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A GENERALIZATION OF COLMEZ-GREENBERG-STEVENS FORMULA

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Abstract

In this paper we study the derivatives of Frobenius and the derivatives of Hodge-Tate weights for families of Galois representations with triangulations. We give a generalization of the Fontaine-Mazur $\mathcal{L}$-invariant and use it to build a formula which is a generalization of the Colmez-Greenberg-Stevens formula.

1. Introduction

In their remarkable paper [10], Mazur, Tate and Teitelbaum proposed a conjectural formula for the derivative at $s=1$ of the p-adic L-function of an elliptic curve E over $\mathbf{Q}$ when p is a prime of split multiplicative reduction. An important quantity in this formula is the so called $\mathcal{L}$-invariant, namely $\mathcal{L}(E)=\log_p(q_E)/v_p(q_E)$ where $q_E\in\mathbf{Q}_p^\times$ is the Tate period for E. This conjectural formula was proved by Greenberg and Stevens [8] using Hida's families. Indeed, for the weight 2 newform f attached to E, there exists a family of p-adic ordinary Hecke eigenforms containing f. A key formula they proved is

$$\mathcal{L}(E)=-2\frac{\alpha'(f)}{\alpha(f)} \tag{1.1}$$

where α is the function of U_p-eigenvalues of the eigenforms in the Hida family. On the other hand, they showed that $-2\frac{\alpha'(f)}{\alpha(f)}$ is equal to $\frac{L_p'(f,1)}{L(f,1)}$. Combining

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these two facts they obtained the conjectural formula.

In this paper we will focus on (1.1) which was later generalized by Colmez [6] to the non-ordinary setting. We state Colmez's result below.

Theorem 1.1 ([6]). *Suppose that, at each closed point z of* $\mathrm{Max}(S)$ *one of the Hodge-Tate weight of* $\mathcal{V}_z$ *is* 0, *and there exists* $\alpha \in S$ *such that* $(\mathbf{B}^{\varphi=\alpha}_{\mathrm{cris},S}\widehat{\otimes}_S\mathcal{V})^{G_{\mathbf{Q}_p}}$ *is locally free of rank* 1 *over* S. *Suppose* z_0 *is a closed point of* $\mathrm{Max}(S)$ *such that* $\mathcal{V}_{z_0}$ *is semistable with Hodge-Tate weights*[1] 0 *and* $k \geq 1$. *Then the differential*

$$\frac{\mathrm{d}\alpha}{\alpha} - \frac{1}{2}\mathcal{L}\mathrm{d}\kappa + \frac{1}{2}\mathrm{d}\delta$$

is zero at z_0, *where* $\mathcal{L}$ *is the Fontaine-Mazur* $\mathcal{L}$*-invariant of* $\mathcal{V}_{z_0}$.

See [6] for the precise meanings of κ and δ. Roughly speaking, $\mathrm{d}\delta$ is the derivative of Frobenius, and $\mathrm{d}\kappa$ is the derivative of Hodge-Tate weights.

The condition that "$(\mathbf{B}^{\varphi=\alpha}_{\mathrm{cris},S}\widehat{\otimes}_S\mathcal{V})^{G_{\mathbf{Q}_p}}$ is locally free of rank 1 over S" in Theorem 1.1 is equivalent to that $\mathcal{V}$ admits a triangulation [5]. So, Theorem 1.1 means that the derivatives of Frobenius and the derivatives of Hodge-Tate weights of a family of 2-dimensional representations of $G_{\mathbf{Q}_p}$ with a triangulation satisfy a non-trivial relation at each semistable (but non-crystalline) point.

Colmez's theorem was generalized by Zhang [14] for families of 2- dimensional Galois representations of G_K (K a finite extension of $\mathbf{Q}_p$) and Pottharst [12] who considered families of (not necessarily étale) (φ,Γ)-modules of rank 2 instead of families of 2-dimensional Galois representations.

In this paper we give a generalization of Colmez's theorem which includes the above generalizations as special cases.

Fix a finite extension K of $\mathbf{Q}_p$. What we work with is a family of K-B-pair (called S-B-pair in our context) that is locally triangulable. We will provide conditions for Fontaine-Mazur $\mathcal{L}$-invariant to be defined. Note that, the $\mathcal{L}$-invariant is now a vector with component number equal to $[K : \mathbf{Q}_p]$.

[1] In this paper, the Hodge-Tate weights are defined to be minus the generalized eigenvalues of Sen's operators. In particular the Hodge-Tate weight of the cyclotomic character χ_{cyc} is -1.

Theorem 1.2. *Let W be an S-B-pair that is semistable at a point $z \in \mathrm{Max}(S)$. Suppose that W is locally triangulable at z with the local triangulation parameters $(\delta_1, \ldots, \delta_n)$. Assume that for D_z, the filtered E-(φ, N)-module attached to W_z, the Fontaine-Mazur $\mathcal{L}$-invariant $\vec{\mathcal{L}}_{s,t}$ (see Definition 6.5) can be defined for $s, t \in \{1, 2, \ldots, n\}$. Then*

$$\frac{1}{[K : \mathbf{Q}_p]} \left(\frac{\mathrm{d}\delta_t(p)}{\delta_t(p)} - \frac{\mathrm{d}\delta_s(p)}{\delta_s(p)} \right) + \vec{\mathcal{L}}_{s,t} \cdot (\mathrm{d}\vec{w}(\delta_t) - \mathrm{d}\vec{w}(\delta_s)) = 0.$$

Here, $\vec{w}(\delta_i)$ is the Hodge-Tate weight of the character δ_i.

In [13] we proved Theorem 1.2 for a special case, where we consider the case of $K = \mathbf{Q}_p$ and demand that the Frobenius is simisimple at z. The motivation and some potential applications of our theorem was also discussed in [13].

Our paper is orginized as follows. In Section 2 we recall the theory of B-pairs built by Berger. Then in Section 3 we extend a part of this theory to families of B-pairs, and discuss the relation between triangulations of semistable B-pairs and refinements of their associated filtered (φ, N)-modules. In Section 4 we compare cohomology groups of (φ, Γ)-modules and those of B-pairs, and then attach a 1-cocycle to each infinitesimal deformation of a B-pair. In Section 5 we use the reciprocity law to build an auxiliary formula for L-invariants. The L-invariant is defined in Section 6. In Section 7 we prove a formula called "projection vanishing property" for the above 1-cocycle. Finally in Section 8 we use the auxiliary formula in Section 5 and the projection vanishing property to deduce Theorem 1.2.

Notations

Let K be a finite extension of $\mathbf{Q}_p$, G_K the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. Let K_0 be the maximal absolutely unramified subfield of K. Let G_K^{ab} denote the maximal abelian quotient of G_K.

Let χ_{cyc} be the cyclotomic character of G_K, H_K the kernel of χ_{cyc} and Γ_K the quotient G_K/H_K. Then χ_{cyc} induces an isomorphism from Γ_K onto an open subgroup of $\mathbf{Z}_p^\times$.

Let E be a finite extension of K such that all embeddings of K into an algebraic closure of E are contained in E, $\mathrm{Emb}(K, E)$ the set of embeddings of K into E. We consider E as a coefficient field and let G_K acts trivially on E.

Let rec_K be the reciprocity map of local class field theory such that $\mathrm{rec}_K(\pi_K)$ is a lifting of the inverse of qth power Frobenius of k, where π_K is a uniformizing element of K and k is the residue field of K with cardinal number q. Note that the image of rec_K coincides with the image of the Weil group $W_K \subset G_K$ by the quotient map $G_K \to G_K^{\mathrm{ab}}$. Let $\mathrm{rec}_K^{-1} : W_K \to K^\times$ be the converse map of rec_K.

2. (φ, Γ_K)-modules and B-pairs

2.1. Fontaine's rings

We recall the construction of Fontaine's period rings. Please consult [7, 2] for more details.

Let $\mathbf{C}_p$ be a completed algebraic closure of $\mathbf{Q}_p$ with valuation subring $\mathfrak{o}_{\mathbf{C}_p}$ and p-adic valuation v_p normalized such that $v_p(p) = 1$.

Let $\widetilde{\mathbf{E}}$ be $\{(x^{(i)})_{i\geq 0} \mid x^{(i)} \in \mathbf{C}_p,\ (x^{(i+1)})^p = x^{(i)}\ \forall\, i \in \mathbf{N}\}$, and let $\widetilde{\mathbf{E}}^+$ be the subset of $\widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathfrak{o}_{\mathbf{C}_p}$. If $x, y \in \widetilde{\mathbf{E}}$, we define $x+y$ and xy by

$$(x+y)^{(i)} = \lim_{j\to\infty} (x^{(i+j)} + y^{(i+j)})^{p^j}, \quad (xy)^{(i)} = x^{(i)} y^{(i)}.$$

Then $\widetilde{\mathbf{E}}$ is a field of characteristic p. Define a function $v_{\mathbf{E}} : \widetilde{\mathbf{E}} \to \mathbf{R} \cup \{+\infty\}$ by putting $v_{\mathbf{E}}((x^{(n)})) = v_p(x^{(0)})$. This is a valuation for which $\widetilde{\mathbf{E}}$ is complete and $\widetilde{\mathbf{E}}^+$ is the ring of integers in $\widetilde{\mathbf{E}}$. If we let $\varepsilon = (\varepsilon^{(n)})$ be an element of $\widetilde{\mathbf{E}}^+$ with $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$, then $\widetilde{\mathbf{E}}$ is a completed algebraic closure of $\mathbf{F}_p((\varepsilon - 1))$. Put $\omega = [\varepsilon] - 1$. Let $\tilde{p}$ be an element of $\widetilde{\mathbf{E}}$ such that $\tilde{p}^{(0)} = p$.

Let $\widetilde{\mathbf{A}}^+$ be the ring $\mathbf{W}(\widetilde{\mathbf{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathbf{E}}^+$, $\widetilde{\mathbf{A}}$ the ring of Witt vectors $\mathbf{W}(\widetilde{\mathbf{E}})$, and $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}[1/p]$. The map

$$\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p, \quad \sum_{n\gg-\infty} p^k [x_k] \mapsto \sum_{n\gg-\infty} p^k x_k^{(0)}$$

is surjective. Let $\mathbf{B}_{\mathrm{dR}}^+$ be the $\ker(\theta)$-adic completion of $\widetilde{\mathbf{B}}^+$. Then $t_{\mathrm{cyc}} = \log[\varepsilon]$ is an element of $\mathbf{B}_{\mathrm{dR}}^+$, and put $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t_{\mathrm{cyc}}]$. There is a filtration $\mathrm{Fil}^\bullet$ on $\mathbf{B}_{\mathrm{dR}}$ such that $\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} = \bigoplus_{j\geq i} \mathbf{B}_{\mathrm{dR}}^+ t_{\mathrm{cyc}}^j$.

Let $\mathbf{B}^+_{\max}$ be the subring of $\widetilde{\mathbf{B}}^+$ consisting of elements of the form $\sum_{n\geq 0} b_n([\tilde{p}]/p)^n$, where $b_n \in \widetilde{\mathbf{B}}^+$ and $b_n \to 0$ when $n \to +\infty$. Put $\mathbf{B}_{\max} = \mathbf{B}^+_{\max}[1/t_{\rm cyc}]$; $\mathbf{B}_{\max}$ is equipped with a φ-action. Put $\mathbf{B}_{\log} = \mathbf{B}_{\max}[\log[\tilde{p}]]$; $\mathbf{B}_{\log}$ is equipped with a φ-action and a monodromy N; $\mathbf{B}_{\log}^{N=0} = \mathbf{B}_{\max}$; $\mathbf{B}_{\log}$ is a subring of $\mathbf{B}_{\rm dR}$. Put $\mathbf{B}_e = \mathbf{B}_{\max}^{\varphi=1}$. We have the following fundamental exact sequence

$$0 \longrightarrow \mathbf{Q}_p \longrightarrow \mathbf{B}_e \longrightarrow \mathbf{B}_{\rm dR}/\mathbf{B}^+_{\rm dR} \longrightarrow 0.$$

If r and s are two elements in $\mathbf{N}[1/p]\cup\{+\infty\}$, we put $\widetilde{\mathbf{A}}^{[r,s]} = \widetilde{\mathbf{A}}^+\{\frac{p}{[\bar{\omega}^r]}, \frac{[\bar{\omega}^s]}{p}\}$ and $\widetilde{\mathbf{B}}^{[r,s]} = \widetilde{\mathbf{A}}^{[r,s]}[1/p]$ with the convention that $p/[\bar{\omega}^{+\infty}] = 1/[\bar{\omega}]$ and $[\bar{\omega}^{+\infty}]/p = 0$. We equip these rings with the p-adic topology. There are natural continuous G_K-actions on $\widetilde{A}_{[r,s]}$ and $\widetilde{B}_{[r,s]}$. Frobenius induces isomorphisms $\varphi : \widetilde{A}_{[r,s]} \xrightarrow{\sim} \widetilde{A}_{[pr,ps]}$ and $\varphi : \widetilde{B}_{[r,s]} \xrightarrow{\sim} \widetilde{B}_{[pr,ps]}$. If $r \leq r_0 \leq s_0 \leq s$, then we have the G_K-equivariant injective natural map $\widetilde{A}_{[r,s]} \hookrightarrow \widetilde{A}_{[r_0,s_0]}$. For $r > 0$ we put $\widetilde{\mathbf{B}}^{\dagger,r}_{\rm rig} = \bigcap_{s\in[r,+\infty)} \widetilde{B}_{[r,s]}$ (equipped with certain Frechet topology) and $\widetilde{\mathbf{B}}^{\dagger}_{\rm rig} = \cup_{r>0}\widetilde{\mathbf{B}}^{\dagger,r}_{\rm rig}$ (equipped with the inductive limit topology). Frobenius induces isomorphisms $\varphi : \widetilde{\mathbf{B}}^{\dagger,r}_{\rm rig} \xrightarrow{\sim} \widetilde{\mathbf{B}}^{\dagger,pr}_{\rm rig}$ and $\varphi : \widetilde{\mathbf{B}}^{\dagger}_{\rm rig} \xrightarrow{\sim} \widetilde{\mathbf{B}}^{\dagger}_{\rm rig}$.

