_2(s)ds}} \right| \int_{-\infty}^t |a_2(u,s)| \, dsdu \\
&\leq \beta_2 M + L_2 K_2 + \alpha_2 Q_2 W_2(M).
\end{aligned}$$

As a consequence of (2.21)

$$\frac{L_2 K_2 + \alpha_2 Q_2 W_2(M)}{1-\mu_2-\beta_2} \leq M,$$

so,

$$L_2 K_2 + \alpha_2 Q_2 W_2(M) \leq (1-\mu_2-\beta_2) M.$$

This implies that

$$\begin{aligned}
|E_2(x,y)(t)| &\leq \mu_2 M + \beta_2 M + L_2 K_2 + \alpha_2 Q_2 W_2(M) \\
&\leq \mu_2 M + \beta_2 M + (1-\mu_2-\beta_2) M = M.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

In the next theorem we relax condition (2.1).

Theorem 2.4. *Suppose* (1.2), (1.3), (2.2) *and* (2.3)-(2.9) *hold. Suppose that*

$$\beta_1 + L_1 \sum_{j=1}^4 N_j + \alpha_1 \sum_{j=1}^2 d_j \leq 1 \text{ and } \beta_2 + L_2 \sum_{j=1}^4 R_j + \alpha_2 \sum_{j=1}^2 q_j \leq 1.$$

In addition, we assume the existence of continuous nondecreasing function

W_1 such that

$$|f_1(x,y)| \leq f_1(x,|y|) \leq Q_1 W_1(|y|), \tag{2.22}$$

for some positive constant Q_1, and for $u>0$ we ask that

$$\frac{W_1(u)}{u} \leq \frac{1-\mu_1-\beta_1-\frac{L_1K_1}{M}}{\alpha_1 Q_1}. \tag{2.23}$$

Then (1.1) *has a T-periodic solution.*

Proof. Set

$$M = \max\left\{\frac{L_1K_1+\alpha_1 Q_1 W_1(M)}{1-\mu_1-\beta_1}, \frac{L_2K_2+\alpha_2 M_2}{1-\mu_1-\beta_1}\right\}. \tag{2.24}$$

Note that due to (2.23) we have

$$M \geq \frac{L_1K_1+\alpha_1 Q_1 W_1(M)}{1-\mu_1-\beta_1},$$

and hence (2.23) is well defined. For any $(x,y)\in\Omega_{x,y}$, we have by the proof of the previous theorem that

$$|E_2(x,y)(t)| \leq M.$$

Thus

$$|B_1(x,y)(t)| \leq \left|\frac{c_1(t)}{1-\tau_1'(t)}\right| |x(t-\tau_1(t))| \leq \mu_1 M,$$

and

$$\begin{aligned}
&|A_1(x,y)(t)| \\
&\leq \int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right| |r_1(u)|\,|x(u-\tau_1(u))|\,du \\
&\quad+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right| |G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))|\,du \\
&\quad+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\right| \int_{-\infty}^t |a_1(u,s)|\, f_1(x(s),|y(s)|)\,dsdu
\end{aligned}$$

$$\leq M \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| |r_1(u)| \, du + K_1 \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| du$$
$$+ Q_1 W_1(M) \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \right| \int_{-\infty}^t |a_1(u,s)| \, ds du$$
$$\leq \beta_1 M + L_1 K_1 + \alpha_1 Q_1 W_1(M).$$

As a consequence of (2.24)

$$\frac{L_1 K_1 + \alpha_1 Q_1 W_1(M)}{1 - \mu_1 - \beta_1} \leq M,$$

so,

$$L_1 K_1 + \alpha_1 Q_1 W_1(M) \leq (1 - \mu_1 - \beta_1) M.$$

This implies that

$$\begin{aligned} |E_1(x,y)(t)| &\leq \mu_1 M + \beta_1 M + L_1 K_1 + \alpha_1 Q_1 W_1(M) \\ &\leq \mu_1 M + \beta_1 M + (1 - \mu_1 - \beta_1) M = M. \end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

