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AN EXPLICIT FORMULA FOR SZEGŐ KERENL ON HIGH-CODIMENSIONAL HEISENBERG GROUP

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Abstract

In this paper, we introduce high codimensional Heisenberg groups and we give an explicit formula for the associated Szegő kernel.

1. Introduction

Let $(X, T^{1,0}X)$ be a CR manifold of dimension $2n+1$, $n \geq 1$, and let $\Box_b^{(q)}$ be the Kohn Lalpacian acting on $(0,q)$ forms. The orthogonal projection $S^{(q)} : L^2_{(0,q)}(X) \to \operatorname{Ker} \Box_b^{(q)}$ onto $\operatorname{Ker} \Box_b^{(q)}$ is called the Szegő projection, while its distribution kernel $S^{(q)}(x,y)$ is called the Szegő kernel. The study of the Szegő projection and kernel is a classical and important subject in several complex variables and CR geometry.

For $p \in X$, let $\mathcal{L}_p$ be the Levi form of a X at p. Given q, $0 \leq q \leq n$, the Levi form is said to satisfy condition $Y(q)$ at $p \in X$ if for any $|J| = q$, $J = (j_1, j_2, \ldots, j_q)$, $1 \leq j_1 < j_2 < \cdots < j_q \leq n-1$, we have

$$\left| \sum_{j \notin J} \lambda_j - \sum_{j \in J} \lambda_j \right| < \sum_{j=1}^{n-1} |\lambda_j| ,$$

where λ_j, $j = 1, \ldots, (n-1)$, are the eigenvalues of $\mathcal{L}_p$. If the Levi form is non-degenerate at p, then $Y(q)$ holds at p if and only if $q \neq n_-, n_+$, where (n_-, n_+) is the signature of $\mathcal{L}_p$, i.e. the number of negative eigenvalues of

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L_p is n_- and $n_+ + n_- = n-1$. When the Levi form satisfies condition $Y(q)$ on X, then Kohn's subelliptic estimates with loss of one derivative for the solutions of $\Box_b^{(q)} u = f$ hold, cf. [2, 14], and hence $S^{(q)}$ is a smoothing operator. When condition $Y(q)$ fails, one is interested in the singularities of the Szegő kernel $\Pi^{(q)}(x, y)$.

A very important case is when X is a compact strictly pseudoconvex CR manifold (in this case $Y(0)$ fails). When X is compact, strongly pseudoconvex and $\Box_b^{(0)}$ has L^2 closed range, Boutet de Monvel-Sjöstrand [1] showed that $S^{(0)}(x, y)$ is a complex Fourier integral operator with complex phase. In particular, $S^{(0)}(x, y)$ is smooth outside the diagonal of $X \times X$ and there is a precise description of the singularity on the diagonal $x = y$, where $S^{(0)}(x, x)$ has a certain asymptotic expansion. Hsiao [6] showed that if X is compact, the Levi form is non-degenerate, $Y(q)$ fails and $\Box_b^{(q)}$ has L^2 closed range for some $q \in \{0, 1, \ldots, n-1\}$, then $S^{(q)}(x, y)$ is a complex Fourier integral operator.

When the Levi form is degenerate, Hsiao and Marinescu [7] obtained Szegő kernel asymptotic expansions on the non-degenerate part under local closed range assumption. Recently, Hsiao and Savale [13] established pointwise Szegő kernel asymptotic expansions on three dimensional weakly pseudoconvex CR manifolds of finite type under the assumption that $\overline{\partial}_b$ has L^2 closed range.

The description of the Szegő kernel had profound impact in Several complex variables, CR and complex geometry [7, 8, 9, 10, 11, 12, 5, 18], to quote just a few. These ideas also partly motivated the introduction of alternative approaches, see [15, 16, 17]. Recently, Fritsch, Herrmann and Hsiao [3] considered CR manifolds of high codimension. They obtained G-equivatiant CR embedding theorem for CR manifolds of high codimension and CR orbifold version of Boutet de Monvel's embedding theorem. Thus, it is very natural and interesting to study Szegő kernels for CR manifolds of high codimension. Moreover, Fritsch, Herrmann and Hsiao also showed that any CR orbifold of hypersurface type comes from the quotient X/G, where X is a CR manifold and G is a compact CR Lie group acting on X. Hence, the study of Szegő kernels for CR manifolds of high codimension could has applications in CR orbifold geometry. In this paper, we consider a high codimensional Heisenberg group, we give explicit formula for the associated Szegő kernel. The main inspiration of this work comes from [4].

We now formulate our main results. Consider $\mathbb{H} = \mathbb{H}^{n,d} = \mathbb{C}^n \times \mathbb{R}^d$. Let $(z,t) = (x+iy,t)$ be coordinats of $\mathbb{H}$, where $z = (z_1,\ldots,z_n) = x+iy = (x_1+iy_1,\ldots,x_n+iy_n)$, be coordinates of $\mathbb{C}^n$, $t = (t_1,\ldots,t_d)$ be coordinates of $\mathbb{R}^d$. Let

$$\varphi = (\varphi_1,\ldots,\varphi_d) \in C^\infty(\mathbb{C}^n,\mathbb{R}^d).$$

Consider

$$T^{1,0}\mathbb{H} := \operatorname{span}\left\{ L_j := \frac{\partial}{\partial z_j} - i\sum_{i=1}^{d} \frac{\partial \varphi}{\partial z_j}\frac{\partial}{\partial t_i}, j = 1,\ldots,d \right\}.$$

Then, $(\mathbb{H}, T^{1,0}\mathbb{H})$ is a CR manifold of codimension d (see Section 2.2). Let

$$d\mathfrak{m} := dx_1dy_1\cdots dx_ndy_ndt_1\cdots dt_d$$

be the flat Lebesgue measure on $\mathbb{H}$ and let

$$\mathcal{H}_b := \left\{ u \in L^2(\mathbb{H});\, \overline{L}_j u = 0, j = 1,\ldots,n \right\}.$$

Let $\langle\cdot\,\cdot\rangle$ be the L^2 inner product of $C_c^\infty(\mathbb{H})$ induced by $d\mathfrak{m}$ and let $L^2(\mathbb{H})$ be the completion of $C_c^\infty(\mathbb{H})$ with respect to $<,>$. Let

$$P : L^2 \to \mathcal{H}_b$$

be the orthonormal projection from $L^2(\mathbb{H})$ onto $\mathcal{H}_b$ with respect to $\langle\cdot\,\cdot\rangle$. We also write (w,θ) to denote (z,t). Let $S = S(w,z,\theta,t) \in \mathscr{D}'(\mathbb{H}\times\mathbb{H})$ denote the distributional kernel of P. We call S the Szegő kernel. Formally,

$$< Pu, \bar{v} >= \int u(z,t)v(w,\theta)S(w,z,\theta,t)d\mathfrak{m}, \quad u,v \in L^2(\mathbb{H}).$$

The goal of this work is to give an explicit formula for S when φ is quadratic. Fix

$$\lambda_j = (\lambda_{1j},\ldots,\lambda_{dj}) \in \mathbb{R}^d, \quad j = 1,\ldots,n.$$

In this paper, we always tale φ of the form:

$$\varphi = \sum_{j=1}^{n} |z_j|^2 \lambda_j \in C^\infty(\mathbb{C}^n,\mathbb{R}^d).$$

Put

$$A_\lambda = \{\eta \in \mathbb{R}^d \mid \eta \cdot \lambda_j > 0, \quad \text{for all } j = 1, 2, \ldots, n\},$$
$$B_\lambda = A_\lambda \cap \{v \in \mathbb{R}^d \mid |v| = 1\}.$$

The main result of this work is as follows:

Theorem 1. *With the notations used above, we have*

$$S(w, z, \theta, t) = \frac{2^{n-d}}{\pi^{n+d}} \int_{B_\lambda} \frac{(n+d-1)!\Pi(v \cdot \lambda_j)}{(iv \cdot [(t-\theta) - i\Phi(w,z)])^{n+d}} d\Omega(v) \tag{1}$$

in sense of Fourier integral, where

$$\Phi(w, z) := \sum_{j=1}^{n} \lambda_j (|w_j - z_j|^2 + w_j \bar{z}_j - \bar{w}_j z_j)$$

and $d\Omega(v)$ *is Euclidean surface measure of the sphere.*

The integral (1) is defined in the sense of oscillatory integral (see also Section 5).

Remark 1. From Theorem 1, we see that if $B_\lambda \neq \emptyset$, then $S(w, z, \theta, t)$ has singularities and hence the space $\mathcal{H}_b$ is non-trivial. On the other hand, if $B_\lambda = \emptyset$, then $\mathcal{H}_b$ is trivial.