Put

$$A_{K_0'} = \{\sum_{k\geq -\infty}^{+\infty} a_k\omega^k \mid a_k \in \mathfrak{o}_{K_0'},\ a_k \to 0 \ \text{ when } k \to -\infty)\}$$

and $B_{K_0'} = A_{K_0'}[1/p]$. Here K_0' is the maximal absolutely unramified subfield of $K_\infty = K(\mu_{p^\infty})$. Then $A_{K_0'}$ is a complete discrete valuation ring with p as a prime element, and $B_{K_0'}$ is the fractional field of $A_{K_0'}$. The G_K-action and φ preserve $A_{K_0'}$: $\varphi(\omega) = (1+\omega)^p - 1$ and $g(\omega) = (1+\omega)^{\chi_{\rm cyc}(g)} - 1$. Let $\mathbf{A}$ be the p-adic completion of the maximal unramified extension of $A_{K_0'}$ in $\widetilde{\mathbf{A}}$, $\mathbf{B}$ its fractional field. Then φ and the G_K-action preserve $\mathbf{A}$ and $\mathbf{B}$.

We put $\mathbf{B}_K = \mathbf{B}^{H_K}$ and $\mathbf{B}_K^{\dagger,r} = \mathbf{B}_K \cap \widetilde{\mathbf{B}}^{\dagger,r}$. Let $\mathbf{B}^{\dagger,r}_{{\rm rig},K}$ be the Frechet completion of $\mathbf{B}_K^{\dagger,r}$ for the topology induced from that on $\widetilde{\mathbf{B}}^{\dagger,r}_{\rm rig}$, and put $\mathbf{B}^{\dagger}_{{\rm rig},K} = \cup_{r>0}\mathbf{B}^{\dagger,r}_{{\rm rig},K}$ equipped with the inductive limit topology. Frobunius induces injections $\mathbf{B}^{\dagger,r}_{{\rm rig},K} \hookrightarrow \mathbf{B}^{\dagger,pr}_{{\rm rig},K}$ and $\mathbf{B}^{\dagger}_{{\rm rig},K} \hookrightarrow \mathbf{B}^{\dagger}_{{\rm rig},K}$; there are continuous Γ_K-actions on $\mathbf{B}^{\dagger,r}_{{\rm rig},K}$ and $\mathbf{B}^{\dagger}_{{\rm rig},K}$.

We end this subsection by the definition of E-(φ,Γ_K)-modules [11].

Definition 2.1. An E-(φ,Γ_K)-module is a finite $\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes_{\mathbf{Q}_p} E$-module M equipped with a Frobenius semilinear action φ_M and a comtinuous semilinear Γ_K-action such that M is free as a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module, that $\mathrm{id}_{\mathbf{B}^\dagger_{\mathrm{rig},K}}\otimes\varphi_M$: $\mathbf{B}^\dagger_{\mathrm{rig},K}\bigotimes_{\varphi,\mathbf{B}^\dagger_{\mathrm{rig},K}} M\rightarrow M$ is an isomorphism, and that φ_M and the Γ_K-action commute with each other.

By [11, Lemma 1.30] if M is an E-(φ,Γ_K)-module, then M is free over $\mathbf{B}^\dagger_{\mathrm{rig},K}\otimes_{\mathbf{Q}_p} E$.

2.2. B-pairs

We recall the theory of E-B-pairs [3, 11].

Put $\mathbf{B}_{e,E}=\mathbf{B}_e\otimes_{\mathbf{Q}_p}E$, $\mathbf{B}^+_{\mathrm{dR},E}=\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}E$ and $\mathbf{B}_{\mathrm{dR},E}=\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}E$. We extend the G_K-actions E-linearly to these rings.

Definition 2.2. An *E-B-pair* of G_K is a couple $W=(W_e,W^+_{\mathrm{dR}})$ such that

- W_e is a finite $\mathbf{B}_{e,E}$-module with a continuous semilinear action G_K-action which is free as a $\mathbf{B}_e$-module.
- $W^+_{\mathrm{dR}}\subset W_{\mathrm{dR}}=\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{B}_e}W_e$ is a G_K-stable $\mathbf{B}^+_{\mathrm{dR},E}$-lattice.

By [11, Remark 1.3] W_e is free over $\mathbf{B}_{e,E}$ and W^+_{dR} is free over $\mathbf{B}^+_{\mathrm{dR},E}$.

If V is an E-representation of G_K, then $W(V)=(\mathbf{B}_{e,E}\otimes_E V,\mathbf{B}^+_{\mathrm{dR},E}\otimes_E V)$ is an E-B-pair, called the *E-B-pair attached to V*.

If S is a Banach E-algebra, we can define S-B-pairs similarly; to each S-representation V of G_K is associated an S-B-pair $W(V)=(\mathbf{B}_{e,E}\otimes_E V,\mathbf{B}^+_{\mathrm{dR},E}\otimes_E V)$.

If $W_1=(W_{1,e},W^+_{1,\mathrm{dR}})$ and $W_2=(W_{2,e},W^+_{2,\mathrm{dR}})$ are two E-B-pairs, we define $W_1\bigotimes W_2$ to be

$$(W_{1,e}\bigotimes_{\mathbf{B}_{e,E}}W_{2,e},W^+_{1,\mathrm{dR}}\bigotimes_{\mathbf{B}^+_{\mathrm{dR},E}}W^+_{2,\mathrm{dR}}).$$

Here, $W_{1,e}\bigotimes_{\mathbf{B}_{e,E}}W_{2,e}$ is equipped with the diagonal G_K-action, and $W^+_{1,\mathrm{dR}}\otimes_{\mathbf{B}^+_{\mathrm{dR},E}}$

$W^+_{2,\mathrm{dR}}$ is naturally considered as a G_K-stable $\mathbf{B}^+_{\mathrm{dR},E}$-lattice of

$$\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} (W_{1,e} \bigotimes_{\mathbf{B}_{e,E}} W_{2,e}) = W_{1,\mathrm{dR}} \bigotimes_{\mathbf{B}_{\mathrm{dR},E}} W_{2,\mathrm{dR}},$$

where $W_{1,\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_{1,e}$ and $W_{2,\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_{2,e}$.

If $W = (W_e, W^+_{\mathrm{dR}})$ is an E-B-pair with $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$, we define the dual of W to be $W^* = (W^*_e, W^{*,+}_{\mathrm{dR}})$, where W^*_e is $\mathrm{Hom}_{\mathbf{B}_e}(W, \mathbf{B}_e)$ equipped with the natural G_K-action, and $W^{*,+}_{\mathrm{dR}}$ is the G_K-stable lattice of $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W^*_e \cong \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}}(W_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}})$ defined by

$$\{\ell \in \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}}(W_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}) : \ell(x) \in \mathbf{B}^+_{\mathrm{dR}} \text{ for all } x \in W^+_{\mathrm{dR}}\}.$$

The relation between (φ, Γ_K)-modules and B-pairs is built by Berger [3]. We recall Berger's construction below.

Let M be a (φ, Γ_K)-module of rank d over the Robba ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Berger [3] showed that

$$W_e(M) := (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} M)^{\varphi=1}$$

is a free $\mathbf{B}_e$-module of rank d and equipped with a continuous semilinear G_K-action.

For sufficiently large $r_0 > 0$ we can take a unique Γ_K-stable finite free $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-submodule $M^r \subset M$ such that

$$\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} M^r = M$$

and

$$\mathrm{id}_{\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}} \otimes \varphi_M : \mathbf{B}^{\dagger,pr}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}} M^r \xrightarrow{\sim} M^{pr}$$

for any $r \geq r_0$. Berger [3] showed that the $\mathbf{B}^+_{\mathrm{dR}}$-module

$$W^+_{\mathrm{dR}}(M) := \mathbf{B}^+_{\mathrm{dR}} \otimes_{i_n, \mathbf{B}^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig},K}} M^{(p-1)p^{n-1}}$$

is independent of any n such that $(p-1)p^{n-1} \geq r_0$, and showed that there is a canonical G_K-equivariant isomorphism $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e(M) \xrightarrow{\sim} \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}(M)$.

Put $W(M) = (W_e(M), W^+_{\mathrm{dR}}(M))$. This is an E-B-pair of rank $d = \mathrm{rank}_{\mathbf{B}^\dagger_{\mathrm{rig},K}} M$.

The following is a variant version of Berger's result [3, Theorem 2.2.7].

Proposition 2.3 ([11], Theorem 1.36)**.** *The functor $M \mapsto W(M)$ is an exact functor and this gives an equivalence of categories between the category of E-(φ, Γ_K)-modules and the category of E-B-pairs of G_K.*

Proposition 2.4. *The functor $M \mapsto W(M)$ respects the tensor products and duals.*

Proof. Let M_1 and M_2 be two E-(φ, Γ_K)-modules. By taking φ-invariants, the isomorphism

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}} M_1) \otimes_{\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} E[1/t]} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}} M_2) \xrightarrow{\sim} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}} (M_1 \otimes M_2)$$

induces a G_K-equivariant injective map

$$W_e(M_1) \otimes_{\mathbf{B}_{e,E}} W_e(M_2) \to W_e(M_1 \otimes M_2).$$

Here, $M_1 \otimes M_2$ denotes the E-(φ, Γ_K)-module $M_1 \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{Q}_p} E} M_2$. Comparing dimensions and using [11, Lemma 1.10] we see that this map is in fact an isomorphism. From the above Berger's construction we see that the natural map

$$W^+_{\mathrm{dR}}(M_1) \otimes_{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E} W^+_{\mathrm{dR}}(M_2) \to W^+_{\mathrm{dR}}(M_1 \otimes M_2)$$

is an isomorphism. This proves that the functor $M \mapsto W(M)$ respects tensor products. The proof of that it respects duals is similar. □

2.3. Semistable E-B-pairs

Definition 2.5. An E-(φ, N)-*module* over K is a $K_0 \otimes_{\mathbf{Q}_p} E$-module D with a $\varphi \otimes 1$-semilinear isomorphism $\varphi_D : D \to D$, and a $K_0 \otimes_{\mathbf{Q}_p} E$-linear map $N_D : D \to D$ such that $N_D \varphi_D = p \varphi_D N_D$. A *filtered* E-(φ, N)-*module* over K is an E-(φ, N)-module with an exhaustive $\mathbf{Z}$-indexed descending filtration $\mathrm{Fil}^\bullet$ on $K \otimes_{K_0} D$.

We have an isomorphism of rings

$$K \otimes_{\mathbf{Q}_p} E \xrightarrow{\sim} \bigoplus_{\tau \in \mathrm{Emb}(K,E)} E_\tau, \quad a \otimes b \mapsto (\tau(a)b)_\tau, \tag{2.1}$$

where E_τ is a copy of E for each $\tau \in \mathrm{Emb}(K, E)$. Let e_τ be the unity of E_τ. Then $1 = \sum_\tau e_\tau$. Put $D_\tau = e_\tau(K \otimes_{K_0} D)$. Then $K \otimes_{K_0} D = \bigoplus_{\tau \in \mathrm{Emb}(K,E)} D_\tau$. Let Fil_τ denote the induced filtration on D_τ.

Definition 2.6. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an E-B-pair. We define $\mathbf{D}_{\mathrm{cris}}(W) = (\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e)^{G_K}$, $\mathbf{D}_{\mathrm{st}}(W) = (\mathbf{B}_{\log} \otimes_{\mathbf{B}_e} W_e)^{G_K}$ and $\mathbf{D}_{\mathrm{dR}}(W) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e)^{G_K}$. Then we have $\dim_{K_0}(\mathbf{D}_?(W)) \leq \mathrm{rank}_{\mathbf{B}_e} W_e$ for $? = \mathrm{cris}, \mathrm{st}$, and $\dim_K(\mathbf{D}_{\mathrm{dR}}(W)) \leq \mathrm{rank}_{\mathbf{B}_e} W_e$. We say that W is *crystalline* (resp. *semistable*) if $\dim_{K_0}(\mathbf{D}_?(W)) := \mathrm{rank}_{\mathbf{B}_e} W_e$ for $? = \mathrm{cris}$ (resp. st).

If W is a semistable E-B-pair, we attach to W a filtered E-(φ, N)-module as follows. The underlying E-(φ, N)-module is $\mathbf{D}_{\mathrm{st}}(W)$; the filtration on $\mathbf{D}_{\mathrm{dR}}(W) = K \otimes_{K_0} \mathbf{D}_{\mathrm{st}}(W)$ is given by $\mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}(W) = t^i W^+_{\mathrm{dR}} \cap \mathbf{D}_{\mathrm{dR}}(W)$.

Proposition 2.7.

(a) *The functor $W \mapsto \mathbf{D}_{\mathrm{st}}(W)$ realizes an equivalence of categories between the category of semistable E-B-pairs of G_K and the category of filtered E-(φ, N)-modules over K.*

(b) *If W_1 and W_2 are semistable, then so is $W_1 \otimes W_2$.*

(c) *The functor $W \mapsto \mathbf{D}_{\mathrm{st}}(W)$ respects the tensor products and duals.*

(d) *If*

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$$

is a short exact sequence of E-B-pairs, and W is semistable, then W_1 and W_2 are semistable.