3. Asymptotic Stability of Periodic Solutions

In this section, we show that under mild conditions one obtains asymptotically periodic solutions. We do not assume the periodicity condition on the functions a_1, a_2, c_1, c_2, τ_1, τ_2, G_1 and G_2, we only assume h_1 and h_2 are T-periodic, and

$$\int_0^T h_1(s)ds = 0, \ \int_0^T h_2(s)ds = 0. \tag{3.1}$$

Since h_1 and h_2 are T-periodic, there are constants m_k and M_k^*, $k = 1,2$ such that

$$m_1 \leq e^{\int_0^t h_1(s)ds} \leq M_1^*, \ m_2 \leq e^{\int_0^t h_1(s)ds} \leq M_2^*. \tag{3.2}$$

Furthermore, we assume that there are positive numbers V_1 and V_2 such that

$$\int_t^\infty \int_{-\infty}^u |a_1(u,s)| \, ds du \leq V_1, \ \int_t^\infty \int_{-\infty}^u |a_2(u,s)| \, ds du \leq V_2. \tag{3.3}$$

In addition, we suppose that

$$\sup_{t\in\mathbb{R}}\left|\frac{c_1(t)}{1-\tau_1'(t)}\right| = \mu_1^*,\ \sup_{t\in\mathbb{R}}\left|\frac{c_2(t)}{1-\tau_2'(t)}\right| = \mu_2^*,\ \max\{\mu_1^*,\mu_2^*\} = \mu^* < 1,$$

$$\lim_{t\to\infty}\frac{c_1(t)}{1-\tau_1'(t)} = 0,\ \lim_{t\to\infty}\frac{c_2(t)}{1-\tau_2'(t)} = 0, \tag{3.4}$$

$$\lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |a_1(u,s)|\,dsdu = 0,\ \lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |a_2(u,s)|\,dsdu = 0, \tag{3.5}$$

$$\lim_{t\to\infty}\int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}}du = 0,\ \lim_{t\to\infty}\int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}}du = 0, \tag{3.6}$$

$$\int_t^\infty |r_1(u)|\,du \to 0,\ \int_t^\infty |r_2(u)|\,du \to 0 \text{ as } t\to\infty, \tag{3.7}$$

and for positive constants M_k^*, $k=\overline{3,6}$ we ask that

$$\int_t^\infty |r_1(u)|\,du \le M_3^*,\ \int_t^\infty |r_2(u)|\,du \le M_4^*, \tag{3.8}$$

and

$$\int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}}du \le M_5^*,\ \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}}du \le M_6^*. \tag{3.9}$$

Finally, we make the assumption that

$$1-\mu_1^* - M_1^*M_3^*m_1^{-1} > 0, \tag{3.10}$$

and

$$1-\mu_2^* - M_2^*M_4^*m_2^{-1} > 0. \tag{3.11}$$

Theorem 3.1. *Assume* (1.2), (1.3). *Then x and y is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) =& \rho_1 e^{\int_0^t h_1(s)ds} + \frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)} + \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\,x(u-\tau_1(u))\,du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))\,du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u a_1(u,s)\,f_1(x(s),y(s))\,dsdu, \end{aligned} \tag{3.12}$$

and

$$y(t) = \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t)y(t-\tau_2(t))}{1-\tau_2'(t)} + \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} r_2(u)y(u-\tau_2(u))du$$
$$- \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} G_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))du$$
$$- \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} \int_{-\infty}^u a_2(u,s) f_2(x(s), y(s)) dsdu, \tag{3.13}$$

where r_1 *and* r_2 *are given by* (1.6) *and* (1.7).

Proof. Let (x, y) be a solution of (1.1). Next we multiply both sides of the first equation in (1.1) by $e^{-\int_0^t h_1(s)ds}$, and then integrate from t to ∞, to obtain

$$\int_t^\infty \left[x(u)e^{-\int_0^u h_1(s)ds}\right]' du$$
$$= \int_t^\infty e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u))) du$$
$$+ \int_t^\infty e^{-\int_0^u h_1(s)ds} c_1(u) x'(u-\tau_1(u)) du$$
$$+ \int_t^\infty e^{-\int_0^u h_1(s)ds} \int_{-\infty}^u a_1(u,s) f_1(x(s), y(s)) dsdu.$$

