A NEW APPROACH FOR THE APPROXIMATE SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, the construction of the analytical solutions of space-time fractional integro-differential equations are studied. We first transform the space-time fractional integro-differential equations into space-time fractional differential equations by means of new transformations then construct the solutions via residual power series method (RPSM). Finally, we give some examples to illustrate how powerful and effective the method is.

1. Introduction

Since the mathematical models including fractional derivative are more realistic then the one including integer order derivatives, the concept of fractional calculus gets increasing attention from many scientists. As a result, this concept has various applications in many fields such as mathematics, physics and engineering [1, 22, 23, 24]. By means of nonsingular and singular kernels, quite a few fractional derivatives are defined to model engineering and scientific phenomena better [7, 8, 9, 10, 11]. Since obtaining an analytical solution of fractional integro-differential equations is impossible or inconvenient, various numerical methods have been developed to construct the solutions of them [1, 2, 5, 6, 12, 13, 14, 15, 16, 17, 21, 25, 26].

RPSM, developed by the Jordan mathematician Abu Arqub [2], is a very useful analytical method to establish the solutions of different types of FPDEs in series form [3, 4, 20, 21, 22, 23, 7, 8, 9].

In this manuscript, we consider the following space-time fractional integro-differential equations:

$$D_t^\alpha u = I_x^\beta u + F(x,t,u), \tag{1}$$

$$u(x,0) = \psi(x), \quad 0 < x \leq l, \tag{2}$$

$$u(0,t) = \mu_1(t), \quad 0 < t \leq T, \tag{3}$$

$$u_x(0,t) = \mu_2(t), \quad 0 < t \leq T, \tag{4}$$

where $0 \leq n < \alpha \leq n+1$, $0 \leq m < \beta \leq m+1$. The RPSM is implemented to construct a new algorithm for determining analytical solutions of the problem above. The existence and uniqueness as well as convergence of the approximate solution of this problem is proved in [27].

2. Preliminaries

We first give the main definitions and various features of the fractional calculus theory in this section. The Riemann-Liouville fractional integral operator of order α $(\alpha \geq 0)$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha > 0, x > 0, \tag{5}$$

$$J^0 f(x) = f(x). \tag{6}$$

The Caputo fractional derivative of order α is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{7}$$
$$m-1 < \alpha \leq m, \ x > 0,$$

where D^m is the classical differential operator of order m.

The time fractional derivative operator of order α of $u(x,t)$ is defined

as [18, 19]

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$$
$$= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\partial^n u(\tau,t)}{\partial t^n} d\tau, & n-1 < \alpha \leq n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n\epsilon N, \end{cases} \tag{8}$$

where n is the smallest integer greater than α.

3. Solving the Fractional Integro-Differential Equations (FIE)

In this section, we achieve the solution of fractional integro-differential equations $(1)-(2)$ with the help of the transformation $u = D_x^\beta v$ and RPSM. We consider the following two cases.

Case 1: Solution of FIE by the transformation $u = D_x^\beta v$ for $n = m = 0$

By the help of the transformation $u = D_x^\beta v$, we eliminate the Riemann-Liouville integration in Eq. (1). As a result, the following problem is obtained:

$$D_t^\alpha(D_x^\beta v) = v - \frac{I_x^\beta \mu_1(t)}{\Gamma(2-\beta)\Gamma(\beta)} + F(x,t,D_x^\beta v), \tag{9}$$

$$v(x,0) = I_x^\beta \psi(x), \qquad 0 < x \leq 1, \tag{10}$$

$$v(0,t) = I_x^\beta \mu_1(t), \qquad 0 < t \leq 1. \tag{11}$$

By the initial condition (10), the 0th-RPS approximate solution $v_0(x,t)$ is established as follows

$$v_0(x,t) = v(x,0) = I_x^\beta \psi(x) = f_0(x), \tag{12}$$

and k^{th}-truncated residual function of Eq. (9) is

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k + I_x^\beta \mu_1(t) \frac{1}{\Gamma(2-\beta)\Gamma(\beta)} - F(x,t,D_x^\beta v_k),$$
$$k = 0,1,2,\ldots \tag{13}$$

Now the steps below are taken:

Step 1: To construct $f_1(x)$, substituting

$$v_1(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{14}$$

into Eq. (13) leads to

$$Res_1 = D_x^\beta f_1(x) - f_0(x) - f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + I_x^\beta \mu_1(t)\frac{1}{\Gamma(2-\beta)\Gamma(\beta)} - F. \tag{15}$$

plugging $t = 0$, we have the following results:

$$f_1(0) = I_x^{2\beta}\mu_1(0), \quad f_1(x) = I_x^\beta f_0(x) + I_x^\beta F|_{t=0}. \tag{16}$$

Step 2: To establish $f_2(x)$, we substitute

$$v_2(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{17}$$

into Eq. (13) and use

$$Res_2 = D_x^\beta f_1(x) + D_x^\beta f_2(x)\frac{t^\alpha}{\Gamma(1+\alpha)} - f_0(x) - f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$- f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + I_x^\beta \mu_1(t)\frac{1}{\Gamma(2-\beta)\Gamma(\beta)} - F, \tag{18}$$

in Eq. (13). Hence $f_2(x)$ becomes

$$f_2(0) = I_x^{2\beta} D_t^\alpha \mu_1(0), \quad f_2(x) = I_x^\beta f_1(x) + I_x^\beta D_t^\alpha F|_{t=0}, \tag{19}$$

and so on.