Remark 2. Our method is inspired on $d = 1$, in [4]. Using distribution theory we get $\mathcal{F} : (L^2, \mathcal{H}_b) \to (L^2_{\eta\cdot\varphi}, H_{\eta\cdot\varphi})$ is an isometry isomorphism, combine the uniqueness of orthonormal projection, we get $P = \mathcal{F}^{-1} K \mathcal{F}$, therefore we can use knowing K_η to calculus Szegő kernel.

$$\begin{array}{ccc} L^2 & \xrightarrow{\mathcal{F}} & L^2_{\eta\cdot\varphi} \\ \big\downarrow P & & \big\downarrow K \\ \mathcal{H}_b & \xrightarrow{\mathcal{F}^{-1}} & H_{\eta\cdot\varphi} \end{array} \tag{2}$$

2. Preliminaries

2.1. Notations

We shall use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. Let $m \in \mathbb{N}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_m$ its norm and by $l(\alpha) = m$ its length. α is strictly increasing if $\alpha_1 < \alpha_2 < \cdots < \alpha_m$.

Let $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. We write

$$
\begin{aligned}
&z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1} \ldots \overline{z}_n^{\alpha_n}, \\
&\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big), \quad j = 1, \ldots, n.
\end{aligned}
$$

For $j, s \in \mathbb{Z}$, set $\delta_{j,s} = 1$ if $j = s$, $\delta_{j,s} = 0$ if $j \neq s$.

Let M be a C^∞ paracompact manifold. We let TM and T^*M denote the tangent bundle of M and the cotangent bundle of M, respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$, respectively. Write $\langle \cdot\,, \cdot \rangle$ to denote the pointwise standard pairing between TM and T^*M. We extend $\langle \cdot\,, \cdot \rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let G be a C^∞ vector bundle over M. The fiber of G at $x \in M$ will be denoted by G_x. Let E be a vector bundle over a C^∞ paracompact manifold M_1. We write $G \boxtimes E^*$ to denote the vector bundle over $M \times M_1$ with fiber over $(x, y) \in M \times M_1$ consisting of the linear maps from E_y to G_x. Let $Y \subset M$ be an open set. From now on, the spaces of distribution sections of G over Y and smooth sections of G over Y will be denoted by $\mathscr{D}'(Y, G)$ and $C^\infty(Y, G)$, respectively.

Let G and E be C^∞ vector bundles over paracompact orientable C^∞ manifolds M and M_1, respectively, equipped with smooth densities of integration. If $A : C_c^\infty(M_1, E) \to \mathscr{D}'(M, G)$ is continuous, we write $A(x, y)$ to denote the distribution kernel of A.

Let $H(x, y) \in \mathscr{D}'(M \times M_1, G \boxtimes E^*)$. We write H to denote the unique continuous operator $C_c^\infty(M_1, E) \to \mathscr{D}'(M, G)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Notations of differential:

(i) If $(x, y) \in \mathbb{R}^{n+m}$. The gradient operator is denoted by ∇, and $\nabla_x u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$, $\nabla_y u = (u_{y_1}, u_{y_2}, \ldots, u_{y_m})$.
(ii) The high order differential operator is denoted by $\nabla^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ with order $|\alpha|$ for all $\alpha \in \mathbb{N}_0^n$, where the absolute value of multi-index is $|\alpha| = \Sigma \alpha_i$.
(iii) Let X_i be vector fields for $i = 1, \ldots, n$. Then $\nabla_X u = (X_1 u, \ldots, X_n u)$.
(iv) The high order differential of vector field X is denoted by $\nabla_X^\alpha = X_i^{\alpha_1} \cdots X_n^{\alpha_n}$.
(v) Let $F = (F_1,\ F_2, \ldots,\ F_m) : \mathbb{C}^n \to \mathbb{C}^m$, $\nabla_X F$ denotes the $m \times n$ matrix with entries $(\nabla_X F)_{ij} = X_j F_i$.

Notations of spaces:

Let $X = \mathbb{C}^n$, $\mathbb{H}$ or other Euclidean subspace.

(i) Let $\mathfrak{m}$ denotes the Lebesgue's measure. For $p \geq 1$, the L^p space is denoted by $L^p(X)$, consisting of all u with $\int_X |u|^p d\mathfrak{m} < \infty$.
(ii) The space $L^p_\rho(X)$ denote the L^p space with respect to weight ρ that consisting of measurable u with $\|u\|^p_{L^p_\rho(X)} := \int_X |u|^p e^{-2\rho} d\mathfrak{m} < \infty$.
(iii) Let H be a inner product space, and its inner product and norm are denoted by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$, in particular, $\| \cdot \| := \| \cdot \|_{L^2}$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2}$.
(iv) The test function space is collection of smooth functions with compact support and denoted by $C_c^\infty(X)$.
(v) The Schwartz space on X is denoted by $\mathscr{S}(X)$, which is collection of smooth functions f with $\sup |p(z,t) \nabla^\alpha f(z,t)| < \infty$ for all multi-index α and polynomial p.

2.2. Cauchy-Riemann manifold

Definition 1 (Cauchy-Riemann manifold)**.** Let X be a smooth manifold with dimension $2n + d$, $n, d \geq 1$, and let $T^{1,0}(X)$ be a subbundle of $\mathbb{C}T(X)$. The pair $(X, T^{1,0}(X))$ is said to be a CR manifold or a CR structure if

(i) $\dim_{\mathbb{C}} T_p^{1,0}(X) = n$, for every $p \in X$.
(ii) $T^{1,0}(X) \cap T^{0,1}(X) = \{0\}$. $(T^{0,1}(X) := \overline{T^{1,0}(X)})$.
(iii) For any tangent vector fields $V,\ W \in C^\infty(X, T^{1,0}(X))$, we have the commutator $[V, W]$ is also in $C^\infty(X, T^{1,0}(X))$.

In this case we say that $(X, T^{1,0}(X))$ or X is a n-dimensional CR manifold with codimension d, or simplify, n-dimensional CR manifold if $d = 1$; if $d > 1$ we say that X is high codimensional CR manifold.

For the convenience for the reader, we recall some basis facts for CR manifolds of codimension one. Let $(X, T^{1,0}(X))$ be an orientable not necessarily compact, paracompact CR manifold of dimension $2n+1$ of codimension one. Fix a smooth Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ so that $\langle\, u \,|\, v \,\rangle$ is real if u, v are real tangent vectors and $T^{1,0}X$ is orthogonal to $T^{0,1}X := \overline{T^{1,0}X}$. Then locally there is a real vector field T of length one which is pointwise orthogonal to $T^{1,0}(X) \oplus T^{0,1}(X)$. T is unique up to the choice of sign. For $v \in \mathbb{C}TX$,

Locally there exists an orthonormal frame $e_1, \ldots, e_n$ of the bundle $T^{*1,0}X$. The real $(2n)$ form $\omega = i^n e_1 \wedge \overline{e}_1 \wedge \ldots \wedge e_n \wedge \overline{e}_n$ is independent of the choice of the orthonormal frame. Thus ω is globally defined. Locally there exists a real 1-form ω_0 of length one which is orthogonal to $T^{*1,0}X \oplus T^{*0,1}X$. The form ω_0 is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$ form Θ on X. Thus, ω_0 can be specified uniquely by requiring that $\omega \wedge \omega_0 = f\Theta$, where f is a positive function. Therefore ω_0, so chosen, is globally defined. We call ω_0 the uniquely determined global real 1-form. We take a vector field T so that

$$|T| = 1\,, \quad \langle\, T \,, \omega_0 \,\rangle = -1\,. \tag{3}$$

Therefore T is uniquely determined. We call T the uniquely determined global real vector field. We have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0;\, \lambda \in \mathbb{C}\}\,, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T;\, \lambda \in \mathbb{C}\}\,. \end{aligned} \tag{4}$$

For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T^{1,0}_p X$ defined as follows. For any $Z,\ W \in T^{1,0}_p X$, pick $\mathcal{Z}, \mathcal{W} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{Z}(p) = Z$, $\mathcal{W}(p) = W$. Set

$$\mathcal{L}_p(Z, \overline{W}) = \frac{1}{2i}\big\langle \big[\mathcal{Z}\ , \overline{\mathcal{W}}\,\big](p)\ , \omega_0(p) \big\rangle\,, \tag{5}$$

where $\left[\mathcal{Z}\ ,\overline{\mathcal{W}}\right]=\mathcal{Z}\ \overline{\mathcal{W}}-\overline{\mathcal{W}}\ \mathcal{Z}$ denotes the commutator of $\mathcal{Z}$ and $\overline{\mathcal{W}}$. Note that $\mathcal{L}_p$ does not depend of the choices of $\mathcal{Z}$ and $\mathcal{W}$.