Consequently, we have

$$\rho_1^* - x(t)e^{-\int_0^t h_1(s)ds}$$
$$= \int_t^\infty e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u))) du$$
$$+ \int_t^\infty e^{-\int_0^u h_1(s)ds} c_1(u) x'(u-\tau_1(u)) du$$
$$+ \int_t^\infty e^{-\int_0^u h_1(s)ds} \int_{-\infty}^u a_1(u,s) f_1(x(s), y(s)) dsdu.$$

Multiply both sides with $e^{\int_0^t h_1(s)ds}$, we obtain

$$x(t) = \rho_1^* e^{\int_0^t h_1(s)ds}$$
$$- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u))) du$$

$$-\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} c_1(u)\, x'(u-\tau_1(u))\, du$$
$$-\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s)\, f_1(x(s), y(s))\, ds du. \tag{3.14}$$

Letting

$$\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} c_1(u)\, x'(u-\tau_1(u))\, du$$
$$= \int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \frac{c_1(u)}{1-\tau_1'(u)} \left(1-\tau_1'(u)\right) x'(u-\tau_1(u))\, du.$$

Performing an integration by parts, we get

$$\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} c_1(u)\, x'(u-\tau_1(u))\, du$$
$$= \left[\frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \frac{c_1(u)\, x(u-\tau_1(u))}{1-\tau_1'(u)}\right]_t^{\infty} - \int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du$$
$$= \rho_1^{**} e^{\int_0^t h_1(s)ds} - \frac{c_1(t)\, x(t-\tau_1(t))}{1-\tau_1'(t)} - \int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du, \tag{3.15}$$

where r_1 is given by (1.6). Substituting (3.15) into (3.14), we obtain

$$x(t) = \rho_1 e^{\int_0^t h_1(s)ds} + \frac{c_1(t)x(t-\tau_1(t))}{1-\tau_1'(t)} + \int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)x(u-\tau_1(u))du$$
$$-\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du$$
$$-\int_t^{\infty} \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^{u} a_1(u,s)\, f_1(x(s), y(s))\, ds du,$$

where $\rho_1 = \rho_1^* - \rho_1^{**}$. The proof of (3.13) is similar and hence we omit it. □

Theorem 3.2. *Suppose that* (2.1), (2.2) *and* (3.1)-(3.11) *hold. Then system* (1.1) *has asymptotically* T*-periodic solution* (x, y) *satisfying*

$$x(t) = x_1(t) + x_2(t),\ \ y(t) = y_1(t) + y_2(t),$$

where

$$x_1(t) = \rho_1 e^{\int_0^t h_1(s)ds},\ y_1(t) = \rho_2 e^{\int_0^t h_2(s)ds},\ t \in \mathbb{R},$$

for arbitrary fixed nonzero constants $\rho_1,\ \rho_2$ *and*

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} y_2(t) = 0.$$

Proof. Define

$$\begin{aligned} P_T^* = &\{(\varphi,\psi) : \varphi = \varphi_1 + \varphi_2,\ \psi = \psi_1 + \psi_2,\ (\varphi_1,\psi_1)(t+T) = (\varphi_1,\psi_1)(t), \\ &\text{and } (\varphi_2,\psi_2)(t) \to (0,0) \text{ as } t \to \infty\}, \end{aligned}$$

where both φ and ψ are real valued bounded continuous functions on $\mathbb{R}$. Then P_T^* is a Banach space when endowed with the maximum norm

$$\|(x,y)\| = \max\left\{\sup_{t\in\mathbb{R}} |x(t)|, \sup_{t\in\mathbb{R}} |y(t)|\right\}.$$

We define a subset $\Omega_{x,y}^*$ of P_T^* as follows. For a constant V^* to be defined later in the proof, let

$$\Omega_{x,y}^* = \{(x,y) \in P_T^* \text{ with } \|(x,y)\| \le V^*\}.$$

Then $\Omega_{x,y}^*$ is a bounded, closed and convex subset of P_T^*. Now for $(x,y) \in \Omega_{x,y}^*$ we can define an operator $F : \Omega_{x,y}^* \to P_T^*$ by