(e) *The functor $W \mapsto \mathbf{D}_{\mathrm{st}}(W)$ is exact.*

Proof. Assertion (a) follows from [3, Proposition 2.3.4]. See also [11, Theorem 1.18 (2)].

Let W_1 and W_2 be two E-B-pairs. The isomorphism

$$(\mathbf{B}_{\log} \otimes_{\mathbf{B}_e} W_1) \otimes_{\mathbf{B}_{\log} \otimes_{\mathbf{Q}_p} E} (\mathbf{B}_{\log} \otimes_{\mathbf{B}_e} W_2) \xrightarrow{\sim} \mathbf{B}_{\log} \otimes_{\mathbf{B}_e} (W_1 \otimes W_2)$$

induces an injective map

$$\mathbf{D}_{\mathrm{st}}(W_1) \otimes_{K_0 \otimes_{\mathbf{Q}_p} E} \mathbf{D}_{\mathrm{st}}(W_2) \to \mathbf{D}_{\mathrm{st}}(W_1 \otimes W_2). \tag{2.2}$$

When W_1 and W_2 are semistable, the dimension of the source over K_0 is $\frac{\mathrm{rank}_{\mathbf{B}_e} W_1 \mathrm{rank}_{\mathbf{B}_e} W_2}{[E:\mathbf{Q}_p]}$. The dimension of the target over K_0 is always equal to or less than $\mathrm{rank}_{\mathbf{B}_e}(W_1 \otimes W_2) = \frac{\mathrm{rank}_{\mathbf{B}_e} W_1 \mathrm{rank}_{\mathbf{B}_e} W_2}{[E:\mathbf{Q}_p]}$. Hence, (2.2) is an isomorphism, and so $W_1 \otimes W_2$ is semistable. This proves (b). Similarly, the isomorphism

$$(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_1) \otimes_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E} (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_2) \xrightarrow{\sim} \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} (W_1 \otimes W_2) \tag{2.3}$$

induces an isomorphism

$$\mathbf{D}_{\mathrm{dR}}(W_1) \otimes_{K \otimes_{\mathbf{Q}_p} E} \mathbf{D}_{\mathrm{dR}}(W_2) \to \mathbf{D}_{\mathrm{dR}}(W_1 \otimes W_2).$$

Via the isomorphism (2.3) the filtration on $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_1) \otimes_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E} (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_2)$ coincides with that on $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} (W_1 \otimes W_2)$. Therefore, the filtration on $\mathbf{D}_{\mathrm{dR}}(W_1) \otimes_{K \otimes_{\mathbf{Q}_p} E} \mathbf{D}_{\mathrm{dR}}(W_2)$ and that on $\mathbf{D}_{\mathrm{dR}}(W_1 \otimes W_2)$ coincide. Indeed, they are the restrictions of the filtrations on $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_1) \otimes_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E} (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_2)$ and $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} (W_1 \otimes W_2)$ respectively. Similarly we can show that $W \mapsto \mathbf{D}_{\mathrm{st}}(W)$ respects duals. This proves (c).

For (d) we have the following exact sequence

$$0 \longrightarrow \mathbf{D}_{\mathrm{st}}(W_1) \longrightarrow \mathbf{D}_{\mathrm{st}}(W) \longrightarrow \mathbf{D}_{\mathrm{st}}(W_2). \tag{2.4}$$

So (d) follows from a dimension argument. Furthermore, when W is semistable, $\mathbf{D}_{\mathrm{st}}(W) \to \mathbf{D}_{\mathrm{st}}(W_2)$ is surjective. For any $i \in \mathbf{Z}$ we write $d_i(W)$ for $\dim_K \mathrm{Fil}^i \mathbf{D}_{\mathrm{st}}(W)$. As the maps in the exact sequence (2.4) respect filtrations, we have $d_i(W) \leq d_i(W_1) + d_i(W_2)$. Similarly, we have $d_{1-i}(W^*) \leq d_{1-i}(W_1^*) + d_{1-i}(W_2^*)$. As $W \mapsto \mathbf{D}_{\mathrm{st}}(W)$ respects duals, we have $d_i(W) = \dim_K(\mathbf{D}_{\mathrm{dR}}(W)) - d_{1-i}(W^*)$. Then

$$\begin{aligned} d_i(W) &= \dim_K(\mathbf{D}_{\mathrm{dR}}(W)) - d_{1-i}(W^*) \\ &\geq (\dim_K(\mathbf{D}_{\mathrm{dR}}(W_1)) - d_{1-i}(W_1^*)) + \dim_K(\mathbf{D}_{\mathrm{dR}}(W_2)) - d_{1-i}(W_2^*) \\ &= d_i(W_1) + d_i(W_2). \end{aligned}$$

Thus we must have $d_i(W) = d_i(W_1) + d_i(W_2)$ for all $i \in \mathbf{Z}$. In other words, the maps in (2.4) are strict for the filtrations, which shows (e). □

By [3, Proposition 2.3.4] the quasi-inverse of the functor $\mathbf{D}_{\mathrm{st}}$ is given by

$$\mathbf{D}_B(D) = ((\mathbf{B}_{\log} \otimes_{K_0} D)^{\varphi=1,N=0}, \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D)). \tag{2.5}$$

For a filtered E-(φ, N)-module D we put

$$\mathbf{X}_{\log}(D) = (\mathbf{B}_{\log} \otimes_{K_0} D)^{\varphi=1,N=0} \text{ and } \mathbf{X}_{\mathrm{dR}}(D) = \mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D/\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D).$$

If $\mathbf{D}_B(D) = (W_e, W_{\mathrm{dR}}^+)$, then $\mathbf{X}_{\log}(D) = W_e$ and $\mathbf{X}_{\mathrm{dR}}(D) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e)/W_{\mathrm{dR}}^+$.

3. S-B-pairs of Rank 1 and Triangulations

3.1. S-B-pairs of rank 1

Let S be a Banach E-algebra.

For any $a \in S^\times$ we define a filtered S-φ-module D_a as follows. As a $K_0 \otimes_{\mathbf{Q}_p} S$-module,

$$D_a = K_0 \otimes_{\mathbf{Q}_p} S = \oplus_{\tau: K_0 \hookrightarrow E} S e_\tau;$$

the $\varphi \otimes 1$-semilinear action φ on D_a satisfies

$$\varphi(e_{\mathrm{id}}) = e_{\varphi^{-1}},\ \varphi(e_{\varphi^{-1}}) = e_{\varphi^{-2}},\ \ldots,\ \varphi(e_{\varphi^{1-f}}) = a e_{\mathrm{id}};$$

the descending filtration on $D_{a,K} = K \otimes_{\mathbf{Q}_p} S$ is given by $\mathrm{Fil}^0 D_{a,K} = D_{a,K}$ and $\mathrm{Fil}^1 D_{a,K} = 0$.

Lemma 3.1. *If $a \in S$ satisfies that $a-1$ is topologically nilpotent, then there exists a unit $u_0 \in \mathbf{B}_{\max} \widehat{\otimes}_{K_0} S$ such that $\varphi^{[K_0:\mathbf{Q}_p]}(u_0) = a u_0$. Consequently*

$$\{x \in \mathbf{B}_{\max} \widehat{\otimes}_{K_0} S : \varphi^{[K_0:\mathbf{Q}_p]}(x) = ax\} = (\mathbf{B}_{e,K_0} \widehat{\otimes}_{K_0} S) u_0.$$

Proof. Let $\mathbf{Q}_p^{\mathrm{ur}}$ be the completed unramified extension of $\mathbf{Q}_p$. Then there exists an inclusion $\mathbf{Q}_p^{\mathrm{ur}} \hookrightarrow \mathbf{B}_{\max}$ that is compatible with φ.

As $\varphi^{[K_0:\mathbf{Q}_p]}-1$ is surjective on $\mathbf{Q}_p^{\mathrm{ur}}$, there exists a sequence $c_0=1, c_1, \cdots$ of elements in $\mathbf{Q}_p^{\mathrm{ur}}$ such that

$$(\varphi^{[K_0:\mathbf{Q}_p]}-1)c_i = c_{i-1}$$

for $i \geq 1$. The image of c_i by the map

$$\mathbf{Q}_p^{\mathrm{ur}} \hookrightarrow \mathbf{B}_{\max} \to \mathbf{B}_{\max}\widehat{\otimes}_{K_0} S$$

is again denoted by c_i. Put

$$u_0 = \sum_{i=0}^{\infty} c_i (a-1)^i.$$

Then u_0 is a unit and we have $\varphi^{[K_0:\mathbf{Q}_p]}u_0 = au_0$. □

Proposition 3.2. *If $a \in S$ satisfies that $a-1$ is topologically nilpotent, then $\mathbf{D}_B(D_a)$ is an S-B-pair of rank 1. Here $\mathbf{D}_B$ is the functor defined by (2.5).*

Proof. For each $z \in \mathbf{B}_{\max}\widehat{\otimes}_{\mathbf{Q}_p} D_a$ we write $z = \sum c_\tau e_\tau$ with $c_\tau \in \mathbf{B}_{\max}\widehat{\otimes}_{K_0,\tau} S$. Then $\varphi(z) = z$ if and only if $\varphi(c_{\varphi^i}) = c_{\varphi^{i-1}}$ $(i = 1, \ldots, [K_0 : \mathbf{Q}_p])$ and $\varphi^{[K_0:\mathbf{Q}_p]}(c_{\mathrm{id}}) = ac_{\mathrm{id}}$. Our assertion follows from Lemma 3.1. □

For any $a \in S^\times$, let $\delta_a : K^\times \to S^\times$ denote the character such that $\delta_a(\pi_K) = a$ and $\delta_a|_{\mathfrak{o}_K^\times} = 1$.

Remark 3.3. In the case of $S = E$, for any $u \in E^\times$, $\mathbf{D}_B(D_u)$ coincides with the E-B-pair $W(\delta_u)$ defined in [11] (see [11, §1.4]). From now on the base change of $W(\delta_u)$ from E to S is again denoted by $W(\delta_u)$.

Let $\delta : K^\times \to S^\times$ be a continuous character such that $\delta(\pi_K)$ is of the form $\delta(\pi_K) = au$, where $u \in E^\times$ and $a \in S$ satisfies that $a-1$ is topologically nilpotent. We call such a character *a good character*. Let W_a be the resulting S-B-pair in Proposition 3.2. Let δ' be the unitary continuous character $K^\times \to E^\times$ such that $\delta'|_{\mathfrak{o}_K^\times} = \delta|_{\mathfrak{o}_K^\times}$ and $\delta'(\pi_K) = 1$. By local class field theory, this induces a continuous character $\widetilde{\delta'} : G_K \to S^\times$ such that $\widetilde{\delta'} \circ \mathrm{rec}_K = \delta'$. Then we put

$$W(\delta) = W(S(\widetilde{\delta'})) \otimes W(\delta_u) \otimes W_a,$$

where $W(S(\widetilde{\delta'}))$ is the S-B-pair attached to the Galois representation $S(\widetilde{\delta'})$.

If δ is a continuous character $\delta : K^\times \to S^\times$, we write $\log(\delta)$ for the logarithmic of $\delta|_{\mathfrak{o}_K^\times}$, which is a $\mathbf{Z}_p$-linear homomorphism $\log(\delta) : K \to S$.

For any $\tau \in \mathrm{Emb}(K, E)$ we use the same notation τ to denote the composition of $\tau : K \hookrightarrow E$ and $E \hookrightarrow S$. Then $\{\tau : K \hookrightarrow S\}$ is a basis of $\mathrm{Hom}_{\mathbf{Z}_p}(E, S)$ over S. Write $\log(\delta) = \sum_\tau k_\tau \tau$, $k_\tau \in S$. We call $(k_\tau)_\tau$ the *weight vector* of δ and denote it by $\vec{w}(\delta)$. We use $w_\tau(\delta)$ to denote k_τ.

Remark 3.4. Let S be an affinoid algebra over E. For any continuous character $\delta : K^\times \to S^\times$ and any point z_0 of $\mathrm{Max}(S)$, there exists an affinoid neighborhood $U = \mathrm{Max}(S')$ of z_0 in $\mathrm{Max}(S)$ such that the restriction of δ to U is good.

Lemma 3.5. *Let δ be a character of $K^\times$ with values in $S = E[Z]/(Z^2)$, $\bar{\delta}$ the character of $K^\times$ with values in E obtained from δ modulo (Z). Write $\delta = \bar{\delta}_S(1 + Z\epsilon)$, where $\bar{\delta}_S$ is the character $K^\times \xrightarrow{\bar{\delta}} E^\times \hookrightarrow S^\times$. Let ϵ' be the additive character of G_K such that $\epsilon' \circ \mathrm{rec}_K(p) = 0$ and $\epsilon' \circ \mathrm{rec}_K|_{\mathfrak{o}_K^\times} = \epsilon|_{\mathfrak{o}_K^\times}$.*

Assume that $W(\bar{\delta})$ is crystalline and $\varphi^{[K_0:\mathbf{Q}_p]}$ acts on $\mathbf{D}_{\mathrm{cris}}(W(\bar{\delta}))$ by α. Then there is a nonzero element

$$x \in (\mathbf{B}_{\max,E} \otimes_{\mathbf{B}_{e,E}} W(\delta)_e)^{\varphi^{[K_0:\mathbf{Q}_p]}=\alpha(1+Zv_p(\pi_K)\epsilon(p)), G_K=(1+Z\epsilon')}$$

whose reduction modulo Z is a basis of $\mathbf{D}_{\mathrm{st}}(W(\bar{\delta}))$ over $K \otimes_{\mathbf{Q}_p} E$.