As a result, the solution of (9)-(11) can be acquired in the following form:

$$v(x,t) = f_0(x) + (I_x^\beta f_0(x) + I_x^\beta F|_{t=0})\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ (I_x^\beta f_1(x) + I_x^\beta D_t^\alpha F|_{t=0})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \tag{20}$$

By the help of the transformation $u(x,t) = D_x^\beta v(x,t)$, the solution of problem

$(1)-(3)$ is constructed in the following form

$$u(x,t) = \psi(x) + (I_x^\beta \psi(x) + F|_{t=0})\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$+ (I_x^{2\beta}\psi(x) + I_x^\beta F|_{t=0} + D_t^\alpha F|_{t=0})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \quad (21)$$

Case 2: Solution of the FIE by the transformation $u = D_x^\beta v$ for $n = 0$ and $m = 1$

By taking the transformation $u = D_x^\beta v$ into account, we eliminate the Riemann-Liouville integration in Eq. (1). As a result, the following problem is obtained

$$D_t^\alpha(D_x^\beta v) = v - \frac{I_x^\beta \mu_2(t)x}{\Gamma(3-\beta)\Gamma(\beta)} - \frac{I_x^\beta \mu_1(t)}{\Gamma(3-\beta)\Gamma(\beta-1)} + F(x,t,D_x^\beta v), \quad (22)$$

$$v(x,0) = I_x^\beta \psi(x), \qquad 0 < x \le 1, \quad (23)$$

$$v(0,t) = I_x^\beta \mu_1(t), \qquad 0 < t \le 1, \quad (24)$$

$$v_x(0,t) = I_x^\beta \mu_2(t), \qquad 0 < t \le 1. \quad (25)$$

The kth-truncated residual function of Eq. (22) is obtained in the following form

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k + \frac{I_x^\beta \mu_2(t)x}{\Gamma(3-\beta)\Gamma(\beta)} + \frac{I_x^\beta \mu_1(t)}{\Gamma(3-\beta)\Gamma(\beta-1)}$$

$$- F(x,t,D_x^\beta v_k), \qquad k = 0,1,2,\ldots. \quad (26)$$

As above, following steps are taken :

Step 1: By the help of the initial condition, initial approximation becomes

$$v_0(x,t) = I_x^\beta \psi(x) = f_0(x). \quad (27)$$

In order to construct $f_1(x)$, we substitute

$$v_1(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (28)$$

into Eq. (26) which leads to the following

$$Res_1 = D_x^\beta f_1(x) - f_0(x) - f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} - f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$- \frac{I_x^\beta \mu_2(t)x}{\Gamma(3-\beta)\Gamma(\beta)} - \frac{I_x^\beta \mu_1(t)}{\Gamma(3-\beta)\Gamma(\beta-1)} - F. \tag{29}$$

As a result, we have

$$f_1(0) = I_x^{2\beta}\mu_1(0), \quad f_1'(0) = I_x^{2\beta}\mu_2(0), \quad f_1(x) = I_x^\beta f_0(x) + I_x^\beta F|_{t=0}. \tag{30}$$

Step 2: To construct $f_2(x)$, plugging

$$v_2(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{31}$$

into Eq. (26) which leads to the following results

$$f_2(0) = I_x^{2\beta}D_t^\alpha\mu_1(0), \quad f_2'(0) = I_x^{2\beta}D_t^\alpha\mu_2(0), \quad f_2(x) = I_x^\beta f_1(x) + I_x^\beta D_t^\alpha F|_{t=0}, \tag{32}$$

and so on.

The solution of $(22)-(25)$ is obtained in the series form as follows:

$$v(x,t) = f_0(x) + (I_x^\beta f_0(x) + I_x^\beta F|_{t=0})\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ (I_x^\beta f_1(x) + I_x^\beta D_t^\alpha F|_{t=0})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \tag{33}$$

Finally, the solution of $(1)-(4)$ is constructed in the following form via the transformation $u(x,t) = D_x^\beta v(x,t)$

$$u(x,t) = \psi(x) + (I_x^\beta \psi(x) + F|_{t=0})\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ (I_x^{2\beta}\psi(x) + I_x^\beta F|_{t=0} + D_t^\alpha F|_{t=0})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots . \tag{34}$$

4. Numerical Examples

In this section, some examples are presented to illustrate the obtained results in Case 1 and Case 2.

Example 1. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u, 0 < \alpha, \beta \le 1 \tag{35}$$

$$u(x,0) = \exp(2x), \tag{36}$$

$$u(0,t) = \exp(\frac{t}{2}), \tag{37}$$

which has the exact solution $u(x,t) = \exp(2x + \frac{t}{2})$. Eqs. (35)-(37) are transformed as follows:

$$D_t^\alpha(D_x^\beta v) = v - \frac{I_x^\beta \exp(\frac{t}{2})}{\Gamma(2-\beta)\Gamma(\beta)}, \tag{38}$$

$$v(x,0) = I_x^\beta \exp(2x) = f_0(x). \tag{39}$$

$$v(0,t) = I_x^\beta \exp(\frac{t}{2}). \tag{40}$$

The kth-truncated residual function is obtained in the following form

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k + \frac{I_x^\beta \exp(\frac{t}{2})}{\Gamma(2-\beta)\Gamma(\beta)}, k = 0, 1, 2, \ldots \tag{41}$$

Based on the RPS procedure, the first two coefficients are obtained as follows:

$$f_1(x) = I_x^\beta \exp(2x),$$
$$f_2(x) = I_x^{2\beta} \exp(2x),$$

and so on.

Hence the approximate solution of (35)−(37) is constructed in the following form:

$$u(x,t) = \exp(2x) + \frac{x^{1+\beta}}{2} E_{1,2+\beta}(2x) \frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ \frac{x^{2+2\beta}}{4} E_{1,3+2\beta}(2x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots . \tag{42}$$