$$F(x,y)(t) = (F_1(x,y)(t), F_2(x,y)(t)),$$

where

$$\begin{aligned} F_1(x,y)(t) = &\rho_1 e^{\int_0^t h_1(s)ds} + \frac{c_1(t)\, x(t-\tau_1(t))}{1-\tau_1'(t)} \\ &+ \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u a_1(u,s)\, f_1(x(s), y(s))\, dsdu, \qquad (3.16) \end{aligned}$$

and

$$\begin{aligned} F_2(x,y)(t) = & \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t)\, y(t-\tau_2(t))}{1-\tau_2'(t)} \\ & + \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} r_2(u)\, y(u-\tau_2(u))\, du \\ & - \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} G_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ & - \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} \int_{-\infty}^u a_2(u,s)\, f_2(x(s),y(s))\, dsdu. \end{aligned} \tag{3.17}$$

We will show that the mapping F has a fixed point in $\Omega^*_{x,y}$. To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is continuous compact. Therefore, we state (3.12) as

$$F_1(x,y)(t) = B_3(x,y)(t) + A_3(x,y)(t),$$

where $B_3, A_3 : \Omega^*_{x,y} \to P_T^*$ are given by

$$B_3(x,y)(t) = \rho_1 e^{\int_0^t h_1(s)ds} + \frac{c_1(t)\, x(t-\tau_1(t))}{1-\tau_1'(t)},$$

and

$$\begin{aligned} A_3(x,y)(t) = & \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du \\ & - \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ & - \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u a_1(u,s)\, f_1(x(s),y(s))\, dsdu. \end{aligned}$$

And we state (3.13) as

$$F_2(x,y)(t) = B_4(x,y)(t) + A_4(x,y)(t),$$

where $B_4, A_4 : \Omega^*_{x,y} \to P_T^*$ are given by

$$B_4(x,y)(t) = \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t)\, y(t-\tau_2(t))}{1-\tau_2'(t)},$$

and

$$\begin{aligned}A_4(x,y)(t) &= \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} r_2(u)\, y(u-\tau_2(u))\, du \\ &\quad - \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} G_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))du \\ &\quad - \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} \int_{-\infty}^u a_2(u,s)\, f_2(x(s),y(s))\, dsdu.\end{aligned}$$

Now for $(x,y) \in \Omega^*_{x,y}$ we can define the operators $B^*, A^* : \Omega^*_{x,y} \to P^*_T$ by

$$B^*(x,y)(t) = (B_3(x,y)(t), B_4(x,y)(t)),$$

$$A^*(x,y)(t) = (A_3(x,y)(t), A_4(x,y)(t)).$$

Set $V^* = \max\{V_1^*, V_2^*\}$, where

$$V_1^* = \frac{K_1 M_5^* m_1^{-1} + M_1^* M_1 V_1 m_1^{-1} + \rho_1 M_1^*}{1 - \mu_1^* - M_1^* M_3^* m_1^{-1}},$$

$$V_2^* = \frac{K_2 M_6^* m_2^{-1} + M_2^* M_2 V_2 m_1^{-1} + \rho_2 M_2^*}{1 - \mu_2^* - M_2^* M_4^* m_2^{-1}}.$$

We note that V^* is well defined due to (3.10) and (3.11). First, we demonstrate that $F\left(\Omega^*_{x,y}\right) \subseteq \Omega^*_{x,y}$. If $(x,y) \in \Omega^*_{x,y}$, then by (3.10) we have

$$\begin{aligned}&\left|F_1(x,y)(t) - \rho_1 e^{\int_0^t h_1(s)ds}\right| \\ &\le \left|\frac{c_1(t)}{1-\tau_1'(t)}\right| |x(t-\tau_1(t))| + \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} |r_1(u)|\, |x(u-\tau_1(u))|\, du \\ &\quad + \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} |G_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))|\, du \\ &\quad + \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u |a_1(u,s)|\, |f_1(x(s),y(s))|\, dsdu, \\ &\le \mu_1^* V^* + M_1^* M_3^* m_1^{-1} V^* + K_1 M_5^* m_1^{-1} + M_1^* M_1 V_1 m_1^{-1},\end{aligned}$$

and in a similar way we have

$$\left|F_2(x,y)(t) - \rho_2 e^{\int_0^t h_2(s)ds}\right| \le \mu_2^* V^* + M_2^* M_4^* m_2^{-1} V^* + K_2 M_6^* m_2^{-1} + M_2^* M_2 V_2 m_2^{-1}.$$