Proof. This follows from the fact that $W(\delta) = W(\bar{\delta}_S) \otimes W_{\delta_{1+Zv_p(\pi_K)\epsilon(p)}} \otimes W(1 + Z\epsilon')$. □

3.2. Triangulations and refinements

Now let S be an affinoid algebra over E. For any open affinoid subset U of S and any S-B-pair W let W_U denote the restriction to U of W.

Definition 3.6. Let W be an S-B-pair of rank n, z_0 a point of $\mathrm{Max}(S)$. If there is

- an affinoid neighborhood $U = \mathrm{Max}(S_U)$ of z_0,
- a strictly increasing filtration

$$\{0\} = \mathrm{Fil}_0 W_U \subset \mathrm{Fil}_1 W_U \subset \cdots \subset \mathrm{Fil}_n W_U = W_U$$

of saturated free sub-S_U-B-pairs, and

- n good continuous characters $\delta_i : \mathbf{Q}_p^\times \to S_U^\times$

such that for any $i = 1, \ldots, n$,

$$\mathrm{Fil}_i W_U / \mathrm{Fil}_{i-1} W_U \simeq W(\delta_i),$$

we say that W is *locally triangulable at* z_0; we call $\mathrm{Fil}_\bullet$ a *local triangulation* of W at z_0, and call $(\delta_1, \ldots, \delta_n)$ the *local triangulation parameters* attached to $\mathrm{Fil}_\bullet$.

Please consult [6, 4] for more knowledge on triangulations.

To discuss the relation between triangulations and refinements, we restrict ourselves to the case of $S = E$.

Let D be a filtered E-(φ, N)-module of rank n. The operator $\varphi^{[K_0:\mathbf{Q}_p]}$ on D is $K_0 \otimes_{\mathbf{Q}_p} E$-linear. We assume that the eigenvalues of $\varphi^{[K_0:\mathbf{Q}_p]} : D \to D$ are all in $K_0 \otimes_{\mathbf{Q}_p} E$, i.e. there exists a basis of D over $K_0 \otimes_{\mathbf{Q}_p} E$ such the matrix of $\varphi^{[K_0:\mathbf{Q}_p]}$ with respect to this basis is upper-triangular.

Following Mazur [9] we define a *refinement* of D to be a filtration on D

$$0 = \mathcal{F}_0 D \subset \mathcal{F}_1 D \subset \cdots \subset \mathcal{F}_n D = D$$

by E-subspaces stable by φ_D and N_D, such that each factor $\mathrm{gr}_i^{\mathcal{F}} D = \mathcal{F}_i D / \mathcal{F}_{i-1} D$ $(i = 1, \ldots, n)$ is of rank 1 over $K_0 \otimes_{\mathbf{Q}_p} E$. Any refinement fixes an ordering $\alpha_1, \ldots, \alpha_n$ of eigenvalues of $\varphi^{[K_0:\mathbf{Q}_p]}$ and an ordering $\vec{k}_1, \ldots, \vec{k}_n$ of Hodge-Tate weights of $K \otimes_{K_0} D$ taken with multiplicities such that the eigenvalue of $\varphi^{[K_0:\mathbf{Q}_p]}$ on $\mathrm{gr}_i^{\mathcal{F}} D$ is α_i and the Hodge-Tate weight of $\mathrm{gr}_i^{\mathcal{F}} \mathcal{D}$ is $\vec{k}_i$.

We have the following analogue of [1, Proposition 1.3.2].

Proposition 3.7. *Let W be a semistable E-B-pair, $D = \mathbf{D}_{\mathrm{st}}(W)$.*

(a) *The equivalence of categories between the category of semistable E-B-pairs and the category of filtered E-(φ, N)-modules induces a bijection between the set of triangulations on W and the set of refinements on D.*

(b) *If* $(\mathrm{Fil}_i W)$ *is a triangulation of* W *with triangulation parameters* $(\delta_1, \ldots, \delta_n)$ *that correspond to a refinement* $\mathcal{F}_\bullet D$ *of* D *with the ordering of Hodge-Tate weights being* $\vec{k}_1, \ldots, \vec{k}_n$*, then* $\delta_i = \tilde{\delta}_i \prod_{\tau \in \mathrm{Emb}(K,E)} \tau(x)^{k_{i,\tau}}$*, where* $\tilde{\delta}_i$ *is a smooth character.*

Proof. Assertion (a) follows from the fact that $\mathbf{D}_{\mathrm{st}}$ is an exact. Assertion (b) follows from [11, Lemma 4.1]. □

4. Cohomology Theory

4.1. Cohomology of (φ, Γ_K)-modules and cohomology of B-pairs

Let M be a (φ, Γ_K)-module. Assume that Γ_K has a topological generator γ. Define the cohomology $H^\bullet_{\Phi\Gamma}(M)$ by the complex $C^\bullet(M)$ defined by

$$C^0(M) = M \xrightarrow{(\gamma-1,\varphi-1)} C^1(M) = M \oplus M \to C^2(M) = M,$$

where the map $C^1(M) \to C^2(M)$ is given by $(x, y) \mapsto (\varphi - 1)x - (\gamma - 1)y$. Denote the kernel of $C^1(M) \to C^2(M)$ by $Z^1(M)$.

There is a one-to-one correspondence between $H^1(M)$ and the set of extensions of M_0 by M in the category of (φ, Γ_K)-modules, where $M_0 = \mathbf{B}^\dagger_{\mathrm{rig},K} e_0$ is the trivial (φ, Γ_K)-module with $\varphi(e_0) = \gamma(e_0) = e_0$. Let $\tilde{M}$ be an extension of M_0 by M, and let $\tilde{e}$ be any lifting of e_0 in $\tilde{M}$. Then the element in $H^1(M)$ corresponding to the extension $\tilde{M}$ is the class of $((\gamma - 1)\tilde{e}, (\varphi - 1)\tilde{e}) \in Z^1(M)$.

In [11] Nakamura introduced a cohomology for B-pairs and use it to compute the cohomology of (φ, Γ_K)-modules.

If $W = (W_e, W^+_{\mathrm{dR}})$ is an E-B-pair, let $C^\bullet(W)$ be the complex of G_K-modules defined by

$$C^0(W) := W_e \to C^1(W) := W_{\mathrm{dR}}/W^+_{\mathrm{dR}}.$$

Here, $W_e \to W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$ is the natural map.

Definition 4.1. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an E-B-pair. We define the *Galois cohomology of* W by $H^i_B(W) := H^i(G_K, C^\bullet(W))$.

By definition there is a long exact sequence

$$\cdots \to H^i_B(W) \to H^i(G_K, W_e) \to H^i(G_K, W_{\mathrm{dR}}/W^+_{\mathrm{dR}}) \to \cdots . \tag{4.1}$$

For a G_K-module M put $C^0(M) = M$ and let $C^i(M)$ be the space of continuous functions from $(G_K)^{\times i}$ to M. Let $\delta_0 : C^0(M) \to C^1(M)$ be the map $x \mapsto (g \mapsto g(x) - x)$ and let $\delta_1 : C^1(M) \to C^2(M)$ be the map $f \mapsto ((g_1, g_2) \mapsto f(g_1g_2) - f(g_1) - g_1f(g_2))$.

Nakamura [11] showed that $H^1_B(W)$ is isomorphic to $\ker(\tilde{\delta}_1)/\mathrm{im}(\tilde{\delta}_0)$, where $\tilde{\delta}_0$ and $\tilde{\delta}_1$ are defined by

$$\begin{aligned} \tilde{\delta}_0 : C^0(W_e) \oplus C^0(W^+_{\mathrm{dR}}) &\to C^1(W_e) \oplus C^1(W^+_{\mathrm{dR}}) \oplus C^0(W_{\mathrm{dR}}) : \\ (x, y) &\mapsto (\delta_0(x), \delta_0(y), x - y), \\ \tilde{\delta}_1 : C^1(W_e) \oplus C^1(W^+_{\mathrm{dR}}) \oplus C^0(W_{\mathrm{dR}}) &\to C^2(W_e) \oplus C^2(W^+_{\mathrm{dR}}) \oplus C^1(W_{\mathrm{dR}}) : \\ (f_1, f_2, x) &\mapsto (\delta_1(f_1), \delta_1(f_2), f_1 - f_2 - \delta_0(x)). \end{aligned}$$

The map $H^1_B(W) \to H^1(G_K, W_e)$ is induced by the forgetful map

$$C^1(W_e) \oplus C^1(W^+_{\mathrm{dR}}) \oplus C^0(W_{\mathrm{dR}}) \to C^1(W_e).$$

There is a one-to-one correspondence between $H^1(G_K, W)$ and the set of extensions of W_0 by W in the category of E-B-pairs. Here, $W_0 = (\mathbf{B}_e \otimes_{\mathbf{Q}_p} E, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} E)$ is the trivial E-B-pair. Let $\tilde{W} = (\tilde{W}_e, \tilde{W}^+_{\mathrm{dR}})$ be an extension of W_0 by W. Let $(\tilde{w}_e, \tilde{w}^+_{\mathrm{dR}})$ be a lifting in $\tilde{W}$ of $(1, 1) \in W_0$. Then the element in $H^1_B(W)$ corresponding to the extension $\tilde{W}$ is just the class of $((\sigma \mapsto (\sigma - 1)\tilde{w}_e), (\sigma \mapsto (\sigma - 1)\tilde{w}^+_{\mathrm{dR}}), \tilde{w}_e - \tilde{w}^+_{\mathrm{dR}}) \in \ker(\tilde{\delta}_1)$.

By Proposition 2.3 there is a one-to-one correspondence between $\mathrm{Ext}(M_0, M)$ and
$\mathrm{Ext}(W_0, W(M))$. It induces a natrual isomorphism

$$i_M : H^1_{\Phi\Gamma}(M) \to H^1_B(W(M)).$$

4.2. 1-cocycles from infinitesimal deformations

Let S be the E-algebra $E[Z]/(Z^2)$, $\tilde{M}$ an S-(φ, Γ_K)-module. Let $\{e_1, \ldots, e_n\}$ be an S-basis of $\tilde{M}$, $\{e_1^*, \ldots, e_n^*\}$ the dual basis of $\tilde{M}^*$. Put $M = \tilde{M} \otimes_S E$

and $M^* = \tilde{M}^* \otimes_S E$. Let $e_{i,z}$ denote e_i mod Z, and $e^*_{j,z}$ denote e^*_j mod Z. Then $\{e_{1,z}, \ldots, e_{n,z}\}$ is an E-basis of M, and $\{e^*_{1,z}, \ldots, e^*_{n,z}\}$ is an E-basis of M^*.

The matrices of φ and γ with respect to $\{e_1, \ldots, e_n\}$ are denote by $\tilde{A}_\varphi$ and $\tilde{A}_\gamma$ respectively, so that $\varphi(e_j) = \sum_i (\tilde{A}_\varphi)_{ij} e_i$ and $\gamma(e_j) = \sum_i (\tilde{A}_\gamma)_{ij} e_i$. Write $\tilde{A}_\varphi = (I_n + ZU_\varphi)A_\varphi$ and $\tilde{A}_\gamma = (I_n + ZU_\gamma)A_\gamma$. Put

$$c_{\Phi\Gamma}(\tilde{M}) = (\sum_{i,j} (U_\varphi)_{i,j} e^*_{j,z} \otimes e_{i,z}, \sum_{i,j} (U_\gamma)_{i,j} e^*_{j,z} \otimes e_{i,z}).$$

Write $\mathbf{D}_B(\tilde{M}) = (\tilde{W}_e, \tilde{W}^+_{\mathrm{dR}})$, $\mathbf{D}_B(M) = W$ and $\mathbf{D}_B(M^*) = W^*$.

Let $f_1, \ldots, f_n$ be a basis of $\tilde{W}_e$ over $\mathbf{B}_{e,E}$, and let $g_1, \ldots, g_n$ be a basis of $\tilde{W}^+_{\mathrm{dR}}$ over $\mathbf{B}^+_{\mathrm{dR},E}$. We write the matrix of $\sigma \in G_K$ with respect to the basis $\{f_1, \ldots, f_n\}$ by $(I_n + ZU_{e,\sigma})A_{e,\sigma}$, and the matrix of σ with respect to the basis $\{g_1, \ldots, g_n\}$ by $(I_n + ZU^+_{\mathrm{dR},\sigma})A^+_{\mathrm{dR},\sigma}$. Here,

$$U_{e,\sigma} \in \mathrm{M}_n(\mathbf{B}_{e,E}),\ U^+_{\mathrm{dR},\sigma} \in \mathrm{M}_n(\mathbf{B}^+_{\mathrm{dR},E}),\ A_{e,\sigma} \in \mathrm{GL}_n(\mathbf{B}_{e,E}),$$

and

$$A^+_{\mathrm{dR},\sigma} \in \mathrm{GL}_n(\mathbf{B}^+_{\mathrm{dR},E}).$$

Write $(f_1, \ldots, f_n) = (g_1, \ldots, g_n)(I_n + ZU_{\mathrm{dR}})A_{\mathrm{dR}}$ and put

$$c_B(\tilde{M}) = \Big((\sigma \mapsto \sum_{i,j} (U_{e,\sigma})_{ij} f^*_{j,z} \otimes f_{i,z}), (\sigma \mapsto \sum_{i,j} (U^+_{\mathrm{dR},\sigma})_{ij} g^*_{j,z} \otimes g_{i,z}), \sum_{i,j} (U_{\mathrm{dR}})_{ij} g^*_{j,z} \otimes g_{i,z}\Big).$$

Proposition 4.2.