From Table 1 and Figures 1−3 , it is clear that as α and β increase to 1 the solution gets closer to the exact solution of FIE with integer orders for all values of x and t.
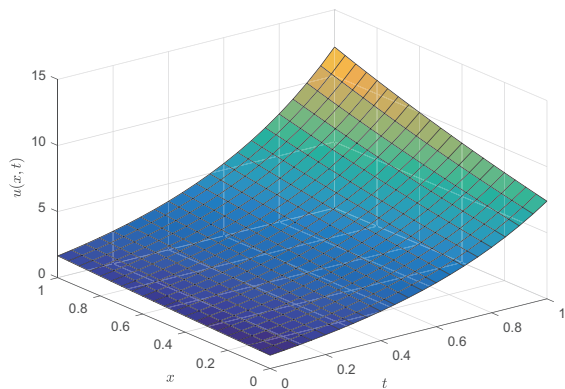
Figure 1:   The exact solution of Ex. 1.
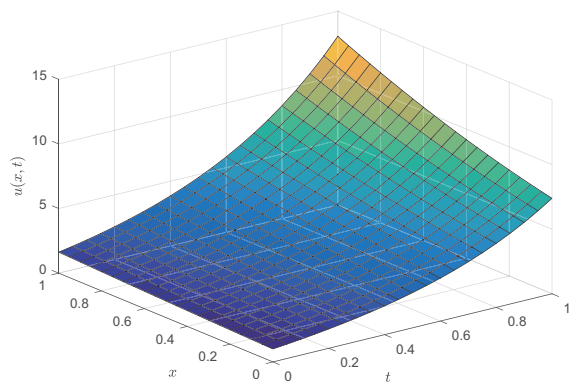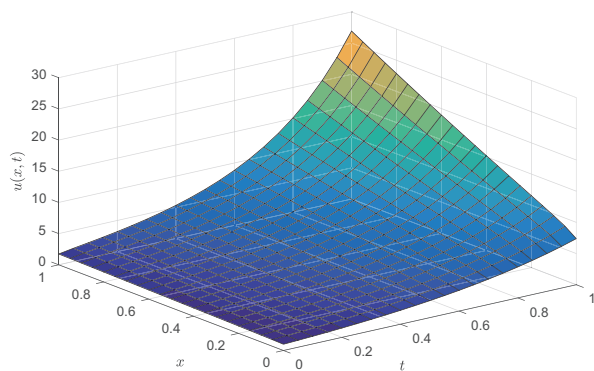

Figure 2:   The RPS solution of Ex. 1 for $\alpha = 1, \beta = 1$.

Figure 3:   The RPS solution of Ex. 1 for $\alpha = 0.75, \beta = 0.75$.

Table 1: The values of the approximate solutions for Example 1.

x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$	$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$
0.2	0.3	1.73325	1.74241	1.80249	1.89331	1.92097	2.02992	2.18494
	0.6	2.01375	2.04664	2.18736	2.39971	2.30467	2.52219	2.83174
	0.9	2.33965	2.40449	2.64645	3.01103	2.70182	3.03740	3.51498
0.4	0.3	2.58571	2.60720	2.70573	2.83398	2.96675	3.16899	3.42195
	0.6	3.00417	3.08450	3.31304	3.60951	3.65261	4.05166	4.54979
	0.9	3.49034	3.65746	4.04747	4.55212	4.37429	4.98516	5.74672
0.6	0.3	3.85743	3.89509	4.03828	4.21271	4.63828	4.99661	5.43093
	0.6	4.48169	4.62396	4.95325	5.35278	5.87339	6.57382	7.42059
	0.9	5.20698	5.50674	6.06502	6.74034	7.18055	8.24612	9.53207
0.8	0.3	5.75460	5.81418	6.01686	6.25631	7.38653	8.04989	8.85895
	0.6	6.68589	6.91171	7.37465	7.91950	9.67309	10.96190	12.53057
	0.9	7.76790	8.24563	9.02642	9.94261	12.09615	14.04878	16.42211

Example 2. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u + (1 - \frac{x}{2})u, 0 < \alpha, \beta \leq 1, \tag{43}$$

$$u(x,0) = 2x, \tag{44}$$

$$u(0,t) = 0, \tag{45}$$

which has the exact solution $u(x,t) = 2x \exp(t)$. Eqs. (43)-(45) are transformed as follows:

$$D_t^\alpha(D_x^\beta v) = v + (1 - \frac{x}{2})D_x^\beta v, \tag{46}$$

$$v(x,0) = I_x^\beta(2x) = f_0(x), \tag{47}$$

$$v(0,t) = 0. \tag{48}$$

The kth-truncated residual function is obtained in the following form

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k - (1 - \frac{x}{2})D_x^\beta v_k, k = 0, 1, 2, \ldots \tag{49}$$

Based on the RPS procedure the first two coefficients are obtained as follows:

$$f_1(x) = I_x^\beta f_0(x) - (1 - \frac{x}{2})D_x^\beta f_0(x),$$
$$f_2(x) = I_x^\beta f_1(x) - (1 - \frac{x}{2})D_x^\beta f_1(x),$$

Table 2: The values of approximate solutions for Example 2.

x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$	$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$
0.2	0.3	0.53994	0.55420	0.57743	0.61513	0.67037	0.72102	0.80152
	0.6	0.72885	0.77680	0.84911	0.96379	0.96242	1.08830	1.28554
	0.9	0.98384	1.06780	1.21502	1.44597	1.29029	1.50927	1.85013
0.4	0.3	1.07989	1.10480	1.14946	1.21161	1.33084	1.41897	1.53975
	0.6	1.45770	1.53920	1.66372	1.83409	1.89686	2.10052	2.37617
	0.9	1.96768	2.10320	2.34279	2.66742	2.52921	2.87113	3.33092
0.6	0.3	1.61983	1.65180	1.71639	1.79757	1.98143	2.10192	2.25190
	0.6	2.18654	2.28720	2.45638	2.66673	2.80334	3.06897	3.39680
	0.9	2.95152	3.10620	3.41997	3.80749	3.71674	4.15163	4.68577
0.8	0.3	2.15977	2.19520	2.27641	2.37008	2.62213	2.76803	2.93530
	0.6	2.91539	3.02080	3.22475	3.45839	3.68185	3.99262	4.34686
	0.9	3.93536	4.07680	4.44501	4.86494	4.85288	5.35180	5.91860

and so on.