This implies that

$$\begin{aligned}|F_1(x,y)(t)| &\le \mu_1^* V^* + M_1^* M_3^* m_1^{-1} V^* + K_1 M_5^* m_1^{-1} + M_1^* M_1 V_1 m_1^{-1} + \rho_1 M_1^* \\ &\le V^*,\end{aligned}$$

and

$$\begin{aligned}|F_2(x,y)(t)| &\le \mu_2^* V^* + M_2^* M_4^* m_2^{-1} V^* + K_2 M_6^* m_2^{-1} + M_2^* M_2 V_2 m_2^{-1} + \rho_2 M_2^* \\ &\le V^*.\end{aligned}$$

Hence, $F\left(\Omega_{x,y}^*\right) \subseteq \Omega_{x,y}^*$ as desired. The work to show that A^* is completely continuous and B^* is a contraction is similar to the corresponding work in Theorem 2.2, and hence we omit it here. Therefore, by Krasnoselskii's fixed point theorem, there exists a fixed point $(x,y) \in \Omega_{x,y}^*$ such that

$$F(x,y)(t) = (F_1(x,y)(t), F_2(x,y)(t)) = (x(t), y(t)).$$

By Theorem 3.1 we know that this fixed point is a solution of (1.1).

For an arbitrary fixed point $(x,y) \in \Omega_{x,y}^*$ of F, we obtain from (3.4)-(3.7)

$$\lim_{t\to\infty} |x(t) - x_1(t)| = \lim_{t\to\infty} |F_1(x,y)(t) - x_1(t)| = 0,$$

and

$$\lim_{t\to\infty} |y(t) - y_1(t)| = \lim_{t\to\infty} |F_2(x,y)(t) - y_1(t)| = 0.$$

By letting

$$\begin{aligned}x_2(t) = {}& \frac{c_1(t)\, x(t-\tau_1(t))}{1-\tau_1'(t)} + \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))du \\ &- \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u a_1(u,s)\, f_1(x(s), y(s))\, ds du,\end{aligned}$$

and

$$\begin{aligned}y_2(t) = {}& \frac{c_2(t)\, y(t-\tau_2(t))}{1-\tau_2'(t)} + \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} r_2(u)\, y(u-\tau_2(u))\, du \\ &- \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} G_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du\end{aligned}$$

$$-\int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^u h_2(s)ds}} \int_{-\infty}^u a_2(u,s) f_2(x(s), y(s))\, dsdu.$$

We see that (x,y) given by

$$x(t) = x_1(t) + x_2(t),\ y(t) = y_1(t) + y_2(t),$$

is an asymptotically T-periodic solution of (1.1). Note that by (3.4)-(3.7)

$$\begin{aligned}\lim_{t\to\infty} |x_2(t)| \leq & V^* \lim_{t\to\infty} \left| \frac{c_1(t)}{1-\tau_1'(t)} \right| + V^* \lim_{t\to\infty} \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} |r_1(u)|\, du \\ & + K_1 \lim_{t\to\infty} \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} du \\ & + M_1 \lim_{t\to\infty} \int_t^\infty \frac{e^{\int_0^t h_1(s)ds}}{e^{\int_0^u h_1(s)ds}} \int_{-\infty}^u |a_1(u,s)|\, dsdu,\end{aligned}$$

Hence

$$\lim_{t\to\infty} x_2(t) = 0.$$

Similarly

$$\lim_{t\to\infty} y_2(t) = 0.$$

Finally, we show that x_1 and y_1 are T-periodic. From (3.1), one can see

$$\begin{aligned}x_1(t+T) =& c_1 e^{\int_0^{t+T} h_1(s)ds} \\ =& c_1 e^{\int_0^t h_1(s)ds} e^{\int_t^{t+T} h_1(s)ds} \\ =& c_1 e^{\int_0^t h_1(s)ds} e^{\int_0^T h_1(s)ds} \\ =& c_1 e^{\int_0^t h_1(s)ds} \\ =& x_1(t).\end{aligned}$$

Similarly, y_1 is T-periodic. □

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