(a) $c_{\Phi\Gamma}(\tilde{M})$ *is in* $Z^1(M^* \otimes M)$.

(b) $c_B(\tilde{M})$ *is in* $\ker(\tilde{\delta}_{1,W^* \otimes W})$.

(c) *We have* $i_M([c_{\Phi\Gamma}(\tilde{M})]) = [c_B(\tilde{M})]$.

Proof. It is easy to verify (a) and (b).

Put $M^*_S = M^* \otimes_E S$. We consider $M^*_S \otimes_S \tilde{M}$ as an extension of $M^* \otimes_E M$ by itself, and form the following commutative diagram

$$\begin{array}{ccccccccc} & & & & & & M_0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M^* \otimes_E M & \longrightarrow & M^*_S \otimes_S \tilde{M} & \longrightarrow & M^* \otimes_E M & \longrightarrow & 0, \end{array}$$

where the vertical map $M_0 \to M^* \otimes_E M$ is given by $1 \mapsto \sum_{i=1}^n e_{i,z}^* \otimes e_{i,z}$, which does not depend on the choice of the basis $\{e_1, \ldots, e_n\}$. Pulling back $M_S^* \otimes_S \tilde{M}$ via $M_0 \to M^* \otimes_E M$ we obtain an extension of M_0 by $M^* \otimes_E M$. Let $\mathcal{M}$ denote the resulting extension. Then $\mathcal{M}$ is a sub-E-B-pair of $M_S^* \otimes_S \tilde{M}$. Put $\mathbf{D}_B(\mathcal{M}) = (\mathcal{W}_e, \mathcal{W}_{\mathrm{dR}}^+)$.

A lifting of 1 in $\mathcal{W}_e$ is $\sum_j f_{j,z}^* \otimes f_j$, and a lifting of 1 in $\mathcal{W}_{\mathrm{dR}}^+$ is $\sum_j g_{j,z}^* \otimes g_j$. We have

$$\begin{aligned}
(\sigma-1) \sum_j f_{j,z}^* \otimes f_j &= \sigma(f_{1,z}^*, \ldots, f_{n,z}^*) \otimes \sigma \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} - (f_{1,z}^*, \ldots, f_{n,z}^*) \otimes \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \\
&= (f_{1,z}^*, \ldots, f_{n,z}^*)(A_{e,\sigma}^t)^{-1} \otimes A_{e,\sigma}^t (1 + z U_{e,\sigma}^t) \\
&= (f_{1,z}^*, \ldots, f_{n,z}^*) \otimes U_{e,\sigma}^t z \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.
\end{aligned}$$

Similarly,

$$(\sigma - 1) \sum_j g_{j,z}^* \otimes g_j = (g_{1,z}^*, \ldots, g_{n,z}^*) \otimes (U_{\mathrm{dR},\sigma}^+)^t z \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$

and

$$\sum_j f_{j,z}^* \otimes f_j - \sum_j g_{j,z}^* \otimes g_j = (g_{1,z}^*, \ldots, g_{n,z}^*) \otimes U_{\mathrm{dR}}^t z \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}.$$

Hence the element in $H_B^1(\mathbf{D}_B(M^* \otimes_E M))$ attached to the extension $\mathbf{D}_B(\mathcal{M})$ is $[c_B(\tilde{M})]$.

A similar computation shows that the element in $H_{\Phi\Gamma}^1(M^* \otimes_E M)$ attached to the extension $\mathcal{M}$ is $[c_{\Phi\Gamma}(\tilde{M})]$. Now (c) follows. □

5. The Reciprocity Law and an Application

5.1. Reciprocity law

In [14, Section 2] using local class field theory Zhang precisely described the perfect pairing

$$H^1(G_K, E) \times H^1(G_K, E(1)) \to H^2(G_K, E(1)).$$

We recall it below.

The Kummer theory gives us a canonical isomorphism so called the Kummer map

$$\begin{aligned}\varprojlim_n (K^\times/(K^\times)^{p^n}) \otimes_{\mathbf{Z}_p} E &\to H^1(G_K, E(1))\\ \sum_i \alpha_i \otimes a_i &\mapsto \sum_i a_i[(\alpha_i)].\end{aligned}$$

Here (α) is the 1-cocycle such that

$$\frac{g(\sqrt[p^n]{\alpha})}{\alpha} = \varepsilon_n^{(\alpha_g)}$$

for $\alpha \in K^\times$ and $g \in G_K$, where $(\sqrt[p^{n+1}]{\alpha})^p = \sqrt[p^n]{\alpha}$. Combining the Kummer map and the exponent map

$$\exp : p\mathfrak{o}_K \to K^\times$$

and extending it by linearity we obtain an embedding from $K \otimes_{\mathbf{Q}_p} E$ to $H^1(G_K, E(1))$, again denoted by exp. Then we have

$$H^1(G_K, E(1)) = \exp(K \otimes_{\mathbf{Q}_p} E) \oplus E \cdot [(p)].$$

Let $\mathrm{Hom}(G_K, E)$ be the group of additive characters of G_K with values in E. As the action of G_K on E is trivial, $H^1(G_K, E)$ is naturally isomorphic to $\mathrm{Hom}(G_K, E)$. Let $\psi_0 : G_K \to E$ be the additive character that vanishes on the inertial subgroup of G_K and maps the geometrical Frobenius to $[K_0 : \mathbf{Q}_p]$. For any $\tau \in \mathrm{Emb}(K, E)$ let ψ_τ be the composition $\tau \circ \log \circ \mathrm{rec}_K^{-1}$[2], where log

[2]Since the character ψ_τ of the Weil group W_K sends any lifting of the qth power Frobenius to 0, it can be extended to a character of G_K which is again denoted by ψ_τ

is normalized such that $\log(p) = 0$. Then $\{\psi_0, \psi_\tau : \tau \in \mathrm{Emb}(K, E)\}$ is an E-basis of $H^1(G_K, E)$.

Lemma 5.1 (Zhang, Proposition 2.1)**.** *The cup product of* $a_0\psi_0 + \sum_{\tau\in\mathrm{Emb}(K,E)} a_\tau\psi_\tau$ $(a_0, a_\tau \in E)$ *and* $b_0[(p)] + \exp(b)$ $(b_0 \in E, b \in K \otimes_{\mathbf{Q}_p} E)$ *is*

$$\Big(a_0b_0 - \mathrm{tr}_{K/\mathbf{Q}_p}((a_\tau)_\tau \cdot b)\Big)(\psi_0 \cup [(p)]).$$

Here, $(a_\tau)_\tau$ is considered as an element in $K \otimes_{\mathbf{Q}_p} E$ via the isomorphism (2.1).

Lemma 5.2. *For $\lambda_0, \lambda_\tau \in E$ $(\tau \in \mathrm{Emb}(K, E))$, the extension of E (as a trivial G_K-module) by E corresponding to the cocycle $\lambda_0\psi_0 + \sum_{\tau\in\mathrm{Emb}(K,E)} \lambda_\tau\psi_\tau$ is de Rham if and only if $\lambda_\tau = 0$ for each τ.*

Proof. By [11, Lemma 4.3], the subspace of extensions of E by E that are de Rham is 1-dimensional, and so consists of those corresponding to the cocycles $\lambda_0\psi_0$ $(\lambda_0 \in E)$. □

5.2. An auxiliary formula

Let $\vec{\mathcal{L}} = (\mathcal{L}_\sigma)_{\sigma:K\hookrightarrow E}$ be a vector. We consider $\vec{\mathcal{L}}$ as an element of $K\otimes_{\mathbf{Q}_p}E$ via the isomorphism (2.1).

Let D be a filtered E-(φ, N)-module: the underlying E-(φ, N)-module D is a $(K_0 \otimes_{\mathbf{Q}_p} E)$-module with a basis $\{f_1, f_2, f_3\}$ such that

$$\varphi^{[K_0:\mathbf{Q}_p]} f_1 = p^{-[K_0:\mathbf{Q}_p]} f_1,\ \varphi^{[K_0:\mathbf{Q}_p]} f_2 = f_2,\ \varphi^{[K_0:\mathbf{Q}_p]} f_3 = f_3,$$

and

$$N(f_1) = 0,\ N(f_2) = -f_1,\ N(f_3) = f_1;$$

the filtration on

$$K \otimes_{K_0} D = (K \otimes_{\mathbf{Q}_p} E)f_1 \oplus (K \otimes_{\mathbf{Q}_p} E)f_2 \oplus (K \otimes_{\mathbf{Q}_p} E)f_3$$

satisfies

$$\mathrm{Fil}^i D = \begin{cases} (K \otimes_{\mathbf{Q}_p} E)(f_2 - \vec{\mathcal{L}}f_1) \oplus (K \otimes_{\mathbf{Q}_p} E)(f_3 + \vec{\mathcal{L}}f_1) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Let π_i be the projection map

$$\mathbf{X}_{\log}(D) \to \mathbf{B}_{\log,E}, \quad \sum_{j=1}^{3} a_j f_j \mapsto a_i.$$

Lemma 5.3. *Let $c : G_K \to \mathbf{X}_{\log}(D)$ be a 1-cocycle whose class in $H^1(G_K, \mathbf{X}_{\log}(D))$ belongs to $\ker(H^1(G_K, \mathbf{X}_{\log}(D)) \to H^1(G_K, \mathbf{X}_{\mathrm{dR}}(D)))$. Then there exist*

$$\gamma_{2,0}, \gamma_{2,\tau}, \gamma_{3,0}, \gamma_{3,\tau} \in E$$

$(\tau \in \mathrm{Emb}(K,E))$ *such that*

$$\pi_2(c) = \gamma_{2,0}\psi_0 + \sum_{\tau\in \mathrm{Emb}(K,E)} \gamma_{2,\tau}\psi_\tau$$

and

$$\pi_3(c) = \gamma_{3,0}\psi_0 + \sum_{\tau\in \mathrm{Emb}(K,E)} \gamma_{3,\tau}\psi_\tau.$$

Furthermore,

$$\gamma_{2,0} - \gamma_{3,0} = \sum_{\tau\in \mathrm{Emb}(K,E)} \mathcal{L}_\tau(\gamma_{2,\tau} - \gamma_{3,\tau}).$$

In our proof of Lemma 5.3 we need the following

Lemma 5.4. *Let D be an E-(φ, N)-module. If Fil_1 and Fil_2 are two filtrations on $K \otimes_{K_0} D$ such that $\mathrm{Fil}^0_1(K \otimes_{K_0} D) = \mathrm{Fil}^0_2(K \otimes_{K_0} D)$, then the kernel of*

$$H^1(G_K, \mathbf{X}_{\log}(D)) \to H^1(G_K, \mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_1))$$

coincides with the kernel of

$$H^1(G_K, \mathbf{X}_{\log}(D)) \to H^1(G_K, \mathbf{X}_{\mathrm{dR}}(D, \mathrm{Fil}_2)).$$

Proof. The proof is similar to that of [13, Proposition 2.5] □

Proof of Lemma 5.3. The argument is similar to the proof of [13, Lemma 5.1]. We only give a sketch.

Write $c_\sigma = \lambda_{1,\sigma} f_1 + \lambda_{2,\sigma} f_2 + \lambda_{3,\sigma} f_3$. As c takes values in $\mathbf{X}_{\log}(D)$, we have $\lambda_{2,\sigma}, \lambda_{3,\sigma} \in E$. This ensures the existence of $\gamma_{2,0}, \gamma_{2,\tau}, \gamma_{3,0}, \gamma_{3,\tau}$.

Let Fil be the filtration on D such that $Fil^{-1}D = D$ and $Fil^iD = \mathrm{Fil}^iD$ if $i \geq 0$. Then (D, Fil) is admissible. Let V be the semistable E-representation of G_K attached to $D_V = (D, Fil)$. By Lemma 5.4, $[c]$ is in the kernel of $H^1(G_K, \mathbf{X}_{\log}(D_V)) \to H^1(G_K, \mathbf{X}_{\mathrm{dR}}(D_V))$ and so there exists a 1-cocycle $c^{(1)} : G_K \to V$ such that the image of $[c^{(1)}]$ by $H^1(G_K, V) \to H^1(G_K, \mathbf{X}_{\log}(D_V))$ is $[c]$.

We form the following commutative diagram

$$\begin{array}{ccccccccc} & & & & V' & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & V_0 & \longrightarrow & V & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \pi_{V,V_1}} & & \downarrow & & \\ 0 & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & T_1 & \longrightarrow & 0 \end{array} \tag{5.1}$$

with the horizontal lines being exact, where V_0 (resp. V') is the subrepresentation of V corresponding to the filtered E-(φ, N)-submodule of D_V generated by f_1 (resp. by $f_2 + f_3$) which is admissible. From (5.1) we obtain the following commutative diagram

$$\begin{array}{ccccc} H^1(G_K, V) & \longrightarrow & H^1(G_K, T) & \longrightarrow & H^2(G_K, V_0) \\ \downarrow{\scriptstyle \pi_{V,V_1}} & & \downarrow & & \| \\ H^1(G_K, V_1) & \longrightarrow & H^1(G_K, T_1) & \longrightarrow & H^2(G_K, V_0), \end{array}$$

where the horizontal lines are exact.