We now come back to our situation. We introduce some notations.

Notation 1.

(i) *The points in* $\mathbb{C}^n\times\mathbb{R}^d$ *will be deboted by* $(z,t)=(x+iy,t)=(x,y,t)$, (w,η), (w,θ), *where* $z\in\mathbb{C}^n$, $t,\eta,\theta\in\mathbb{R}^d$ $x,y\in\mathbb{R}^n$ *and* $w\in\mathbb{C}^n$.

(ii) *The vector fields* Z_i *and* T_i *denote* $\frac{\partial}{\partial z_i}=\frac{1}{2}(\frac{\partial}{\partial x_i}-i\frac{\partial}{\partial y_i})$ *and* $-i\frac{\partial}{\partial t_i}$, *respectively.*

(iii) *Let* $\varphi:\mathbb{C}^n\to\mathbb{R}^d$. *Define vector field* L_i, *by* $L_i=Z_i+(Z_i\varphi)\cdot\nabla_T$.

Definition 2. Let $\mathbb{H}=\mathbb{H}^{n,d}=\mathbb{C}^n\times\mathbb{R}^d$ be the Heisenberg group, and let $T^{1,0}\mathbb{H}=span_{\mathbb{C}}\{L_i\}$, then $(\mathbb{H},\ T^{1,0}\mathbb{H})$ is a high codimensional CR manifold. Let $\lambda_j=(\lambda_{1j},\ldots,\lambda_{dj})\in\mathbb{R}^d$ for $j=1,\ldots,n$. Define $\varphi:\mathbb{C}^n\to\mathbb{R}^d$ by

$$\varphi(z)=\Sigma|z_j|^2\lambda_j, \tag{6}$$

in this paper, the CR structure of Heisenberg group $\mathbb{H}$ always defined with respect to φ.

The space $L^2(\mathbb{H})$ is defined in Notation of space (i). Moreover, for $u\in L^2(\mathbb{H})$,

$$\int_{\mathbb{H}}|u|^2d\mathbf{m}(z,t)=\int_{\mathbb{H}}|u|^2d\mathbf{m}(x,y,t)=\int_{\mathbb{H}}|u|^2d\mathbf{m}(x)d\mathbf{m}(y)d\mathbf{m}(t). \tag{7}$$

Define the Hermitian metric on $\mathbb{C}T\mathbb{H}$ such that

$$T\perp T^{1,0}\mathbb{H}\perp T^{0,1}\mathbb{H}$$

$$\langle L_i,L_j\rangle=\langle\bar{L}_i,\bar{L}_j\rangle:=\frac{1}{2}\delta_{i,j}.$$

The Hermitian metric on $\mathbb{C}T^*\mathbb{H}$ is induced by duality of $\mathbb{C}T\mathbb{H}$ as follows: For a given point $p\in\mathbb{H}$ an anti-linear map $\Gamma:\mathbb{C}T_p\mathbb{H}\to\mathbb{C}T_p^*\mathbb{H}$ is defined by

$$(\Gamma v)(u)=\langle u,v\rangle,\ \text{for}\ u,v\in\mathbb{C}T_p\mathbb{H}. \tag{8}$$

For $\omega,\ \nu\in\mathbb{C}T_p^*\mathbb{H}$

$$\langle\omega,\nu\rangle:=\langle\Gamma^{-1}\omega,\Gamma^{-1}\nu\rangle. \tag{9}$$

For analogous, we define an inner product on $\Lambda^p(\mathbb{C}T^*\mathbb{H})$ by

$$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_p, v_1 \wedge v_2 \wedge \cdots \wedge v_p\rangle = \det(\langle u_i, v_j\rangle)_{i,j} \tag{10}$$

where u_i and v_i are 1-forms, $i = 1, \dots, p$.
Recall the vector field $T_i := -i\frac{\partial}{\partial t_i}$ and the complex tangent bundle

$$\mathbb{C}T\mathbb{H} = T^{1,0}\mathbb{H} \oplus T^{0,1}\mathbb{H} \oplus \mathbb{C}T, \tag{11}$$

where $\mathbb{C}T := span_{\mathbb{C}}\{T_i\}$.
Define $\Lambda^{1,0}T^*\mathbb{H} := \Gamma(T^{1,0}\mathbb{H})$, $\Lambda^{0,1}T^*\mathbb{H} := \Gamma(T^{0,1}\mathbb{H})$ and the space of (p,q)-forms $\Lambda^{p,q}T^*\mathbb{H} := \Lambda^p(\Lambda^{1,0}T^*\mathbb{H}) \wedge (\Lambda^q(\Lambda^{0,q}T^*\mathbb{H}))$ (or $T^{*p,q}\mathbb{H}$).

Put

$$\omega_i = dt_i + \sum_j iZ_j\varphi_i dz_j - i\bar{Z}_j\varphi_i d\bar{z}_j, \tag{12}$$

then ω_i are orthogonal to $\Lambda^{1,0}T^*\mathbb{H} \oplus \Lambda^{0,1}T^*\mathbb{H}$, and hence

$$\mathbb{C}T^*\mathbb{H} = \Lambda^{1,0}T^*\mathbb{H} \oplus \Lambda^{0,1}T^*\mathbb{H} \oplus span_{\mathbb{C}}\{\omega_i\}. \tag{13}$$

Let dL_j and $d\bar{L}_j$ be the dual vectors of L_j and $\overline{L}_j$ respectively, $j = 1, \dots, n$. The volume form dv is define by

$$dv = 2^{-n}|dL_1 \wedge \cdots dL_n \wedge d\bar{L}_1 \wedge \cdots \wedge d\bar{L}_n \wedge d\omega_1 \wedge \cdots d\omega_d| = d\mathfrak{m}. \tag{14}$$

2.3. Cauchy-Riemann complex

There are many way to introduce the Cauchy-Riemann operator, in this paper, we will use the distributional way to introduce Cauchy-Riemann operator, for the reason that firstly, we define the test function space Ω^q and $\Omega^{0,q}$ for smooth q-forms and $(0,q)$-forms, that is

$$\Omega^q := C_c^\infty(\mathbb{H}, \Lambda^q T^*\mathbb{H}) \text{ and } \Omega^{0,q} := C_c^\infty(\mathbb{H}, \Lambda^{0,q}T^*\mathbb{H}). \tag{15}$$

In fact we have the cochain complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n+1} \longrightarrow 0. \tag{16}$$

consider the orthogonal projections $\pi^q : \Lambda^q \mathbb{C}T^*\mathbb{H} \to \Lambda^{0,q}T^*\mathbb{H}$, we have the following property.

Proposition 1. *$\pi^{q+2} \circ d^2 = \pi^{q+1}(d)\pi^{q+1}(d)$, in particular $(\pi^{q+1} \circ d)^2 = 0$, for all $q \geq 0$.*

Proof. This a consequence of the observation $\pi^{q+1} \circ d = \pi^{q+1} \circ d \circ \pi^q$ for all $q \geq 0$. □

Proposition 1 leads that the projections π^q $q = 0, 1, \ldots, 2n+1$ induce a functor $\pi : (\Omega^{\cdot}, d^{\cdot}) \to (\Omega^{0,\cdot}, \pi(d)^{\cdot})$.

Definition 3. The tangential Cauchy-Riemann operator is defined by

$$\bar{\partial}_b := \pi \circ d, \tag{17}$$

that is

$$\bar{\partial}_b u = \pi^{q+1} \circ du,\ u \in \Lambda^{0,q}T^*\mathbb{H}, \tag{18}$$

Proposition 1 shows $\bar{\partial}_b$ is closed.

Since $\dim_{\mathbb{C}} T^{1,0}\mathbb{H}$ is n, we have the diagram of cochain complexes

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \xrightarrow{d} & \cdots \\ & & \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & \downarrow \pi & & \\ 0 & \longrightarrow & \Omega^{0,0} & \xrightarrow{\bar{\partial}_b} & \Omega^{0,1} & \xrightarrow{\bar{\partial}_b} & \cdots & \xrightarrow{\bar{\partial}_b} & \Omega^{0,n} & \longrightarrow & 0 & & \end{array} \tag{19}$$

Now, we want to extend $\bar{\partial}_b$ form $\Omega^{0,q}$ to L^2 space, by using the following proposition.