Hence the approximate solution of $(43)-(45)$ is constructed in the following form:

$$
\begin{aligned}
u(x,t) = 2x + \Big(2\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 2x - x^2 \Big) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \Big(2\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} + 2\frac{x^{\beta}}{\Gamma(1+\beta)} \\
- 2\frac{x^{2+\beta}}{\Gamma(3+\beta)} + (2-x)\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 2x - 2x^2 + \frac{x^3}{2} \Big) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots .
\end{aligned}
\tag{50}
$$

From Table 2 and Figures $4-6$, it is clear that as α and β increase to 1 the solution gets closer to the exact solution of FIE with integer orders for all values of x and t. However as t increases, the solution gets away from the exact solution of FIE with integer orders.
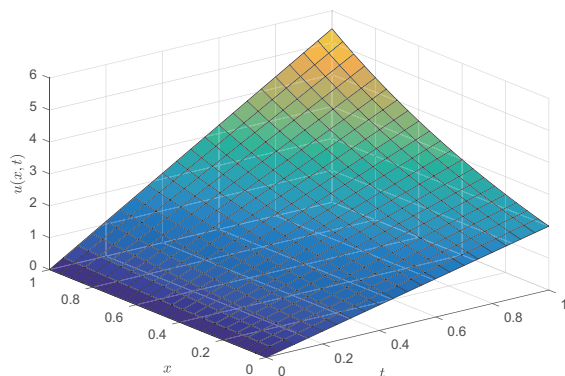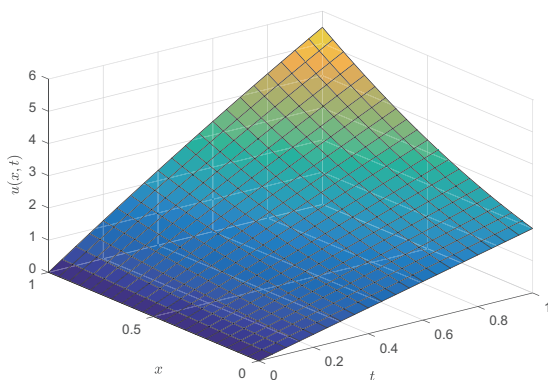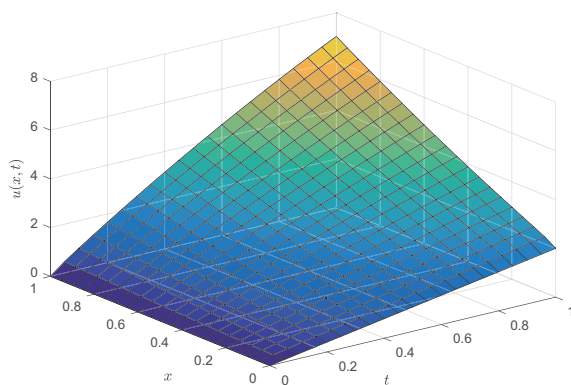
Figure 4: The exact solution of Ex. 2.

Figure 5: The RPS solution of Ex. 2 for $\alpha = 1, \beta = 1$.

Figure 6: The RPS solution of Ex. 2 for $\alpha = 0.75, \beta = 0.75$.

Example 3. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u + u^2 - 4x(1-4t) - x^2(5-18t+52t^2-64t^3+64t^4), 0<\alpha,\beta\le 1, \quad (51)$$
$$u(x,0) = 2x, \quad (52)$$
$$u(0,t) = 0, \quad (53)$$

which has the exact solution $u(x,t) = 2x(1 - 2t + 4t^2)$. Applying transformations to Eqs. (51)−(53), we have :

$$D_t^\alpha(D_x^\beta v) = v + (D_x^\beta v)^2 - 4x(1-4t) - x^2(5-18t+52t^2-64t^3+64t^4), \quad (54)$$
$$v(x,0) = 2\frac{x^{1+\beta}}{\Gamma(2+\beta)} = f_0(x), \quad (55)$$
$$v(0,t) = 0. \quad (56)$$

The kth-truncated residual function becomes

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k - D_x^\beta v D_x^{\beta+1} v + \frac{x^2}{1-2t}, k = 0, 1, 2, \ldots \quad (57)$$

Based on the RPS procedure, the first two coefficients are obtained as follows:

$$f_1(x) = 2\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} + 8\frac{x^{2+\beta}}{\Gamma(3+\beta)} - 4\frac{x^{1+\beta}}{\Gamma(2+\beta)} - 10\frac{x^{2+\beta}}{\Gamma(3+\beta)},$$
$$f_2(x) = 2\frac{x^{1+3\beta}}{\Gamma(2+3\beta)} - 2\frac{x^{2+2\beta}}{\Gamma(3+2\beta)} - 4\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} + 8\frac{\Gamma(3+\beta)}{\Gamma(2+\beta)}\frac{x^{2+2\beta}}{\Gamma(3+2\beta)}$$
$$- 8\frac{x^{3+\beta}}{\Gamma(4+\beta)} - 32\frac{x^{2+\beta}}{\Gamma(3+\beta)},$$

and so on.

Hence the approximate solution of (51)−(53) is constructed in the following form:

$$u(x,t) = 2x + \left(2\frac{x^{1+\beta}}{\Gamma(2+\beta)} - 4x - x^2\right)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left(2\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} - 2\frac{x^{2+\beta}}{\Gamma(3+\beta)}\right.$$
$$\left. - 4\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 8\frac{x^{2+\beta}}{\Gamma(3+\beta)} - 4\frac{x^3}{3} - 16x^2\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots. \quad (58)$$