Write $c^{(2)}$ for the 1-cocycle $G_K \xrightarrow{c^{(1)}} V \to T \to T_1$. By a simple computation we obtain

$$[c^{(2)}] = [\Big((\gamma_{2,0} - \gamma_{3,0})\psi_0 + \sum_{\tau \in \mathrm{Emb}(K,E)} (\gamma_{2,\tau} - \gamma_{3,\tau})\psi_\tau\Big)\bar{f}_2],$$

where $\bar{f}_2$ is the image of $f_2 \in V$ in T_1. Note that T_1 is isomorphic to E, and V_0 is isomorphic to $E(1)$. Being the image of $[\pi_{V,V_1}(c^{(1)})]$ in $H^1(T_1)$, $[c^{(2)}]$ lies in the kernel of $H^1(G_K, T_1) \to H^2(G_K, V_0)$. By [14, Lemma 5.5], as an extension of E by $E(1)$, V_1 corresponds to the element $[(p)] + \exp(\vec{\mathcal{L}})$. Now Lemma 5.1 yields our second assertion. $\square$

6. L-invariants

Let D be a filtered E-(φ, N)-module of rank n. Fix a refinement $\mathcal{F}$ of D. Then $\mathcal{F}$ fixes an ordering $\alpha_1, \ldots, \alpha_n$ of the eigenvalues of $\varphi^{[K_0:\mathbf{Q}_p]}$ and an ordering $\vec{k}_1, \ldots, \vec{k}_n$ of the Hodge-Tate weights.

6.1. The operator $N_{\mathcal{F}}$

The operator φ induces a $K_0 \otimes_{\mathbf{Q}_p} E$-semilinear operator $\varphi_{\mathcal{F}}$ on $\mathrm{gr}^{\mathcal{F}}_{\bullet} D = \bigoplus_{i=1}^{n} \mathcal{F}_i D/\mathcal{F}_{i-1} D$.

We define a $K_0 \otimes_{\mathbf{Q}_p} E$-linear operator $N_{\mathcal{F}}$ on $\mathrm{gr}^{\mathcal{F}}_{\bullet} D$. The definition is similar to the one defined in [13], so we omit some details.

For any $i \in \{1, \ldots, n\}$, if $N(\mathcal{F}_i D) = N(\mathcal{F}_{i-1} D)$, we demand that $N_{\mathcal{F}}$ maps $\mathrm{gr}^{\mathcal{F}}_i D$ to zero.

Now we assume that $N(\mathcal{F}_i D) \supsetneq N(\mathcal{F}_{i-1} D)$. Let j be the minimal integer such that

$$N(\mathcal{F}_i D) \subseteq N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D.$$

Proposition 6.1. $N(\mathcal{F}_{i-1} D) \cap \mathcal{F}_j D = N(\mathcal{F}_{i-1} D) \cap \mathcal{F}_{j-1} D$.

Proof. Note that $\mathcal{F}_j D$, $\mathcal{F}_{j-1} D$, $N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D$ and $N(\mathcal{F}_{i-1} D) + \mathcal{F}_{j-1} D$ are stable by φ. Thus $(N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D)/(N(\mathcal{F}_{i-1} D) + \mathcal{F}_{j-1} D)$ is a φ-module, and so must be free over $K_0 \otimes_{\mathbf{Q}_p} E$. Hence the map

$$\mathcal{F}_j D/\mathcal{F}_{j-1} D \to (N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D)/(N(\mathcal{F}_{i-1} D) + \mathcal{F}_{j-1} D) \tag{6.1}$$

is an isomorphism. It follows that $N(\mathcal{F}_{i-1} D) \cap \mathcal{F}_j D = N(\mathcal{F}_{i-1} D) \cap \mathcal{F}_{j-1} D$. $\square$

The operator N induces a $K_0 \otimes_{\mathbf{Q}_p} E$-linear map

$$\mathcal{F}_i D/\mathcal{F}_{i-1} D \to (N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D)/(N(\mathcal{F}_{i-1} D) + \mathcal{F}_{j-1} D).$$

We define the map $N_{\mathcal{F}} : \mathrm{gr}^{\mathcal{F}}_i D \to \mathrm{gr}^{\mathcal{F}}_j D$ to be the composition of this map and the inverse of (6.1).

Finally we extend $N_{\mathcal{F}}$ to the whole $\mathrm{gr}^{\mathcal{F}}_{\bullet} D$ by $K_0 \otimes_{\mathbf{Q}_p} E$-linearity. Note that $N_{\mathcal{F}} \varphi_{\mathcal{F}} = p \varphi_{\mathcal{F}} N_{\mathcal{F}}$. By definition, for any i we have either $N(\mathrm{gr}^{\mathcal{F}}_i D) = 0$ or $N(\mathrm{gr}^{\mathcal{F}}_i D) = \mathrm{gr}^{\mathcal{F}}_j D$ for some j.

Definition 6.2. For $j \in \{1, \ldots, n-1\}$ we say that j is *marked* (or a *marked index*) for $\mathcal{F}$ if there is some $i \in \{2, \ldots, n\}$ such that $N_{\mathcal{F}}(\mathrm{gr}_i^{\mathcal{F}} D) = \mathrm{gr}_j^{\mathcal{F}} D$.

Note that i and j in the above definition are determined by each other. We write $i = t_{\mathcal{F}}(j)$ and $j = s_{\mathcal{F}}(i)$.

Proposition 6.3. *The following two assertions are equivalent:*

(a) *s is marked and $t = t_{\mathcal{F}}(s)$.*

(b) *$N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D = N\mathcal{F}_{t-1}D \cap \mathcal{F}_{s-1}D$ and $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneq N\mathcal{F}_t D \cap \mathcal{F}_{s-1}D$.*

Proof. We have already seen that, if (a) holds, then (b) holds. Conversely, we assume that (b) holds. Then $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneq N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D$. Thus $N\mathcal{F}_t D \supsetneq N\mathcal{F}_{t-1}D$.

We show that $N\mathcal{F}_t D \subsetneq N\mathcal{F}_{t-1}D + \mathcal{F}_{s-1}D$. If it is not true, then there exists $y \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1}D$ which is a lifting of a basis of $\mathrm{gr}_t^{\mathcal{F}} D$ over $K_0 \otimes_{\mathbf{Q}_p} E$ such that $N(y) \in \mathcal{F}_{s-1}D$. For any $z \in \mathcal{F}_t D$, write $z = w + \lambda y$ with $w \in \mathcal{F}_{t-1}D$ and $\lambda \in K_0 \otimes_{\mathbf{Q}_p} E$. If $N(z)$ is in $\mathcal{F}_s D$, then $N(w)$ is also in $\mathcal{F}_s D$. But $N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D = N\mathcal{F}_{t-1}D \cap \mathcal{F}_{s-1}D$. Thus $N(w)$ is in $\mathcal{F}_{s-1}D$, which implies that $N(z) = N(w) + \lambda N(y)$ is also in $\mathcal{F}_{s-1}D$. So, $N\mathcal{F}_t D \cap \mathcal{F}_s D = N\mathcal{F}_t D \cap \mathcal{F}_{s-1}D$, a contradiction.

From $N\mathcal{F}_t D \cap \mathcal{F}_s D \supsetneq N\mathcal{F}_{t-1}D \cap \mathcal{F}_s D$ we see that there is $x \in \mathcal{F}_t D \backslash \mathcal{F}_{t-1}D$ such that $N(x) \in \mathcal{F}_s D$. We must have $N\mathcal{F}_t D \subseteq N\mathcal{F}_{t-1}D + \mathcal{F}_s D$. Otherwise, let j be the smallest integer such that $N\mathcal{F}_t D \subseteq N\mathcal{F}_{t-1}D + \mathcal{F}_j D$ and assume that $j > s$. Then $N_{\mathcal{F}}(x + \mathcal{F}_{t-1}D) = 0$, which contradicts the fact that $N_{\mathcal{F}} : \mathrm{gr}_t^{\mathcal{F}} D \to \mathrm{gr}_j^{\mathcal{F}} D$ is an isomorphism. □

6.2. Strongly marked indices and $\mathcal{L}$-invariants

Assume that s is marked for $\mathcal{F}$ and $t = t_{\mathcal{F}}(s)$. We consider the decompositions

$$\mathcal{F}_t D / \mathcal{F}_{s-1} D = (K_0 \otimes_{\mathbf{Q}_p} E) \cdot \bar{e}_s \oplus L \oplus (K_0 \otimes_{\mathbf{Q}_p} E)\bar{e}_t$$

that satisfy the following conditions:

- $\overline{\mathcal{F}}_1(\mathcal{F}_t D/\mathcal{F}_{s-1}D) = (K_0 \otimes_{\mathbf{Q}_p} E)\bar{e}_s$ and $\overline{\mathcal{F}}_{t-s}(\mathcal{F}_t D/\mathcal{F}_{s-1}D) = (K_0 \otimes_{\mathbf{Q}_p} E)\bar{e}_s \oplus L$, where $\overline{\mathcal{F}}$ is the refinement on $\mathcal{F}_t D/\mathcal{F}_{s-1}D$ induced by $\mathcal{F}$.

- Both L and $(K_0 \otimes_{\mathbf{Q}_p} E)\bar{e}_s \oplus (K_0 \otimes_{\mathbf{Q}_p} E)\bar{e}_t$ are stable by φ and N; $\varphi^{[K_0:\mathbf{Q}_p]}(\bar{e}_t) = \alpha_t \bar{e}_t$ and $N(\bar{e}_t) = \bar{e}_s$.

Such a decomposition is called an *s-decomposition.*

Remark 6.4. s-decompositions may be not exist. However, if φ is semisimple, then s-decompositions always exist (see [13]).

Let dec denote an s-decomposition $\mathcal{F}_t D/\mathcal{F}_{s-1}D = E\bar{e}_s \oplus L \oplus E\bar{e}_t$.

There is a natural isomorphism $E\bar{e}_s \oplus E\bar{e}_t \to (\mathcal{F}_t D/\mathcal{F}_{s-1}D)/L$ of (φ, N)-modules. Usually the filtration on the filtered E-(φ, N)-submodule $E\bar{e}_s \oplus E\bar{e}_t$ and that on $(\mathcal{F}_t D/\mathcal{F}_{s-1}D)/L$ are different.

When these two filtrations satisfy certain compatible condition, we say the decomposition dec is perfect. Precisely, we say that dec is *perfect* if for any $\tau : K \hookrightarrow E$ we have $k_{s,\tau} < k_{t,\tau}$, and if there exist $k'_{s,\tau}, k'_{t,\tau}$ and $\mathcal{L}_{\mathrm{dec},\tau} \in E$ satisfying $k_{s,\tau} \le k'_{s,\tau} < k'_{t,\tau} \le k_{t,\tau}$ such that the following conditions hold.

- The filtration on the filtered E-(φ, N)-submodule $E\bar{e}_s \oplus E\bar{e}_t$ satisfies

$$\mathrm{Fil}^i_\tau(E\bar{e}_s \oplus E\bar{e}_t) = \begin{cases} E\bar{e}_{s,\tau} \oplus E\bar{e}_{t,\tau} & \text{if } i \le k_{s,\tau}, \\ E(\bar{e}_{t,\tau} + \mathcal{L}_{\mathrm{dec},\tau}\bar{e}_{s,\tau}) & \text{if } k_{s,\tau} < i \le k'_{t,\tau}, \\ 0 & \text{if } i > k'_{t,\tau}, \end{cases}$$

- The filtration on the quotient of $\mathcal{F}_t D/\mathcal{F}_{s-1}D$ by L satisfies

$$\mathrm{Fil}^i_\tau \mathcal{F}_t D/\mathcal{F}_{s-1}D = \begin{cases} E\bar{e}_{s,\tau} \oplus E\bar{e}_{t,\tau} & \text{if } i \le k'_{s,\tau}, \\ E(\bar{e}_t + \mathcal{L}_{\mathrm{dec},\tau}\bar{e}_s) & \text{if } k'_{s,\tau} < i \le k_{t,\tau}, \\ 0 & \text{if } i > k_{t,\tau}, \end{cases}$$

  where the images of $\bar{e}_s$ and $\bar{e}_t$ in $\mathcal{F}_t D/\mathcal{F}_{s-1}D$ are again denoted by $\bar{e}_s$ and $\bar{e}_t$.

Definition 6.5. If there exists a perfect s-decomposition, we say that s is *strongly marked* (or a *strongly marked index*). In this case we attached to each pair (s,t) with $t = t_{\mathcal{F}}(s)$ an invariant $\vec{\mathcal{L}}_{\mathcal{F},s,t} = (\mathcal{L}_{\mathrm{dec},\tau})_\tau$, where dec is a perfect s-decomposition. Proposition 6.6 below tells us that $\vec{\mathcal{L}}_{\mathcal{F},s,t}$ is independent of the choice of perfect s-decompositions. We call $\vec{\mathcal{L}}_{\mathcal{F},s,t}$ the *Fontaine-Mazur $\mathcal{L}$-invariant* associated to $(\mathcal{F}, s, t)$, and denote $\mathcal{L}_{\mathrm{dec},\tau}$ by $\mathcal{L}_{\mathcal{F},s,t,\tau}$.

In the case of $t = s + 1$, s is strongly marked if and only if $k_{s,\tau} < k_{t,\tau}$ for all τ.

Proposition 6.6. *If* dec_1 *and* dec_2 *are two perfect s-decompositions, then* $\mathcal{L}_{\mathrm{dec}_1,\tau} = \mathcal{L}_{\mathrm{dec}_2,\tau}$ *for any* τ.

Proof. The argument is similar to the proof of [13, Proposition 4.9]. □

Let D^* be the filtered E-(φ, N)-module that is the dual of D. Let $\check{\mathcal{F}}$ be the refinement on D^* such that

$$\check{\mathcal{F}}_i D^* := (\mathcal{F}_{n-i} D)^\perp = \{y \in D^* : \langle y, x\rangle = 0 \text{ for all } x \in \mathcal{F}_{n-i} D\}.$$

We call $\check{\mathcal{F}}$ the *dual refinement* of $\mathcal{F}$.