Proposition 2. *Let $A : C_c^\infty \to C_c^\infty$ be a linear operator, then A can be extended a closed linear operator on L^2.*

Proof. Let A^* be the adjoint operator with respect to to L^2-inner product, then A^{**} is a closed extension of A on L^2, and

$$Dom(A^{**}) = \{u \in L^2 | v \mapsto \langle u, A^*v\rangle \text{ is continuous for all } v \in C_c^\infty\}, \tag{20}$$

clearly, $Au = A^{**}u$ if $u \in C_c^\infty$, we still denote A^{**} by A, and a equivalent expression in (20) is

$$Dom(A) = \{u \in L^2 | Au \in L^2\}. \tag{21}$$

□

Denote L^2_q by the L^2-norm space for $(0,q)$-forms on $\mathbb{H}$, extend $\bar{\partial}_b$ to L^2_q in the sense of the distribution $q = 0, 1, \ldots, n$, and let

$$Dom(\bar{\partial}_b) = \{u \in \oplus \Lambda^{0,q} T^* \mathbb{H} | u \in L^2_q \text{ and } \bar{\partial}_b u \in L^2_{q+1}\}, \tag{22}$$

let $Dom_q = L^2_q \cap Dom(\bar{\partial}_b)$ and that we have the cochain complex

$$0 \longrightarrow Dom_0 \xrightarrow{\bar{\partial}_b} Dom_1 \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} Dom_n \xrightarrow{\bar{\partial}_b} 0 \tag{23}$$

Similarly, define $\bar{\partial}^*_b$ by the adjoint operator of $\bar{\partial}_b$ such that

$$\langle \bar{\partial}^*_b u, v \rangle = \langle u, \bar{\partial}_b v \rangle \quad \text{for } u,\ v \in \Omega^{0,q}, \tag{24}$$

extend $\bar{\partial}^*_b$ to L^2 by the same way, and let

$$Dom(\bar{\partial}^*_b) = \{u \in \oplus \Lambda^{0,q} T^* \mathbb{H} | u \in L^2_q \text{ and } \bar{\partial}^*_b u \in L^2_{q-1}\}, \tag{25}$$

let $Dom^*_q = L^2_q \cap Dom(\bar{\partial}^*_b)$ and that we have the chain complex

$$0 \longrightarrow Dom^*_n \xrightarrow{\bar{\partial}_b} Dom^*_{n-1} \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} Dom^*_0 \xrightarrow{\bar{\partial}_b} 0 \tag{26}$$

2.4. The Kohn Laplacain for functions

Kohn laplacain is defined by the tangential Cauchy-Riemann operator,

$$\Box_b = \bar{\partial}_b \bar{\partial}^*_b + \bar{\partial}^*_b \bar{\partial}_b \tag{27}$$

$$Dom(\Box_b) = \{u \in \oplus_{q=0}^{n} \Lambda^{0,q} T^* \mathbb{H} | u \in Dom_q \cap Dom^*_q \text{ and } \Box_b u \in L^2_q\}. \tag{28}$$

In this paper, we focus on the $(0,0)$-form, that is function case, we will straightforwardly define $\Box_b = \Box_b^{(0)}$ on $(0,0)$-form, and that

$$\Box_b = \bar{\partial}^*_b \bar{\partial}_b = -2\Sigma L_i \bar{L}_i. \tag{29}$$

Notation 2.

(i) *The domain of* $\Box_b$ *is therefore that* $Dom(\Box_b) = \{u \in \cap Dom(\bar{L}_i)\ | L_i \bar{L}_i u \in L^2$ *for all* $i\}$.

(ii) *The kernel of Kohn Laplacian* $\Box_b$ *is denoted by* $\mathcal{H}_b := \ker \Box_b$*, since* $\langle \Box_b u, u\rangle = \Sigma \|\bar{L}_i u\|^2$*, we have* $\mathcal{H}_b = \cap \ker \bar{L}_i$ *is a closed subspace.*

(iii) *The Szegő projection* $P : L^2 \to \mathcal{H}_b$ *is the orthonormal projection from* L^2 *onto* $\mathcal{H}_b$*.*

(iv) *The Szegő kernel* S *is the distributional kernel of* $(u, v) \mapsto \langle Pu, \bar{v}\rangle$*, that is* $S \in \mathscr{D}'(\mathbb{H} \times \mathbb{H}, L^2(\mathbb{H}) \boxtimes L^2(\mathbb{H}))$ *and*

$$\langle Pu, \bar{v}\rangle = \int u(z, t)v(w, \theta)S(w, z, \theta, t)d\mathbf{m}.$$

Note that the kernel may not be represented as an integration.

3. Partial Fourier Transformation

3.1. Definition and Basic Propositions

Let $u \in L^2$. We are going to define the (inverse) partial Fourier transform with respect to real variable t. Choose $\chi(\theta) \in C_c^\infty(\mathbb{R}^d)$ so that $\chi(\theta) = 1$ when $|\theta| < 1$ and $\chi(\theta) = 0$ when $|\theta| > 2$ and set $\chi_j(\theta) = \chi(\theta/j)$, $j \in \mathbb{N}$. Let

$$\hat{u}_j(z, \eta) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(z, \theta)\chi_j(\theta)e^{-i\theta\eta}d\theta \in C^\infty(\mathbb{H}),\ j = 1, 2, \ldots. \tag{30}$$

From Parseval's formula, we have

$$\begin{aligned}&\int_{\mathbb{H}} |\hat{u}_j(z, \eta) - \hat{v}_k(z, \eta)|^2 \, d\mathbf{m} \\ &= \int_{\mathbb{H}} |u(z, \theta)|^2 \, |\chi_j(\theta) - \chi_k(\theta)|^2 \, d\mathbf{m} \to 0,\ j, k \to \infty.\end{aligned}$$

Thus, there is $\hat{u}(z, \eta) \in L^2$ such tht $\hat{u}_j(z, \eta) \to \hat{u}(z, \eta)$ in $L^2_{(0,q)}(H_n, \Phi_0)$. We call $\hat{u}(z, \eta)$ the partial Fourier transform of $u(z, \theta)$ with respect to θ. Similarly, we can define $\check{u}(z, \eta)$ the inverse Fourier transform of $u(z, \theta)$ with respect to θ. We write

$$\begin{aligned}\hat{u}(z, \eta) :=& \frac{1}{(2\pi)^{\frac{d}{2}}} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} \chi(\epsilon\theta)u(z, \theta)e^{-i\eta\cdot\theta}d\mathbf{m}(\theta) \\ \check{u}(z, \eta) :=& \frac{1}{(2\pi)^{\frac{d}{2}}} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} \chi(\epsilon\theta)u(z, \theta)e^{i\eta\cdot\theta}d\mathbf{m}(\theta)\end{aligned}. \tag{31}$$

Proposition 3. *The partial Fourier transform is an well defined isomorphism.*

Proof. Let $u \in \mathscr{S}(\mathbb{H})$, be a Schwartz function, by Plancherel theorem with respect to real variable t that

$$\|\hat{u}\|_{L^2}^2 = \int_{\mathbb{C}^n} \int_{\mathbb{R}^d} |\hat{u}(z,\eta)|^2 d\mathbf{m}(\eta) d\mathbf{m}(x,y) = \|u\|_{L^2}^2. \tag{32}$$

On the other hand $\hat{\cdot} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is an isomorphism (with respect to L^2−norm), so $(\hat{u}(z,\cdot))\check{} = u(z,\cdot)$ and $\hat{u}(z,\eta) \in \mathscr{S}(\mathbb{H})$ by symmetry, we have $\hat{\cdot} : \mathscr{S}(\mathbb{H}) \to \mathscr{S}(\mathbb{H})$ is an isometric isomorphism that extend to an isometric automorphism on $L^2(\mathbb{H})$. □

Proposition 4. *The relation of differential with respect to real variable t and partial Fourier transform is the same as differential and Fourier transform, and given by*

(i) $(\nabla_T^\alpha u(z,\cdot))\hat{}(\eta) = (\eta)^\alpha \hat{u}(z,\eta)$,

(ii) $(\nabla_T^\alpha u(z,\cdot))\check{}(\eta) = (-\eta)^\alpha \check{u}(z,\eta)$,

(iii) $\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle = \langle \check{u}, \check{v}\rangle$ *(Plancherel theorem),*

for all $\alpha \in \mathbb{N}_0^d$, where $\eta^\alpha = \eta_1^{\alpha_2} \cdots \eta_d^{\alpha_d}$, and $u, v \in \mathscr{S}(\mathbb{H})$. In particular, (iii) *hold for all $u,\ v\ \in L^2(\mathbb{H})$.*

3.2. The Space with Weight

Notation 3. *Let $\rho : X \to \mathbb{R}$ be a weight function, $X = \mathbb{C}^n$, $\mathbb{H}$ or other Euclidean subspace.*