From Table 3 and Figures 7−9, it is clear that as α and β increase to 1 the solution gets closer to the exact solution of FIE with integer orders for all values of x . However as t increases, the solution is get away from the exact solution of FIE with integer orders.
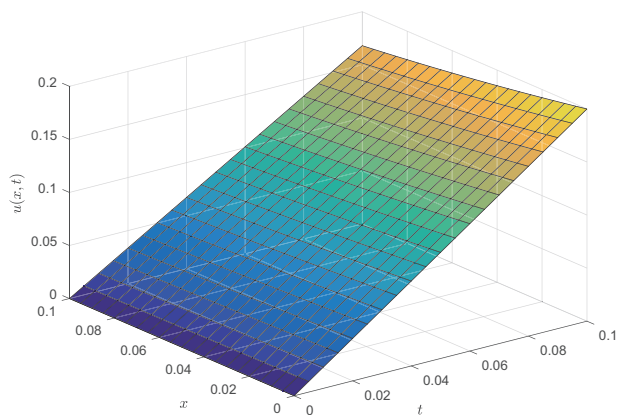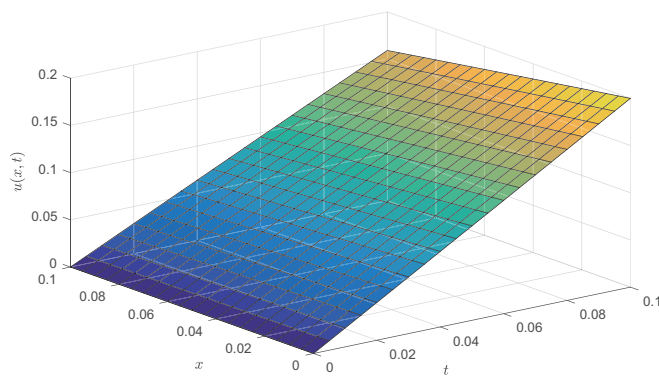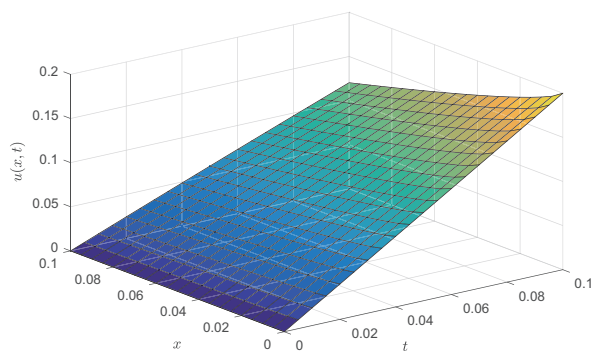
Figure 7: The exact solution of Ex. 3.

Figure 8: The RPS solution of Ex. 3 for $\alpha = 1, \beta = 1$.

Figure 9: The RPS solution of Ex. 3 for $\alpha = 0.75, \beta = 0.75$.

Table 3: The values of the approximate solutions for Example 3.

x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$	$\beta = 1$	$\beta = 0.75$	$\beta = 0.5$
0.2	0.03	0.37744	0.37569	0.37670	0.37848	0.33452	0.33708	0.34161
	0.06	0.35776	0.35074	0.35274	0.35626	0.28675	0.29089	0.29829
	0.09	0.34096	0.32517	0.32812	0.33334	0.24278	0.24820	0.25798
0.4	0.03	0.75488	0.75078	0.75348	0.75740	0.66392	0.67095	0.68126
	0.06	0.71552	0.69912	0.70541	0.71238	0.55899	0.57076	0.58822
	0.09	0.68192	0.64503	0.65310	0.66493	0.45891	0.47480	0.49857
0.6	0.03	1.13232	1.12534	1.12985	1.13567	0.98868	1.00080	1.01657
	0.06	1.07328	1.04537	1.05450	1.06632	0.81815	0.83904	0.86652
	0.09	1.02288	0.96009	0.97394	0.99195	0.65100	0.67993	0.71830
0.8	0.03	1.50976	1.49943	1.50563	1.51290	1.30932	1.32646	1.34676
	0.06	1.43104	1.38972	1.40243	1.41739	1.06563	1.09598	1.13227
	0.09	1.36384	1.27087	1.29039	1.31348	0.82163	0.86459	0.91632

Illustrative examples for Case 2 are given below:

Example 4. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u + 2u, \qquad 0 < \alpha \le 1, 1 < \beta \le 2, \tag{59}$$

$$u(x,0) = \exp(2x), \tag{60}$$

$$u(0,t) = \exp(\frac{t}{4}), \tag{61}$$

$$u_x(0,t) = 2\exp(\frac{t}{4}), \tag{62}$$

which has the exact solution $u(x,t) = \exp(2x+\frac{t}{4})$. Applying transformations to Eqs. $(59)-(62)$, we have :

$$D_t^\alpha(D_x^\beta v) = v - \frac{I_x^\beta 2\exp(\frac{t}{4})x}{\Gamma(3-\beta)\Gamma(\beta)} - \frac{I_x^\beta \exp(\frac{t}{4})}{\Gamma(3-\beta)\Gamma(\beta-1)} + 2D_x^\beta v, \tag{63}$$

$$v(x,0) = I_x^\beta \exp(2x) = f_0(x), \tag{64}$$

$$v(0,t) = I_x^\beta \exp(\frac{t}{4}), \tag{65}$$

$$v_x(0,t) = 2I_x^\beta \exp(\frac{t}{4}). \tag{66}$$

Table 4: The values of approximate solutions for Example 4.

x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$	$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$
0.2	0.3	1.60801	1.57778	1.58497	1.59779	1.62770	1.64272	1.66872
	0.6	1.73325	1.67188	1.68847	1.71803	1.73424	1.76375	1.81483
	0.9	1.86825	1.77412	1.80232	1.85256	1.83691	1.88195	1.95993
0.4	0.3	2.39888	2.34096	2.36367	2.39796	2.43343	2.48527	2.56175
	0.6	2.58571	2.47337	2.52561	2.60440	2.60880	2.71025	2.85986
	0.9	2.78710	2.62279	2.71136	2.84487	2.78498	2.93944	3.16717
0.6	0.3	3.57870	3.49656	3.54295	3.60693	3.70125	3.82151	3.98572
	0.6	3.85743	3.70789	3.81421	3.96072	4.04099	4.27549	4.59551
	0.9	4.15786	3.95409	4.13390	4.38151	4.39212	4.74827	5.23415
0.8	0.3	5.33880	5.23739	5.31775	5.42247	5.70419	5.95203	6.27766
	0.6	5.75460	5.58752	5.77116	6.01022	6.39297	6.87479	7.50742
	0.9	6.20280	6.00342	6.31327	6.71629	7.11462	7.84491	8.80337