If $L \subset M$ are submodules of D, then $M^\perp \subset L^\perp$. The pairing $\langle\cdot,\cdot\rangle$: $L^\perp \times M$ induces a non-degenerate pairing on $L^\perp/M^\perp \times M/L$, so that we can identify $L^\perp/M^\perp$ with the dual of M/L naturally. In particular, $\mathrm{gr}_i^{\check{\mathcal{F}}} D^*$ is naturally isomorphic to the dual of $\mathrm{gr}_{n+1-i}^{\mathcal{F}} D$. Thus $\mathrm{gr}_\bullet^{\check{\mathcal{F}}} D^*$ is naturally isomorphic to the dual of $\mathrm{gr}_\bullet^{\mathcal{F}} D$.

Proposition 6.7.

(a) $N_{\check{\mathcal{F}}}$ *is dual to* $-N_{\mathcal{F}}$.

(b) s *is marked for* $\mathcal{F}$ *if and only if* $n + 1 - t_{\mathcal{F}}(s)$ *is marked for* $\check{\mathcal{F}}$.

(c) s *is strongly marked for* $\mathcal{F}$ *if and only if* $n+1-t_{\mathcal{F}}(s)$ *is strongly marked for* $\check{\mathcal{F}}$.

Proof. The proof of (a) is similar to that of [13, Proposition 4.14]. The proof of (b) is similar to that of [13, Proposition 4.13]. The proof of (c) is similar to that of [13, Proposition 4.15 (a)]. □

7. Projection Vanishing Property

Put $S = E[Z]/(Z^2)$. Let z be the closed point defined by the maximal ideal (Z) of S.

Let $W = (W_e, W_{\mathrm{dR}}^+)$ be an S-B-pair. Let $\{w_1, \ldots, w_n\}$ be a $\mathbf{B}_{e,S}$-basis of W_e. Suppose that W admits a triangulation $\mathrm{Fil}_\bullet$. Let $(\delta_1, \ldots, \delta_n)$ be the corresponding triangulation parameters. Then for each $i = 1, \ldots, n$ there exists a continuous additive character ϵ_i of $K^\times$ with values in E such that $\delta_i = \delta_{i,z}(1 + Z\epsilon_i)$.

Suppose that W_z, the evaluation of W at z, is semistable, and let D_z be the filtered E-(φ, N)-module attached to W_z. Let $\mathcal{F}$ be the refinement of D_z corresponding to the induced triangulation of W_z, and let $\{e_{1,z}, e_{2,z}, \dots, e_{n,z}\}$ be a $(K_0 \otimes_{\mathbf{Q}_p} E)$-basis of D_z that is compatible with $\mathcal{F}$ i.e. $\mathcal{F}_i D = (K_0 \otimes_{\mathbf{Q}_p} E)e_{1,z} \oplus \cdots \oplus (K_0 \otimes_{\mathbf{Q}_p} E)e_{i,z}$. Let $\alpha_{i,z} \in E$ be such that $\varphi^{[K_0:\mathbf{Q}_p]}(e_{i,z}) = \alpha_{i,z} e_{i,z} \bmod \mathcal{F}_{i-1}$.

Let $x_{ij} \in \mathbf{B}_{\log,E}$ $(i, j = 1, \dots, n)$ be such that

$$e_{i,z} = x_{1i} w_{1,z} + \cdots + x_{ni} w_{n,z}. \tag{7.1}$$

Then $X = (x_{ij})$ is in $\mathrm{GL}_n(\mathbf{B}_{\log,E})$. Write the matrix of $\sigma \in G_K$ with respect to the basis $\{w_1, \dots, w_n\}$ by $(I_n + ZU_{e,\sigma})A_{e,\sigma}$. As $e_{1,z}, \dots, e_{n,z}$ are fixed by G_K, we have $X^{-1} A_{e,\sigma} \sigma(X) = I_n$ for all $\sigma \in G_K$.

For $i = 1, \dots, n$ put $e_i = x_{1i} w_1 + \cdots + x_{ni} w_n$. Then $\{e_1, \dots, e_n\}$ is a basis of $\mathbf{B}_{\log,S} \otimes_S W_e$ over $\mathbf{B}_{\log,S}$.

Lemma 7.1. *If T is the matrix of φ_{D_z} for the basis $\{e_{1,z}, \dots, e_{n,z}\}$, then T is also the matrix of $\varphi_{\mathbf{B}_{\log,S} \otimes_S W_e}$ for the basis $\{e_1, \dots, e_n\}$.*

Proof. The assertion follows from the definition of $\{e_1, \dots, e_n\}$ and the fact that $w_{1,z}, \dots, w_{n,z}, w_1, \dots, w_n$ are fixed by φ. $\square$

In Section 4.1 we attach to W an element $c_B(W)$ in $H^1_B(W_z^* \otimes W_z)$. Consider the composition

$$H^1_B(W_z^* \otimes W_z) \to H^1(G_K, W_{e,z}^* \otimes_{\mathbf{B}_{e,E}} W_{e,z}) \to H^1(G_K, \mathbf{B}_{\log,E} \otimes_E (D_z^* \otimes D_z)).$$

As the matrix of $\sigma \in G_K$ for the basis $\{e_1, \dots, e_n\}$ is $I_n + ZX^{-1}U_{e,\sigma}X$, from the discussion in Section 4 we see that the image of c_B in $H^1(G_K, \mathbf{B}_{\log,E} \otimes_E (D_z^* \otimes D_z))$ is the class of the 1-cocycle

$$(U_{e,\sigma})_{ij} w_{j,z}^* \otimes w_{i,z} = (X^{-1} U_{e,\sigma} X)_{ij} e_{j,z}^* \otimes e_{i,z}.$$

Let $\pi_{h\ell}$ be the projection

$$\mathbf{B}_{\log,E} \otimes_E (D_z^* \otimes D_z) \to \mathbf{B}_{\log,E}, \quad \sum_{j,i} b_{ji} e_{j,z}^* \otimes e_{i,z} \mapsto b_{h\ell}. \tag{7.2}$$

For $h = 1, \ldots, n$, let ϵ'_h be the additive character of G_K such that $\epsilon'_h \circ \mathrm{rec}_K(p) = 0$ and $\epsilon'_h \circ \mathrm{rec}_K|_{\mathfrak{o}_K^\times} = \epsilon_h|_{\mathfrak{o}_K^\times}$.

Theorem 7.2.

(a) *For any pair of integers (h, ℓ) such that $h < \ell$ we have $\pi_{h\ell}([c]) = 0$.*

(b) *For any $h = 1, \ldots, n$, $\pi_{h,h}([c])$ coincides with the image of $[\epsilon'_h]$ in $H^1(G_K, \mathbf{B}_{\log,E})$.*

We call (a) the *projection vanishing property.*

Proof. The filtered E-(φ, N)-module attached to $W_z/\mathrm{Fil}_{h-1}W_z$ is $D_z/\mathcal{F}_{h-1}D_z$. We denote the image of $e_{\ell,z}$ $(\ell \geq h)$ in $D_z/\mathcal{F}_{h-1}D_z$ again by $e_{\ell,z}$.

Let δ'_h be the character of G_K such that $\delta'_h = 1 + Z\epsilon'_h$. By Lemma 3.5 there exists an element

$$x \in (\mathbf{B}_{\max,E} \otimes_{\mathbf{B}_{e,E}} (W/\mathrm{Fil}_{h-1}W)_e)^{G_K=\delta'_h, \varphi^{[K_0:\mathbf{Q}_p]}=\alpha_{i,z}(1+Zv_p(\pi_K)\epsilon_h(p))}$$

whose image in $D_z/\mathcal{F}_{h-1}D_z$ is $e_{h,z}$. Write $x = e_h + Z\sum_{\ell \geq h} \lambda_\ell e_\ell$ with $\lambda_\ell \in \mathbf{B}_{\log,E}$.

As the matrix of $\sigma \in G_K$ for the basis $\{e_1, \ldots, e_n\}$ is $I_n + ZX^{-1}U_{e,\sigma}X$, we have

$$\begin{aligned}[1 + Z\epsilon'_h(\sigma)]x &= [1 + Z\epsilon'_h(\sigma)](e_h + Z\sum_{\ell \geq h} \lambda_\ell e_\ell) \\ &= \sigma(x) = e_h + Z\sum_{\ell \geq h}(X^{-1}U_{e,\sigma}X)_{\ell h} e_\ell + Z\sum_{\ell \geq h} \sigma(\lambda_\ell) e_\ell.\end{aligned}$$

For $\ell > h$, comparing the coefficients of e_ℓ we obtain

$$(X^{-1}U_{e,\sigma}X)_{\ell h} = (1 - \sigma)\lambda_\ell,$$

which shows (a). Similarly, comparing coefficients of e_h we obtain

$$(X^{-1}U_{e,\sigma}X)_{hh} - \epsilon'_h(\sigma) = (1 - \sigma)\lambda_h, \tag{7.3}$$

which implies (b). □

8. The proof of Theorem 1.2

We will need the following lemmas.

Lemma 8.1. *The inclusion* $E \hookrightarrow \mathbf{B}_{e,E}$ *induces an isomorphism*

$$H^1(G_K, E) \xrightarrow{\sim} \ker(N : H^1(G_K, \mathbf{B}_{e,E}) \to H^1(G_K, \mathbf{B}_{\log,E})).$$

Proof. The proof is identical to that of [13, Corollary 1.4]. □

Lemma 8.2. *The map* $N : \mathbf{B}_{\log,E}^{\varphi=p} \to \mathbf{B}_{\log,E}^{\varphi=1}$ *is surjective.*

Proof. The proof is identical to that of [13, Lemma 1.2]. □

For the proof of Theorem 1.2 we may assume that $S = E[Z]/(Z^2)$, and z is the closed point defined by the maximal ideal (Z). Let W be as in Theorem 1.2. Replacing W by the E-B-pair $\mathcal{F}_t W/\mathcal{F}_{s-1}W$ and replacing $\mathcal{F}$ by the induced refinement on $\mathcal{F}_t W/\mathcal{F}_{s-1}W$, we may assume that $s = 1$ and $t = n = \mathrm{rank}_{\mathbf{B}_{e,E}}(W_e)$. Let $e_{1,z}, e_{2,z}, \ldots, e_{n,z}$ be a $K_0 \otimes_{\mathbf{Q}_p} E$-basis of D_z such that

$$(K_0 \otimes_{\mathbf{Q}_p} E)e_{1,z} \bigoplus L \bigoplus (K_0 \otimes_{\mathbf{Q}_p} E)e_{n,z} \tag{8.1}$$

with $L = \oplus_{i=2}^{n-1}(K_0 \otimes_{\mathbf{Q}_p} E)e_{i,z}$ a perfect 1-decomposition of D_z for $\mathcal{F}$ (see §6.2 for the meaning of perfect decompositions). Let $e^*_{1,z}, e^*_{2,z}, \ldots, e^*_{n,z}$ be the dual basis of D^*_z over $K_0 \otimes_{\mathbf{Q}_p} E$.

Let D_1 be the quotient of D_z by L, D^*_2 the quotient of D^*_z by $\oplus_{i=2}^{n-1}(K_0 \otimes_{\mathbf{Q}_p} E)e^*_{i,z}$. Put $\mathscr{D} = D^*_2 \otimes D_1$. The images of $e_{1,z}$ and $e_{n,z}$ in D_1 are again denoted by $e_{1,z}$ and $e_{n,z}$, and the images of $e^*_{1,z}$ and $e^*_{n,z}$ in D^*_2 are again denoted by $e^*_{1,z}$ and $e^*_{n,z}$ respectively. So $e^*_{1,z} \otimes e_{1,z}$, $e^*_{1,z} \otimes e_{n,z}$, $e^*_{n,z} \otimes e_{1,z}$, $e^*_{n,z} \otimes e_{n,z}$ form a $K_0 \otimes_{\mathbf{Q}_p} E$-basis of $\mathscr{D}$. Let $\mathscr{D}_0$ be the filtered E-(φ, N)-submodule of $\mathscr{D}$ with a $K_0 \otimes_{\mathbf{Q}_p} E$-basis $\{e^*_{1,z} \otimes e_{1,z},\ e^*_{n,z} \otimes e_{1,z},\ e^*_{n,z} \otimes e_{n,z}\}$. Let $\mathscr{W} = (\mathscr{W}_e, \mathscr{W}^+_{\mathrm{dR}})$ (resp. $\mathscr{W}_0$) be the E-B-pair attached to $\mathscr{D}$ (resp. $\mathscr{D}_0$). Note that

$$\varphi^{[K_0:\mathbf{Q}_p]}(e^*_{1,z} \otimes e_{1,z}) = e^*_{1,z} \otimes e_{1,z},\ \varphi^{[K_0:\mathbf{Q}_p]}(e^*_{n,z} \otimes e_{n,z}) = e^*_{n,z} \otimes e_{n,z},$$

$$\varphi^{[K_0:\mathbf{Q}_p]}(e^*_{n,z} \otimes e_{1,z}) = p^{-[K_0:\mathbf{Q}_p]} e^*_{n,z} \otimes e_{1,z},$$

and

$$-N(e^*_{1,z} \otimes e_{1,z}) = N(e^*_{n,z} \otimes e_{n,z}) = e^*_{n,z} \otimes e_{1,z},\ N(e^*_{n,z} \otimes e_{1,z}) = 0$$

Let $\vec{\mathcal{L}}_{\mathcal{F}}=\vec{\mathcal{L}}_{\mathcal{F},s,t}$ be the $\mathcal{L}$-invariant defined in Definition 6.5. As (8.1) is a prefect decomposition, we have

$$\begin{aligned}\mathrm{Fil}^0(K\otimes_{K_0}\mathscr{D}) &= Ee^*_{n,z}\otimes(e_{n,z}+\vec{\mathcal{L}}_{\mathcal{F}}e_{1,z})\oplus E(e^*_{1,z}-\vec{\mathcal{L}}_{\mathcal{F}}e^*_{n,z})\otimes e_{1,z}\\ &\quad\oplus E(e^*_{1,z}-\vec{\mathcal{L}}_{\mathcal{F}}e^*_{n,z})\otimes(e_{n,z}+\vec{\mathcal{L}}_{\mathcal{F}}e_{1,z}).\end{aligned}$$

and

$$\mathrm{Fil}^0(K\otimes_{K_0}\mathscr{D}_0)=Ee^*_{n,z}\otimes(e_{n,z}+\vec{\mathcal{L}}_{\mathcal{F}}e_{1,z})\oplus E(e^*_{1,z}-\vec{\mathcal{L}}_{\mathcal{F}}e^*_{n,z})\otimes e_{1,z}.$$

Consider W as an infinitesimal deformation of W_z. In Section 4.2 we attach to this infinitesimal deformation an element $c_B(W)$ in $H^1_B(W_z^*\otimes W_z)$. Let $[c]$ be the image of $c_B(W)$ by the composition

$$H^1_B(W_z^*\otimes W_z)\to H^1(G_K,W^*_{e,z}\otimes_{\mathbf{B}_{e,E}}W_{e,z})\to H^1(G_K,\mathbf{B}_{\log,E}\otimes_{K_0\otimes_{\mathbf{Q}_p}E}(D_z^*\otimes D_z)),$$

and choose a 1-cocyle c representing $[c]$. Write c in the form

$$c=\sum_{j,i}c_{j,i}e^*_{j,z}\otimes e_{i,z}$$

with $c_{i,j}$ being a 1-cocycle of G_K with values in $\mathbf{B}_{\log,E}$. By the projection vanishing property (Theorem 7.2 (a)) we have $[c_{1,n}]=0$.