(i) *The space $L^2_\rho(X)$ denotes the L^2 space with respect to the weight ρ consisting of measurable u with $\|u\|^2_{L^2_\rho(X)} := \int_X |u|^2 e^{-2\rho} d\mathbf{m} < \infty$.*

(ii) *The closed subspace consisting of all weakly* **holomorphic** *functions in $L^2_\rho(\mathbb{C}^n)$ is denoted by $H_\rho(\mathbb{C}^n)$. Moreover, $H_{\eta\cdot\varphi}(\mathbb{H})$ is collection of all functions, for which weakly* **holomorphic** *with respect to z variable in $L^2_{\eta\cdot\varphi}(\mathbb{H})$.*

Definition 4. Let $E(z,\eta) = e^{\eta\cdot\varphi(z)}$, and define $\mathcal{F} : L^2(\mathbb{H}) \to L^2_{\eta\cdot\varphi}(\mathbb{H})$ by $(\mathcal{F}u)(z,\eta) = \hat{u}(z,\eta)E(z,\eta)$. Clearly, $\hat{u}E \in L^2_{\eta\cdot\varphi}(\mathbb{H})$, if $u \in L^2(\mathbb{H})$. Therefore $\mathcal{F}$ is an isometric isomorphism.

Theorem 2. *$\mathcal{F}: \mathcal{H}_b \to H_{\eta\cdot\varphi}(\mathbb{H})$ is an isometric isomorphism. This transform determine the relation between the kernel of $\Box_b$ and holomorphic functions in $L^2_{\eta\cdot\varphi}(\mathbb{H})$.*

Proof. Let $f,\ g \in \mathscr{S}(\mathbb{H})$ be Schwartz functions, and let $\bar{Z}_j^*$ be the adjoint operator of $\bar{Z}_j$ with respect to the weight $\eta \cdot \varphi$, then

$$\langle f, \bar{Z}_j^* g\rangle_{\eta\cdot\varphi} = \langle \bar{Z}_j f, gE^{-1}\rangle = \langle f, -(Z_j - 2(Z_j\varphi)\cdot\eta)g\rangle_{\eta\cdot\varphi}. \tag{33}$$

on the other hand, for $u \in \mathcal{H}_b$ and $v \in \mathscr{S}(\mathbb{H})$, we have

$$\langle u, (-Z_j + (Z_j\varphi)\nabla_T)v\rangle = \langle u, \bar{L}_j^* v\rangle = 0, \tag{34}$$

and Plancherel theorem gives

$$\begin{aligned}\langle u, \bar{L}_j^* v\rangle &= \langle \hat{u}, (\bar{L}_j^* v)\hat{}\ \rangle = \langle \hat{u}, (-Z_j + (Z_j\varphi)\cdot\eta)\hat{v}\rangle \\ &= \langle \hat{u}, (-Z_j + 2(Z_j\varphi\cdot\eta))(\hat{v}E)E^{-1}\rangle \\ &= \langle \mathcal{F}u, \bar{Z}_j^* \mathcal{F}v\rangle_{\eta\cdot\varphi}. \end{aligned} \tag{35}$$

Now, let $g \in C_c^\infty(\mathbb{H})$, then $gE^{-1} \in C_c^\infty(\mathbb{H})$, hence $\mathcal{F}^{-1}g = g\check{E}^{-1} \in \mathscr{S}(\mathbb{H})$. Combining (33), (34) and (35) we have

$$\langle \mathcal{F}u, \bar{Z}_j^* g\rangle_{\eta\cdot\varphi} = \langle u, \bar{L}_j^*(\mathcal{F}^{-1}g)\rangle = 0 \quad \text{for all } g \in \mathscr{S}(\mathbb{H}). \tag{36}$$

It follows that $\mathcal{F}u$ is weakly holomorphic with respect to variable z, therefore $\mathcal{F}u \in H_{\eta\cdot\varphi}(\mathbb{H})$. The converse also follows from (33), (34) and (35), this proved the theorem. ☐

4. Weighted Holomorphic space and Bergman Kernel

Theorem 2 shows the relation between the kernel of $\Box_b$ and weighted holomorphic functions, now we study Bergman kernel instead of Szegő kernel, and via the transform $\mathcal{F}$ to get Szegő kernel.

Proposition 5. *Let $f \in H_{\eta\cdot\varphi}(\mathbb{C}^n)$. Then f is holomorphic. Here, of course, means that f has a version g is holomorphic, that is, $f = g$ almost every where and g is holomorphic.*

Proof. Let $\mu \in C_c^\infty(\mathbb{C}^n, [0,1])$ be a radical function and $\mu \geq 1$ on $B(0,1)$. Put $\mu_\epsilon(z) = \epsilon^{-2n}\mu(\frac{z}{\epsilon})$, for $f \in H_{\eta\cdot\varphi}(\mathbb{C}^n)$, $f^\epsilon := f * \mu_\epsilon$. Then $\{f^\epsilon\}$ from an approximate of identity, and each of f^ϵ is holomorphic. On the other hand, by mean value property,

$$\begin{aligned}|f^\epsilon(z) - f^\delta(z)| &= |\fint_{B(z,r)} (f^\epsilon(w) - f^\delta(w))d\mathbf{m}(w)| \\ &\leq C_r \|f^\epsilon - f^\delta\|_{L^2_{\eta\cdot\varphi}(\mathbb{C}^n)} \longrightarrow 0, \qquad (37)\end{aligned}$$

as $\epsilon, \delta \longrightarrow 0$. Hence, $\{f^\epsilon\}$ is compactly Cauchy sequence and therefore converges to a holomorphic function. The result follows. □

Proposition 6. *Let $f \in H_{\eta\cdot\varphi}(\mathbb{H})$. Then for almost η, $z \mapsto f(z,\eta)$ is holomorphic. Hence, weakly holomorphic and strongly holomorphic are equivalent in $L^2_{\eta\cdot\varphi}(\mathbb{H})$.*

Proof. From Fubini's theorem,

$$\int_{\mathbb{C}^n} |f(z,\eta)|^2 \, e^{-2\eta\cdot\varphi} d\mathbf{m}(z) < \infty$$

for almost all $\eta \in \mathbb{R}^d$. More precisely, there is a negligible set $A_0 \subset \mathbb{R}^d$ such that

$$\int_{\mathbb{C}^n} |f(z,\eta)|^2 \, e^{-2\eta\cdot\varphi} d\mathbf{m}(z) < \infty,$$

for every $\eta \notin A_0$. Let $g \in C_c^\infty(\mathbb{C}^n)$. Fix $j = 1, \dots, n$, put

$$h(\eta) = \int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* g}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z)$$

if $\eta \notin A_0$, $h(\eta) = 0$ if $\eta \in A_0$. We can check that

$$|h(\eta)|^2 \leqslant \int_{\mathbb{C}^n} |f(z,\eta)|^2 \, e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) \int_{\mathbb{C}^n} \left|\bar{Z}_j^* g(z)\right|^2 e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z). \qquad (38)$$

For $R > 0$, put

$$\phi_R(\eta) = \tau\Big(\frac{\eta}{R}\Big)\overline{h}(\eta),$$

where $\tau \in C_c^\infty(\mathbb{R}^d)$, $\tau = 1$ on $|\eta| \leq R$, $\tau = 0$ outside $|\eta| > R$. From (38), we have

$$\int |\phi_R(\eta)|^2 \, d\mathbf{m}(\eta) \leq \int_{|\eta|<R} |h(\eta)|^2 \, d\mathbf{m}(\eta)$$
$$\leqslant C \iint |f(z,\eta)|^2 \, e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(\eta) d\mathbf{m}(z) < \infty \,, \tag{39}$$

where $C > 0$. Let $\phi_{R,k}(\eta) \in C_c^\infty(\mathbb{R}^d)$, $k = 1, 2, \ldots$, with

$$\lim_{k\to+\infty} \int |\phi_{R,k}(\eta) - \phi_R(\eta)|^2 \, d\mathbf{m}(\eta) = 0. \tag{40}$$

Fix $k = 1, 2, \ldots$. From Theorem 2, we have

$$\begin{aligned}\int_{\mathbb{R}^d} h(\eta)\varphi_{R,k}(\eta) d\mathbf{m}(\eta) &= \int\int f(z,\eta)\overline{\bar{Z}_j^* g(z)}\phi_{R,k}(\eta) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) d\mathbf{m}(\eta) \\ &= \int\int f(z,\eta)\overline{\bar{Z}_j^*(\overline{\phi_{R,k}(\eta)} g(z))} e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) d\mathbf{m}(\eta) \\ &= 0. \end{aligned} \tag{41}$$

From (41) and (40), we have

$$\begin{aligned}\int_{\mathbb{R}^d} |h(\eta)|^2 \, \tau(\frac{\eta}{R}) d\mathbf{m}(\eta) &= \int_{\mathbb{R}^d} h(\eta)\phi_R(\eta) d\mathbf{m}(\eta) \\ &= \lim_{k\to+\infty} \int_{\mathbb{R}^d} h(\eta)\varphi_{R,k}(\eta) d\mathbf{m}(\eta) \\ &= 0. \end{aligned} \tag{42}$$

Letting $R \to \infty$, we get $h(\eta) = 0$ almost everywhere. We have proved that for a given $f(z) \in C_c^\infty(\mathbb{C}^n)$,

$$\int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* g}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) = 0$$

almost everywhere.