The kth-truncated residual function becomes

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k + \frac{I_x^\beta 2\exp(\frac{t}{4})x}{\Gamma(3-\beta)\Gamma(\beta)} + \frac{I_x^\beta \exp(\frac{t}{4})}{\Gamma(3-\beta)\Gamma(\beta-1)}, k = 0, 1, 2, \ldots .(67)$$

Based on the RPS procedure, the first two coefficients are obtained as follows:

$$f_1(x) = I_x^\beta \exp(2x),$$
$$f_2(x) = I_x^{2\beta} \exp(2x),$$

and so on.

Hence the approximate solution of $(59)-(62)$ is constructed in the following form:

$$u(x, t) = \exp(2x) + \frac{x^{2+\beta}}{4} E_{1,3+\beta}(2x) \frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ \frac{x^{2+2\beta}}{16} E_{1,3+2\beta}(2x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots . \tag{68}$$

From Table 4 and Figures $10-12$, as it can be seen from the table that for $\alpha = 0.75$ and $\beta = 1.5$ the solution is the best approximation of the exact solution of FIE with integer orders for all values of x and t .
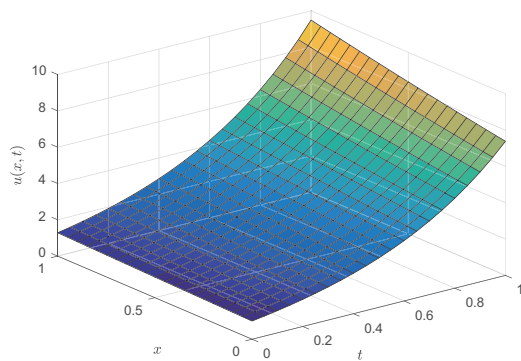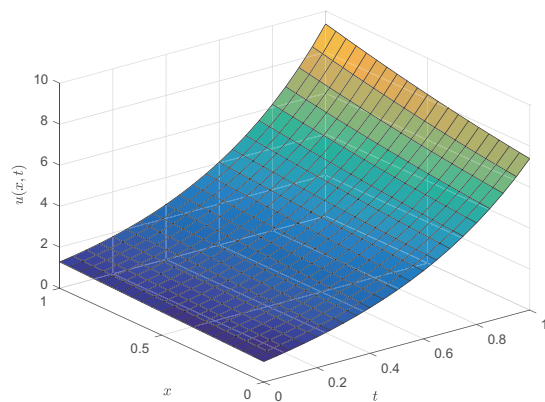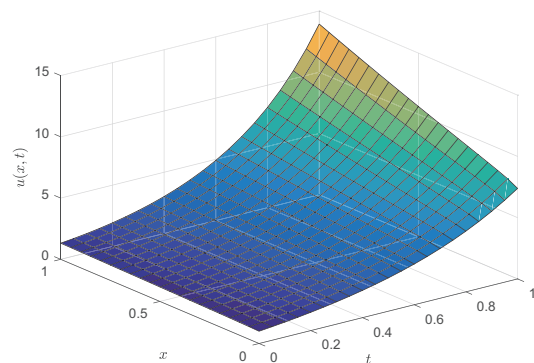
Figure 10:   The exact solution of Ex. 4.

Figure 11:   The RPS solution of Ex. 4 for $\alpha = 1, \beta = 2$.

Figure 12:   The RPS solution of Ex. 4 for $\alpha = 0.75, \beta = 1.75$.

Example 5. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u + (1 - \frac{x^2}{20})u, 0 < \alpha \leq 1, 1 < \beta \leq 2 \tag{69}$$

$$u(x,0) = 20x^3, \tag{70}$$

$$u(0,t) = 0, \tag{71}$$

$$u_x(0,t) = 0 \tag{72}$$

which has the exact solution $u(x,t) = 20x^3 \exp(t)$. Applying transformations to Eqs. (69) and (72), we have the followings:

$$D_t^\alpha(D_x^\beta v) = v + (1 - \frac{x^2}{20})D_x^\beta v, \tag{73}$$

$$v(x,0) = I_x^\beta(20x^3) = f_0(x), \tag{74}$$

$$v(0,t) = 0, \tag{75}$$

$$v_x(0,t) = 0. \tag{76}$$

The kth-truncated residual function is

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k - (1 - \frac{x^2}{20})D_x^\beta v_k, k = 0, 1, 2, \ldots \tag{77}$$

Based on the RPS procedure the first two coefficients are obtained as follows:
$f_1(x) = I_x^\beta(20x^3) + 20x^3 - x^5,$
$f_2(x) = I_x^\beta(f_1(x) + (1 - \frac{x^2}{20})D_x^\beta f_1(x)),$
and so on.

Hence the approximate solution (69)−(72) is constructed in the following form:

$$u(x,t) = 20x^3 + \left(20\Gamma(4)\frac{x^{3+\beta}}{\Gamma(4+\beta)} + 20x^3 - x^5\right)\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ \left(20\Gamma(4)\frac{x^{3+2\beta}}{\Gamma(4+2\beta)} + 20\Gamma(4)\frac{x^{3+\beta}}{\Gamma(4+\beta)} - \Gamma(6)\frac{x^{5+\beta}}{\Gamma(6+\beta)}\right.$$
$$+ (20 - x^2)6\frac{x^{3+\beta}}{\Gamma(4+\beta)} + 20x^3 - 2x^3 + \left.\frac{x^7}{20}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots . \tag{78}$$