Lemma 8.3. *There exist* $\xi_1,\xi_n\in\mathbf{B}_{e,E}$ *and* $\gamma_{1,0},\gamma_{1,\tau},\gamma_{n,0},\gamma_{n,\tau}$ $(\tau\in\mathrm{Emb}(K,E))$ *such that*

$$c_{1,1}(\sigma)=(\sigma-1)\xi_1+\gamma_{1,0}\psi_0(\sigma)+\sum_{\tau\in\mathrm{Emb}(K,E)}\gamma_{1,\tau}\psi_\tau(\sigma)$$

and

$$c_{n,n}(\sigma)=(\sigma-1)\xi_n+\gamma_{n,0}\psi_0(\sigma)+\sum_{\tau\in\mathrm{Emb}(K,E)}\gamma_{n,\tau}\psi_\tau(\sigma)$$

for any $\sigma\in G_K$.

Proof. Let $\bar{c}_B$ be the image of c_B in $H^1_B(\mathscr{W})$, and let $\bar{c}$ be the 1-cocycle

$$\bar{c}=\sum_{j,i\in\{1,n\}}c_{j,i}e^*_{j,z}\otimes e_{i,z}$$

of G_K with values in $\mathbf{B}_{\log,E} \otimes_{K_0\otimes_{\mathbf{Q}_p}E} \mathscr{D}$. Then the image of $\bar{c}_B$ in

$$H^1(G_K, \mathbf{B}_{\log,E} \otimes_{K_0\otimes_{\mathbf{Q}_p}E} \mathscr{D})$$

is $[\bar{c}]$.

Note that $\bar{c}$ has values in $\mathscr{W}_e = (\mathbf{B}_{\log,E} \otimes_{K_0\otimes_{\mathbf{Q}_p}E} \mathscr{D})^{\varphi=1,N=0}$. So, in particular $c_{1,1}$ and $c_{n,n}$ have values in $\mathbf{B}_{e,E}$. As $N\bar{c}=0$, we have

$$N(c_{n,1}) = c_{1,1} - c_{n,n}, \quad -N(c_{1,1}) = N(c_{n,n}) = c_{1,n}.$$

As $[c_{1,n}]=0$, the statement follows from Lemma 8.1. □

Write $\delta_i = \delta_{i,z}(1+Z\epsilon_i)$. Let ϵ'_i be the additive character of G_K with values in E such that $\epsilon'_i \circ \mathrm{rec}_K(p) = 0$ and $\epsilon'_i \circ \mathrm{rec}_K|_{\mathfrak{o}_K^\times} = \epsilon_i|_{\mathfrak{o}_K^\times}$. Then there are $\epsilon_{i,\tau}$ ($\tau \in \mathrm{Emb}(K,E)$) such that $\epsilon'_i = \sum_{\tau\in\mathrm{Emb}(K,E)} \epsilon_{i,\tau}\psi_\tau$.

Lemma 8.4. *For $h=1,n$ we have $[K_0:\mathbf{Q}_p]\gamma_{h,0} = -v_p(\pi_K)\epsilon_h(p)$ and $\gamma_{h,\tau} = \epsilon_{h,\tau}$.*

Proof. We keep to use notations in the proof of Theorem 7.2. By (7.3) and Lemma 8.3 we have

$$\begin{aligned}(\sigma-1)(\lambda_h) &= -(X^{-1}U_\sigma X)_{hh} + \sum_{\tau\in\mathrm{Emb}(K,E)} \epsilon_{h,\tau}\psi_\tau(\sigma) \\ &= -(\sigma-1)\xi_h - \gamma_{h,0}\psi_0(\sigma) + \sum_{\tau\in\mathrm{Emb}(K,E)} (\epsilon_{h,\tau}-\gamma_{h,\tau})\psi_\tau(\sigma).\end{aligned}$$

Note that there exists $\omega \in \mathrm{W}(\overline{\mathbf{F}}_p)$ such that $\varphi(\omega)-\omega = 1$, where $\mathrm{W}(\overline{\mathbf{F}}_p)$ is the ring of Witt vectors with coefficients in the algebraic closure of $\mathbf{F}_p$. Then $(\sigma-1)\omega = \psi_0(\sigma)$. Hence

$$\sum_{\tau\in\mathrm{Emb}(K,E)} (\epsilon_{h,\tau}-\gamma_{h,\tau})\psi_\tau(\sigma) = (\sigma-1)(\lambda_h+\xi_h+\gamma_{h,0}\omega).$$

In other words, the cocycle $\sum_{\tau\in\mathrm{Emb}(K,E)} (\epsilon_{h,\tau}-\gamma_{h,\tau})\psi_\tau(\sigma)$ is de Rham. By Lemma 5.2 we have $\gamma_{h,\tau} = \epsilon_{h,\tau}$ and $\lambda_h+\xi_h+\gamma_{h,0}\omega \in E$. Then

$$(\varphi^{[K_0:\mathbf{Q}_p]}-1)\lambda_h = -(\varphi-1)\xi_h - \gamma_{h,0}(\varphi^{[K_0:\mathbf{Q}_p]}-1)\omega = -[K_0:\mathbf{Q}_p]\gamma_{h,0}. \quad (8.2)$$

By our choice of the basis $\{e_{1,z},\dots,e_{n,z}\}$, $Y_1=\oplus_{i=2}^n Ze_{i,z}$ is stable by φ. Put $Y_n=0$. Let x be as in the proof of Theorem 7.2. By Lemma 7.1 we have $\varphi^{[K_0:\mathbf{Q}_p]}e_{h,z}=\alpha_{h,z}e_{h,z}$. Thus for $h=1,n$ we have

$$\varphi^{[K_0:\mathbf{Q}_p]}(x)=(1+Z\varphi^{[K_0:\mathbf{Q}_p]}(\lambda_h))\alpha_{h,z}e_h \pmod{Y_h}.$$

On the other hand,

$$\begin{aligned}\varphi^{[K_0:\mathbf{Q}_p]}(x) &= (1+Zv_p(\pi_K)\epsilon_h(p))\alpha_{h,z}x\\ &= (1+Zv_p(\pi_K)\epsilon_h(p))\alpha_{h,z}(1+Z\lambda_h)e_h \pmod{Y_h}.\end{aligned}$$

Hence we obtain

$$(\varphi^{[K_0:\mathbf{Q}_p]}-1)\lambda_h=v_p(\pi_K)\epsilon_h(p). \tag{8.3}$$

By (8.2) and (8.3) we have

$$[K_0:\mathbf{Q}_p]\gamma_{h,0}=-(\varphi^{[K_0:\mathbf{Q}_p]}-1)\lambda_h=-v_p(\pi_K)\epsilon_h(p),$$

as wanted. □

By Lemma 8.2 there exists some $y\in\mathbf{B}_{\log,E}^{\varphi=p}$ such that $N(y)=\xi_1-\xi_n$. Let $\vec{c}'$ be the 1-cocycle of G_K with values in $\mathbf{B}_{\log,E}\otimes_{K_0\otimes_{\mathbf{Q}_p}E}\mathscr{D}_0$ such that

$$\vec{c}'=c'_{1,1}e^*_{1,z}\otimes e_{1,z}+c'_{n,n}e^*_{n,z}\otimes e_{n,z}+c'_{n,1}e^*_{n,z}\otimes e_{1,z}$$

with

$$c'_{1,1}=\gamma_{1,0}\psi_0+\sum_{\tau\in\mathrm{Emb}(K,E)}\gamma_{1,\tau}\psi_\tau,\quad c'_{n,n}=\gamma_{n,0}\psi_0+\sum_{\tau\in\mathrm{Emb}(K,E)}\gamma_{n,\tau}\psi_\tau$$

and

$$c'_{n,1}(\sigma)=c_{n,1}(\sigma)-(\sigma-1)y,\quad \sigma\in G_K.$$

It is easy to check that $\varphi(\vec{c}')=\vec{c}'$ and $N(\vec{c}')=0$. Hence $\vec{c}'$ is a 1-cocycle of G_K with values in $\mathbf{X}_{\log}(\mathscr{D}_0)$.

Proposition 8.5. *The image of $[\vec{c}']$ in $H^1(G_K,\mathbf{X}_{\log}(\mathscr{D}_0))$ belongs to the kernel of*

$$H^1(G_K,\mathbf{X}_{\log}(\mathscr{D}_0))\to H^1(G_K,\mathbf{X}_{\mathrm{dR}}(\mathscr{D}_0)).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} H^1(G_K, \mathbf{X}_{\log}(\mathscr{D}_0)) & \longrightarrow & H^1(G_K, \mathbf{X}_{\mathrm{dR}}(\mathscr{D}_0)) \\ \downarrow & & \downarrow \\ H^1(G_K, \mathbf{X}_{\log}(\mathscr{D})) & \longrightarrow & H^1(G_K, \mathbf{X}_{\mathrm{dR}}(\mathscr{D})). \end{array}$$

The right vertical arrow in the above diagram is injective (see [13, Corollary 2.4]). So we only need to show that the image of $[\vec{c}']$ in $H^1(G_K, \mathbf{X}_{\mathrm{dR}}(\mathscr{D}))$ is zero. Note that

$$[\vec{c}'] = [\vec{c}] - [c_{1,n} e^*_{1,z} \otimes e_{n,z}] = -[c_{1,n} e^*_{1,z} \otimes e_{n,z}]$$

in $H^1(G_K, \mathbf{X}_{\mathrm{dR}}(\mathscr{D}))$. As the image of $[c_{1,n}]$ in $H^1(G_K, \mathbf{B}_{\log,E})$ is zero, so is its image in $H^1(G_K, \mathbf{B}_{\mathrm{dR},E}/\mathrm{Fil}^f \mathbf{B}_{\mathrm{dR},E})$, where f is the smallest integer such that $e^*_{1,z} \otimes e_{n,z} \in \mathrm{Fil}^{-f} \mathscr{D}_K$. Hence, the image of $[\vec{c}']$ in $H^1(G_K, \mathbf{X}_{\mathrm{dR}}(\mathscr{D}))$ is zero. □

Now, applying Lemma 5.3 to $\mathscr{D}_0$ with $f_1 = e^*_{n,z} \otimes e_{1,z}$, $f_2 = e^*_{1,z} \otimes e_{1,z}$ and $f_3 = e^*_{n,z} \otimes e_{n,z}$, we get

$$\gamma_{n,0} - \gamma_{1,0} = \sum_{\tau \in \mathrm{Emb}(K,E)} \mathcal{L}_\tau (\gamma_{n,\tau} - \gamma_{1,\tau}).$$

Hence, by Lemma 8.4 we have

$$\frac{v_p(\pi_K)}{[K_0 : \mathbf{Q}_p]} (\epsilon_n(p) - \epsilon_1(p)) + \sum_{\tau \in \mathrm{Emb}(K,E)} \mathcal{L}_\tau (\epsilon_{n,\tau} - \epsilon_{1,\tau}) = 0.$$

As $\frac{\mathrm{d}\delta_h(p)}{\delta_h(p)} = \epsilon_h(p) \mathrm{d}Z$ and $\mathrm{d}\vec{w}(\epsilon_h) = (\epsilon_{h,\tau} \mathrm{d}Z)_\tau$, we obtain

$$\frac{1}{[K : \mathbf{Q}_p]} \left(\frac{\mathrm{d}\delta_n(p)}{\delta_n(p)} - \frac{\mathrm{d}\delta_1(p)}{\delta_1(p)} \right) + \vec{\mathcal{L}}_{\mathcal{F}} \cdot (\mathrm{d}\vec{w}(\delta_n) - \mathrm{d}\vec{w}(\delta_1)) = 0,$$

as desired. This finishes the proof of Theorem 1.2.

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