Let us consider the Sobolev space $H^1(\mathbb{C}^n)$ of distributions in $\mathbb{C}^n$ whose derivatives of order $\leqslant 1$ are in L^2. Since $H^1(\mathbb{C}^n)$ is separable and $C_c^\infty(\mathbb{C}^n)$ is dense in $H^1(\mathbb{C}^n)$, we can find $g_k \in C_c^\infty(\mathbb{C}^n)$, $k = 1, 2, \ldots$, such that $\{v_1, v_2, \ldots\}$ is a dense subset of $H^1(\mathbb{C}^n)$. Moreover, we can take $\{g_1, g_2, \ldots\} \subset$

Span $\{v_i\}$ so that for all $g \in C_c^\infty(\mathbb{C}^n)$ with

$$\operatorname{supp} g \subset B_r := \{z \in \mathbb{C}^n;\, |z| < r\}, \quad r > 0,$$

$\operatorname{supp} g_k \subset B_r$, $k = 1, 2, \ldots$, such that $g_k \to g$ for $k \to \infty$ in $H^1(\mathbb{C}^n)$.

Now, for each k, we can repeat the method above and find a measurable set $A_k \supset A_0$, $|A_k| = 0$ (A_0 is as in the beginning of the proof), such that

$$\int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* v_k}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) = 0$$

for all $\eta \notin A_k$. Put $A = \bigcup_k A_k$. Then, $|A| = 0$ and for all $\eta \notin A$,

$$\int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* v_k}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) = 0$$

for all k. Let $g \in C_c^\infty(\mathbb{C}^n)$ with $\operatorname{supp} g \subset B_r$. From the discussion above, we can find $g_1, g_2, \ldots$, $\operatorname{supp} g_k \subset B_r$, $k = 1, 2, \ldots$, such that $g_k \to g$ in $H^1(\mathbb{C}^n)$, $k \to \infty$. Then, for $\eta \notin A$,

$$\begin{aligned}
&\int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* g}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) \\
&= \int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* (g - g_k)}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) \\
&\quad + \int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* (g_k)}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) \\
&= \int_{\mathbb{C}^n} f(z,\eta)\overline{\bar{Z}_j^* (g - g_k)}(z) e^{-2\eta\cdot\varphi(z)} d\mathbf{m}(z) \\
&\longrightarrow 0 \text{ as } k \longrightarrow +\infty.
\end{aligned} \tag{43}$$

The theorem follows. ☐

Proposition 7. *Let $K_\rho : L^2_\rho(\mathbb{C}^n) \to H_\rho(\mathbb{C}^n)$ be the orthogonal projection from $L^2_\rho(\mathbb{C}^n)$ onto $H_\rho(\mathbb{C}^n)$. If e^ρ is locally bounded, then the distributional kernel exists such that*

$$(K_\rho f)(z) = \int f(w) K_\rho(w,z) d\mathbf{m}(w) \text{ for all } f \in L^2_\rho(\mathbb{C}^n). \tag{44}$$

*We say that the kernel K_ρ is called the **Bergman kernel**.*

Proof. Note that for ever given compact set K, we have

$$\sup_K |f| \leq C_K \sup_K |e^{\rho}| \|f\|_{L^2_\rho(\mathbb{C}^n)}, \text{ for all holomorphic function } f. \tag{45}$$

Let $z \in \mathbb{C}^n$, the evaluation functional $f \mapsto K_\rho f(z)$ is dominated by L^2_ρ-norm, by Riesz representation Theorem, there is $g_z(w) \in L^2_\rho(\mathbb{C}^n)$ such that

$$(K_\rho f)(z) = \int f(w) g_z(w) d\mathbf{m}(w). \tag{46}$$

We rewrite $K_\rho(w,z) := g_z(w)$, then the result proved. □

Proposition 8. *Assume that $\eta \cdot \varphi$ is locally bounded, then the distributional kernel of $K : L^2_{\eta\cdot\varphi}(\mathbb{H}) \to H_{\eta\cdot\varphi}(\mathbb{H})$ exists such that*

$$(Kf)(z,\eta) = \int f(w,\eta) K(w,z,\eta) d\mathbf{m}(w). \tag{47}$$

*for all $u \in L^2_{\eta\cdot\varphi}(\mathbb{H})$. Moreover, this kernel is also called the **Bergman kernel**.*

Proof. Let $f \in L^2_{\eta\cdot\varphi}(\mathbb{H})$, then for almost every η, $f_\eta := f(\cdot,\eta) \in L^2_{\eta\cdot\varphi}(\mathbb{C}^n)$, by Proposition 7, $K_{\eta\cdot\varphi}(f_\eta)$ is holomorphic with respect to z and

$$\|K_{\eta\cdot\varphi}(f_\eta)\|_{L^2_{\eta\cdot\varphi}(\mathbb{C}^n)} \leq \|f_\eta\|_{L^2_{\eta\cdot\varphi}(\mathbb{C}^n)}. \tag{48}$$

Integrating this inequality with respect to η obtains

$$\int |(K_{\eta\cdot\varphi} f_\eta)(z)|^2 e^{-2\eta\cdot\varphi} d\mathbf{m}(z,\eta) \leq \int |f(z,\eta)|^2 e^{-2\eta\cdot\varphi} d\mathbf{m}(z,\eta). \tag{49}$$

On the other hand, if $f \in H_{\eta\cdot\varphi}(\mathbb{H})$, then for almost every η, $f_\eta \in H_{\eta\cdot\varphi}(\mathbb{C}^n)$. Hence, $(Kf)(z,\eta) = (K_{\eta\cdot\varphi} f_\eta)(z)$ for all z and almost every η and therefore

$$(Kf)(z,\eta) = \int f(w,\eta) K(w,z,\eta) d\mathbf{m}(w). \tag{50}$$

□

Lemma 1. *Let K_ρ be the Bergman kernel with respect to $\rho(z) := \Sigma\lambda_j |z_j|^2$ in space $H_\rho(\mathbb{C}^n)$, then we have*

$$K_\rho(w,z) = \lambda_1 \cdots \lambda_n \left(\frac{2}{\pi}\right)^n e^{-\Sigma\lambda_j(|w_j - z_j|^2 + w_j \bar{z}_j - \bar{w}_j z_j) + \Sigma\lambda_j(|z_j|^2 - |w_j|^2)}$$

for every scalar $\lambda_j > 0$.

Proof. Note that

$$K_\rho(w,z) = \sum f_i(z)\bar{f}_i(w)e^{-2\rho(w)}$$

for every orthonormal basis $\{f_i\}$ in $H_\rho(\mathbb{C}^n)$. Since every holomorphic function is compactly limit of linearly combination of z^m, $m = (m_1, m_{2,} \ldots, m_n) \in (N \cup \{0\})^n$, $e^{-2\rho(z)}$ is a radical function and $z^{2m}e^{-2\rho(z)}$ is integrable, we have $\{z^m\}$ from an orthogonal basis in $H_\rho(\mathbb{C}^n)$. Therefore

$$K_\rho(w,z) = \sum \frac{1}{\|z^{|m|}\|^2_{H_\rho(\mathbb{C}^n)}} \bar{w}^m z^m e^{-2\rho(w)}, \tag{51}$$

where the summation takes over all nonnegative multi-index m.