In Table 5 and Figures 13−15, as it is obvious that for $\alpha = 1$ and $\beta = 1.5$ the solution is the best approximation of the exact solution of FIE with integer orders for all values of x and t.
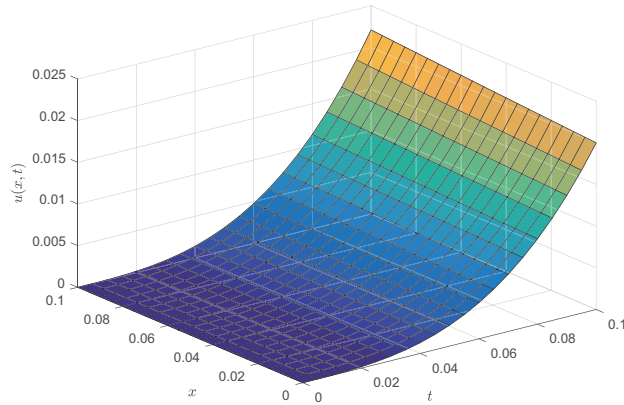
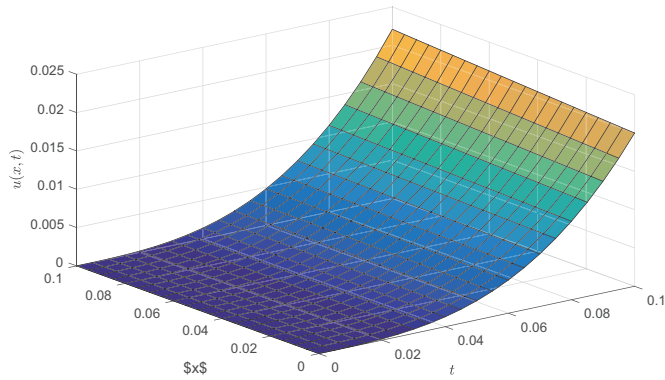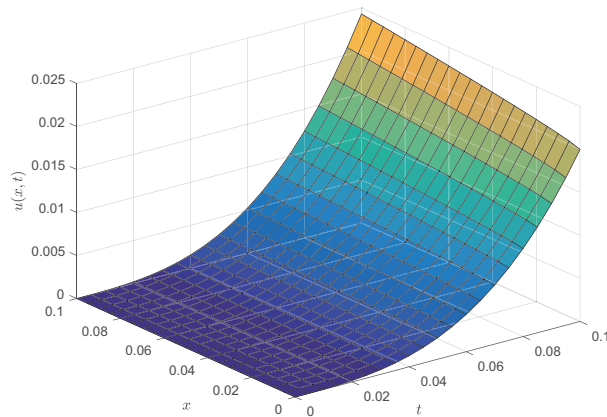Figure 13:   The exact solution of Ex. 5.

Figure 14:   The RPS solution of Ex. 5 for $\alpha = 1, \beta = 2$.

Figure 15:   The RPS solution of Ex. 5 for $\alpha = 0.75, \beta = 1.75$.

Table 5: The values of approximate solutions for Example 5.

x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$	$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$
0.2	0.3	0.21598	0.21451	0.21467	0.21502	0.24845	0.24873	0.24936
	0.6	0.29154	0.28204	0.28243	0.28330	0.32925	0.32984	0.33116
	0.9	0.39354	0.36258	0.36328	0.36484	0.41376	0.41470	0.41679
0.4	0.3	1.72782	1.71676	1.72042	1.72726	1.98948	1.99593	2.00802
	0.6	2.33231	2.25905	2.26805	2.28490	2.63938	2.65289	2.67822
	0.9	3.14829	2.90685	2.92288	2.95294	3.31997	3.34144	3.38172
0.6	0.3	5.83139	5.79796	5.82012	5.85751	6.72518	6.76429	6.83040
	0.6	7.87155	7.63983	7.69438	7.78661	8.93812	9.02000	9.15866
	0.9	10.62549	9.84563	9.94280	10.10731	11.26040	11.39053	11.61110
0.8	0.3	13.82255	13.75621	13.83423	13.95653	15.97661	16.11430	16.33068
	0.6	18.65850	18.16084	18.35289	18.65479	21.28689	21.57515	22.02929
	0.9	25.18634	23.45390	23.79597	24.33477	26.87547	27.33355	28.05632

Example 6. Let us consider the problem including FIE

$$D_t^\alpha u = I_x^\beta u + uu_x - \frac{24x}{(1-2t)^2} - \frac{x^3}{(1-2t)}, \quad 0 < \alpha \le 1, 1 < \beta \le 2 \quad (79)$$

$$u(x,0) = 6x, \quad (80)$$

$$u(0,t) = 0, \quad (81)$$

$$u_x(0,t) = \frac{6}{(1-2t)}, \quad (82)$$

which has the exact solution $u(x,t) = \frac{6x}{1-2t}$. Applying transformations to Eqs. (79) and (82), we have the followings:

$$D_t^\alpha(D_x^\beta v) = v + D_x^\beta v D_x^{\beta+1} v - \frac{24x}{(1-2t)^2} - \frac{x^3}{(1-2t)} - \frac{I_x^\beta \frac{6}{(1-2t)}x}{\Gamma(3-\beta)\Gamma(\beta)}, \quad (83)$$

$$v(x,0) = 6\frac{x^{1+\beta}}{\Gamma(2+\beta)} = f_0(x), \quad (84)$$

$$v(0,t) = 0, \quad (85)$$

$$v_x(0,t) = I_x^\beta \frac{6}{(1-2t)}, \quad (86)$$

The kth-truncated residual function is

$$Res_k = D_t^\alpha(D_x^\beta v_k) - v_k - D_x^\beta v_k D_x^{\beta+1} v_k$$

$$- \frac{24x}{(1-2t)^2} - \frac{x^3}{(1-2t)} - \frac{I_x^\beta \frac{6}{(1-2t)}x}{\Gamma(3-\beta)\Gamma(\beta)}, \quad k = 0, 1, 2, \dots. \quad (87)$$

Based on the RPS procedure the first two coefficients are obtained as follows:

$$f_1(x) = 6\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} + 12\frac{x^{1+\beta}}{\Gamma(2+\beta)} - 6\frac{x^{3+\beta}}{\Gamma(4+\beta)}$$

$$f_2(x) = 6\frac{x^{1+3\beta}}{\Gamma(2+3\beta)} + 48\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} - 6\frac{x^{3+2\beta}}{\Gamma(4+2\beta)} + 36\frac{\Gamma(2+\beta)}{\Gamma(1+\beta)}\frac{x^{1+2\beta}}{\Gamma(2+2\beta)}$$
$$+ 72\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 144\frac{x^{2+\beta}}{\Gamma(3+\beta)} - 36\frac{x^{3+\beta}}{\Gamma(4+\beta)} - 144\frac{x^{4+\beta}}{\Gamma(5+\beta)},$$

and so on.