On the other hand, for $c > 0$, $m \in \mathbb{N} \cup \{0\}$,we have

$$\int_0^\infty r^{2m+1}e^{-2cr^2}dr = \frac{1}{(2c)^{m+1}}\int r^{2m+1}e^{-r^2}dr = \frac{1}{(2c)^{m+1}}\frac{m!}{2}. \tag{52}$$

Thus,

$$\begin{aligned}\|z^{|m|}\|^2_{H_\rho(\mathbb{C}^n)} &= (2\pi)^n\int_{(\mathbb{R}^+)^n} r_1\cdots r_n r^{2m}e^{-2\rho}d\mathbf{m}(r)\\ &= \frac{1}{\lambda_1\cdots\lambda_n}\Big(\frac{\pi}{2}\Big)^n\frac{m!}{(2\lambda)^m},\end{aligned} \tag{53}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$. Hence,

$$\begin{aligned}K_\rho(w,z) &= \lambda_1\cdots\lambda_n\Big(\frac{2}{\pi}\Big)^n\sum\frac{(2\lambda)^m}{m!}\bar{w}^m z^m e^{-2\rho(w)}\\ &= \lambda_1\cdots\lambda_n\Big(\frac{2}{\pi}\Big)^n e^{2\Sigma\lambda_j(\bar{w}_j z_j - |w_j|^2)} \qquad (54)\\ &= \lambda_1\cdots\lambda_n\Big(\frac{2}{\pi}\Big)^n e^{-\Sigma\lambda_j(|w_j-z_j|^2+w_j\bar{z}_j-\bar{w}_j z_j)+\Sigma\lambda_j(|z_j|^2-|w_j|^2)}. \qquad \square\end{aligned}$$

Corollary 1. *Recall* (2) *that* $\varphi(z) = \Sigma|z_j|^2\lambda_j$ *with* $\lambda_j \in \mathbb{R}^d$ *for* $j = 1, \ldots, n$. *The Bergman kernel with respect to* $\eta\cdot\varphi$ *in space* $H_{\eta\cdot\varphi}(\mathbb{C}^n)$ *is*

$$K_{\eta\cdot\varphi}(w,z) := \Big(\frac{2}{\pi}\Big)^n\prod_{j=1}^n\lambda_j\cdot\eta e^{-\eta\cdot\Phi(w,z)+\eta\cdot(\varphi(z)-\varphi(w))}. \tag{55}$$

where $\Phi(w,z) := \varphi(w-z) + \Sigma(w_j\bar{z}_j - \bar{w}_j z_j)\lambda_j$, *and* $\eta\cdot\lambda_j > 0$.

Theorem 3.

$$\begin{aligned} K_\eta(w,z) &= K_{\eta\cdot\varphi}(w,z)\chi_{A_\lambda}(\eta) \\ &= \Big(\frac{2}{\pi}\Big)^n \chi_{A_\lambda}(\eta)\prod_{j=1}^{n}\lambda_j\cdot\eta e^{-\eta\cdot\Phi(w,z)+\eta\cdot(\varphi(z)-\varphi(w))} \end{aligned}$$

is the Bergman kernel K with respect to $\eta\cdot\varphi(z)$ in space $H_\varphi(\mathbb{H})$, where χ_{A_λ} is the characteristic function of A_λ , and $A_\lambda = \{\eta\in\mathbb{R}^d | \eta\cdot\lambda_j > 0,\ j = 1,2,\ldots,n\}$.

Proof. By Proposition 8, we only need to check $K_\eta = K_\eta\chi_{A_\lambda}$ or equivalently $f = f\chi_{A_\lambda}$ for every $f\in H_{\eta\cdot\varphi}(\mathbb{H})$. Let $f\in H_{\eta\cdot\varphi}(\mathbb{H})$, consider sets $E_j = \{\eta\cdot\lambda_j \le 0,\}$, since $f(\cdot,\eta)$ is holomorphic for a.e. η, we have sub-mean inequality:

$$|f(z_0,\eta)|^2 \le \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} |f((r_1e^{i\theta_1}, r_2e^{i\theta_2},\ldots,r_ne^{i\theta_n}) + z_0,\eta)|^2 d\mathbf{m}(\theta), \tag{56}$$

for every $r_j\ge 0$. We multiply both sides of (56) by $r_1r_2\cdots r_ne^{-2\eta\cdot\varphi(r)}$, then integrate it from 0 to R in each coordinates, and yields that

$$|f(z_0,\eta)|^2 \le \frac{\int_{D_R(0)^n} |f(z+z_0,\eta)|^2 e^{-2\eta\cdot\varphi(z)}d\mathbf{m}(z)}{\int_{z_0+D_R(0)^n} e^{-2\eta\cdot\varphi(z)}d\mathbf{m}(z)}, \tag{57}$$

where $D_R(0)^n$ is the n-dimensional unit polydisc centered at the origin.

Since $f\in L^2_{\eta\cdot\varphi}(\mathbb{H})$, for almost η, $|f(\cdot,\eta)|^2e^{-2\eta\cdot\varphi(\cdot)}$ is integrable, and hence for almost η, $\int_{\mathbb{C}^n}|f(z,\eta)|^2e^{-2\eta\cdot\varphi(z)}d\mathbf{m}(z) < \infty$, then right hand side of (57) tends to zero as R tends to ∞, if $\eta\in E_j$. Hence, $f = 0$ a.e. in $\mathbb{C}^n\times E_j$ for all j, that is $f = f\chi_{A_\lambda}$ a.e. for every $f\in H_\varphi$, it follows that the orthogonal projection

$$K : L^2_{\eta\cdot\varphi}(\mathbb{H}) \to H_{\eta\cdot\varphi}(\mathbb{H}) \quad \text{is} \quad (Kf)(z,\eta) = < f, \bar{K}_\eta(\cdot,z) > . \tag{58}$$

□

5. The Proof of Main Theorem

Consider the figure (2), the Szegő projection represent as $\mathcal{F}^{-1}K\mathcal{F}$, formally,

$$(Pu)(z,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d}\int_{\mathbb{C}^d}\int_{\mathbb{R}^d} e^{-it\cdot\eta}e^{-\eta\cdot\varphi(z)}K_\eta(w,z)e^{\eta\cdot\varphi(w)}e^{i\eta\cdot\theta}u(w,\theta)d\mathbf{m}(\theta,w,\eta). \tag{59}$$

If Fubini's Theorem can be worked, then Pu will become

$$\frac{1}{(2\pi)^d}\int_{\mathbb{H}} u(w,\theta)\Big(\int_{\mathbb{R}^d} K_\eta(w,z)e^{-i\eta\cdot(t-\theta)-\eta\cdot(\varphi(z)-\varphi(w))}d\mathbf{m}(\eta)\Big)d\mathbf{m}(w,\theta). \tag{60}$$

Therefore,

$$S(w,z,\theta,t) = \int_{\mathbb{R}^d}\tilde{S}d\mathbf{m}(\eta) \tag{61}$$

in sense of oscillatory integral, where

$$\begin{aligned}\tilde{S} &= \frac{1}{(2\pi)^d}K_\eta(w,z)e^{-i\eta\cdot(t-\theta)-\eta\cdot(\varphi(z)-\varphi(w))}\\ &= \frac{2^{n-d}}{\pi^{n+d}}\chi_{A_\lambda}\prod_{j=1}^{n}\lambda_j\cdot\eta e^{-i\eta\cdot[(t-\theta)-i\eta\cdot\Phi(w,z)]}.\end{aligned} \tag{62}$$

Note that

$$K_\eta(w,z) = \Big(\frac{2}{\pi}\Big)^n\chi_{A_\lambda}\prod_{j=1}^{n}\lambda_j\cdot\eta e^{-\eta\cdot\Phi(w,z)+\eta\cdot(\varphi(z)-\varphi(w))} \tag{63}$$

by Theorem 3.

Proposition 9. *If $u\in L^2(\mathbb{H})\cap L^1(\mathbb{H})$ and $K\mathcal{F}u\in L^1_{\eta\cdot\varphi}(\mathbb{H})$. Then the Fourier transform can be write down for exactly integral formula, and therefore we have* (59)

$$(Pu)(z,t) = \int_{\mathbb{H}\times\mathbb{R}^d} u(w,\theta)K_\eta(w,z)e^{-i\eta\cdot(t-\theta)-\eta\cdot(\varphi(z)-\varphi(w))}d\mathbf{m}(\theta,w,\eta), \tag{64}$$

for a.e. (z,t).

Theorem 4. *For $u \in C_c^\infty$ and χ be any suitable smooth cutoff, then*

$$(Pu)(z,t) = \lim_{\epsilon\to 0^+} \int_{\mathbb{H}} u(w,\theta) \int_{\mathbb{R}^d} \chi_\epsilon(\eta)\tilde{S}(w,z,\theta,t,\eta)d\mathbf{m}(\eta)d\mathbf{m}(w,\theta), \quad (65)$$

where $\Phi(w,z) := \varphi(w-z) + \Sigma(w_j\bar{z}_j - \bar{w}_j z_j)\lambda_j$ and $\chi(\epsilon\eta) = \chi_\epsilon(\eta)$ is the scaling of χ.