Hence the approximate solution $(79)-(82)$ is constructed in the following form:

$$u(x,t) = 6x + \left(6\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 12x - x^3\right)\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$+ \left(6\frac{x^{1+2\beta}}{\Gamma(2+2\beta)} + 48\frac{x^{1+\beta}}{\Gamma(2+\beta)} - 6\frac{x^{3+\beta}}{\Gamma(4+\beta)}\right.$$
$$\left. + 36\frac{x^{1+\beta}}{\Gamma(2+\beta)} + 72x + 72x^2 - 6x^3 - 6x^4\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots. \quad (88)$$

In Table 6 and Figures $16-18$, as it is obvious that for $\alpha = 1$ and $\beta = 2$ the solution is the best approximation of the exact solution of FIE with integer orders for all values of x.

Table 6: The values of approximate solutions for Example 6.

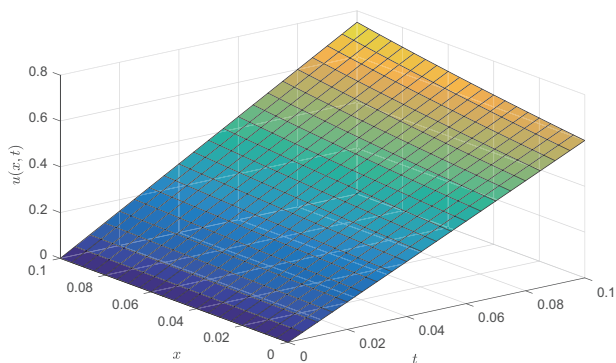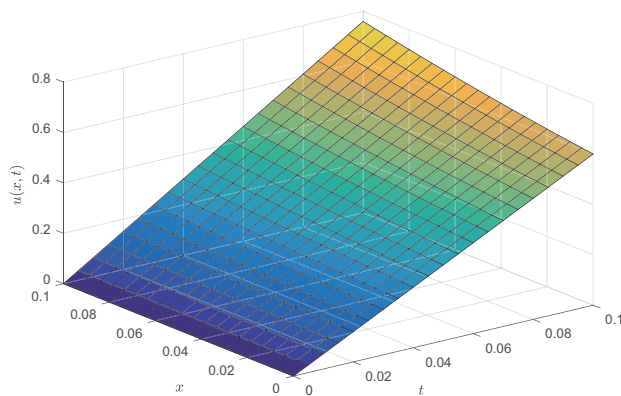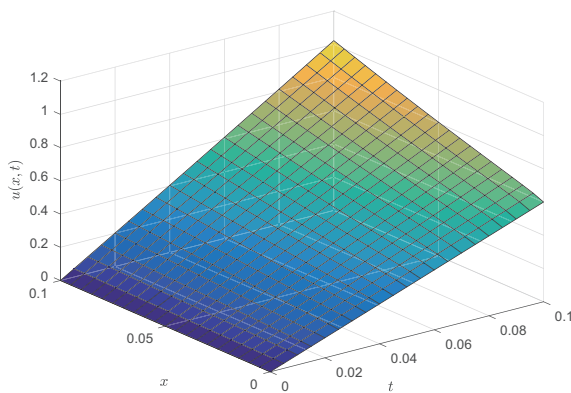| x	t	Exact	$\alpha = 1$			$\alpha = 0.75$		
			$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$	$\beta = 2$	$\beta = 1.75$	$\beta = 1.5$
0.2	0.03	1.27660	1.27984	1.28018	1.28081	1.45637	1.45776	1.46037
	0.06	1.36364	1.37537	1.37621	1.37780	1.70928	1.71247	1.71841
	0.09	1.46341	1.48659	1.48810	1.49095	1.98312	1.98844	1.99833
0.4	0.03	2.55319	2.56265	2.56446	2.56736	2.93848	2.94598	2.95773
	0.06	2.72727	2.76260	2.76713	2.77430	3.49138	3.50852	3.53514
	0.09	2.92683	2.99986	3.00802	3.02084	4.09999	4.12861	4.17285
0.6	0.03	3.82979	3.84870	3.85335	3.86001	4.44874	4.46786	4.49467
	0.06	4.09091	4.16279	4.17439	4.19081	5.35307	5.39664	5.45715
	0.09	4.39024	4.54228	4.56312	4.59239	6.36310	6.43571	6.53604
0.8	0.03	5.10638	5.13816	5.14694	5.15854	5.98864	6.02444	6.07065
	0.06	5.45455	5.57663	5.59843	5.62685	7.29862	7.37988	7.48369
	0.09	5.85366	6.11542	6.15446	6.20495	8.78025	8.91540	9.08712

Figure 16:   The exact solution of Ex. 6.

Figure 17:   The RPS solution of Ex. 6 for $\alpha = 1, \beta = 2$.

Figure 18:   The RPS solution of Ex. 6 for $\alpha = 0.75, \beta = 1.75$.

5. Conclusion

In order to obtain the analytic solution of FIE, the RPSM is used with new transformation which makes the implementation of RPSM easier. Examples for different values of α and β illustrate that this method is feasible and convenient to establish the analytic solution for this kind of problems.

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