Proof. Let $u_\epsilon(w,\theta,\eta) := u(w,\theta)\chi(\epsilon\eta) \in C^\infty(\mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d)$. By Theorem 3,

$$K_\eta(w,z) = \left(\frac{2}{\pi}\right)^n \chi_{A_\lambda}(\eta)\prod_{j=1}^n \lambda_j \cdot \eta e^{-\eta\cdot\Phi(w,z)+\eta\cdot(\varphi(z)-\varphi(w))}, \quad (66)$$

and therefore

$$\tilde{S}(w,z,\theta,t,\eta) = \frac{2^{n-d}}{\pi^{n+d}}\chi_{A_\lambda}(\eta)\prod_{j=1}^n \lambda_j \cdot \eta e^{-i\eta\cdot[(t-\theta)-i\Phi(w,z)]}. \quad (67)$$

On the other hand, the real part of phase function is

$$\begin{aligned} \operatorname{Re}(-i\eta\cdot[(t-\theta)-i\Phi(w,z)]) &= -\operatorname{Re}(\eta\cdot\Phi(w,z)) = -\eta\cdot\varphi(w-z) \\ &= -\Sigma\eta\cdot\lambda_j|w_j-z_j|^2 \approx -O(|\eta|) \end{aligned} \quad (68)$$

if $\chi_{A_\lambda}(\eta) > 0$. Hence, $\tilde{S}$ is bounded and integrable with respect to η out side the diagonal $\{w_j = z_j, \text{ for all } j\}$, and therefore $u_\epsilon\tilde{S}$ is integrable over $\mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d$, according Fubini's Theorem, we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{H}} \tilde{u}_\epsilon\tilde{S}d\mathbf{m}(w,\theta)d\mathbf{m}(\eta) = \int_{\mathbb{H}} u(w,\theta)\int_{\mathbb{R}^d}\chi_\epsilon\tilde{S}d\mathbf{m}(\eta)d\mathbf{m}(w,\theta). \quad (69)$$

On the other hand, the continuity of projections gives that

$$\mathcal{F}^{-1}(K\mathcal{F}u_\epsilon) = \mathcal{F}^{-1}(\rho_\epsilon K\mathcal{F}u) \to Pu \text{ in } L^2. \quad (70)$$

Since $u(w,\cdot) \in L^1(\mathbb{R}^d)$, and $e^{\eta\cdot\varphi}\chi_\epsilon K\mathcal{F}u$ (compact support with respect to $\eta) \in L^1(\mathbb{R}^d)$, and by Proposition 9,

$$Pu = \lim \mathcal{F}^{-1}(\chi_\epsilon K\mathcal{F}u) = \lim_{\epsilon\to 0^+}\int_{\mathbb{H}} u(w,\theta)\int_{\mathbb{R}^d}\chi_\epsilon\tilde{S}d\mathbf{m}(\eta)d\mathbf{m}(w,\theta). \quad (71)$$

□

Remark 3. Since $\tilde{S}$ is bounded a.e., χ can be replace by any function in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $\chi_\epsilon \to 1$ a.e.

Next, we study S with respect to polar coordinates in formal sense:

$$\int_{\mathbb{R}^d} \tilde{S} = \int_{A_\lambda} \tilde{S} = \int_{\substack{|v|=1 \\ v\cdot\lambda_j>0}} \int_0^\infty r^{d-1} \tilde{S} dr d\Omega(v), \tag{72}$$

where $d\Omega(v)$ is Euclidean surficial measure.

Theorem 5. *For a suitable χ, we have for every $u \in C_c^\infty$,*

$$(Pu)(z,t) = \lim_{\epsilon\to 0^+} \int_{\mathbb{H}} u(w,\theta) S_\epsilon(w,z,\theta,t) d\mathbf{m}(w,\theta), \tag{73}$$

where

$$S_\epsilon = \int_{\mathbb{R}^d} \chi_\epsilon \tilde{S} = \frac{2^{n-d}}{\pi^{n+d}} \int_{B_\lambda} \frac{(n+d-1)!\Pi v\cdot\lambda_j}{(iv\cdot[(t-\theta)-i\Phi(w,z)]+\epsilon)^{n+d}} d\Omega(v), \tag{74}$$

and $B_\lambda = A_\lambda \cap \{|v|=1\}$ the intersection of A_λ and $d-1$ dimensional sphere.

Proof. Let Ψ be the part of phase function $\Psi(w,z,\theta,t) = (t-\theta) - i\Phi(w,z)$, then

$$\tilde{S} = \frac{2^{n-d}}{\pi^{n+d}} \chi_{A_\lambda} \prod_{j=1}^n \lambda_j \cdot \eta e^{-i\eta\cdot\Psi}. \tag{75}$$

Formally, if the order of integrals can be changed, then

$$\int r^{n+d-1} (\Pi\lambda_j \cdot v) e^{-irv\cdot\Psi} = \int (\Pi v\cdot\lambda_j) \int r^{n+d-1} e^{-irv\cdot\Psi}. \tag{76}$$

Since $\mathrm{Re}(irv\cdot\Psi) = \Sigma v\cdot\lambda_j |w_j - z_j|^2 > 0$ when $w \neq z$ and v lies in integral area. By the formula:

$$\int_0^\infty t^m e^{-xt} dt = m! x^{-(m+1)}, \text{ for } m \geq 0 \text{ and } \mathrm{Re}(x) > 0, \tag{77}$$

we have

$$\int r^{n+d-1} e^{-irv\cdot\Psi} dr = \frac{(n+d-1)!}{(iv\cdot\Psi)^{n+d}}, \tag{78}$$

and hence

$$\int_{A_\lambda} \tilde{S} = \frac{2^{n-d}}{\pi^{n+d}} \int_{B_\lambda} \frac{(n+d-1)!\Pi v \cdot \lambda_j}{(iv \cdot \Psi)^{n+d}} d\Omega(v). \tag{79}$$

However the order of singularity of (79) is d, the surficial integral is therefore does not exists. Now we put

$$\tilde{S}_\epsilon = \tilde{S}\chi_\epsilon := \tilde{S}e^{-\epsilon|\eta|}, \tag{80}$$

then $\tilde{S}_\epsilon$ apply to (79) with out singularity

$$S_\epsilon = \int_{\mathbb{R}^d} \tilde{S}_\epsilon = \frac{2^{n-d}}{\pi^{n+d}} \int_{B_\lambda} \frac{(n+d-1)!\Pi v \cdot \lambda_j}{(iv \cdot \Psi + \epsilon)^{n+d}}. \tag{81}$$

Hence, we conclude that:

$$(Pu)(z,t) = \frac{2^{n-d}}{\pi^{n+d}} \lim_{\epsilon \to 0^+} \int_{\mathbb{H}} u(w,\theta) \int_{B_\lambda} S_\epsilon d\Omega(v) d\mathbf{m}(w,\theta). \tag{82}$$

□

Theorem 6. *Put $P_\epsilon u = \int u S_\epsilon$, for $u \in L^2(\mathbb{H})$, then*

$$P_\epsilon u \to Pu \text{ in } L^2(\mathbb{H}), \tag{83}$$

thus, the Szegő kernel

$$S(w,z,\theta,t) = \frac{2^{n-d}}{\pi^{n+d}} \int_{B_\lambda} \frac{(n+d-1)!\Pi v \cdot \lambda_j}{(iv \cdot [(t-\theta) - i\Phi(w,z)])^{n+d}} d\Omega(v) \tag{84}$$

in sense of Fourier integral, where $\Phi(w,z) := \varphi(w-z) + \Sigma(w_j\bar{z}_j - \bar{w}_j z_j)\lambda_j = \Sigma(|w_j - z_j|^2 + w_j\bar{z}_j - \bar{w}_j z_j)\lambda_j$.

Proof. We first suppose that

$$P_\epsilon u = \mathcal{F}^{-1}(\chi_\epsilon K \mathcal{F} u), \tag{85}$$

since $K\mathcal{F}u \in L^2_{\eta\cdot\varphi}(\mathbb{H})$, by continuity that

$$\chi_\epsilon K\mathcal{F}u \to K\mathcal{F}u \quad \Rightarrow \quad P_\epsilon u \to Pu. \tag{86}$$

To see (85), let $\rho_\delta(w,\theta) = e^{-\delta(|w|+|\theta|)}$, then

$$\mathcal{F}^{-1}(\chi_\epsilon K\mathcal{F}\rho_\delta u) \to \mathcal{F}^{-1}(\chi_\epsilon K\mathcal{F}u) \quad \text{as} \quad \delta \to 0^+. \tag{87}$$

On the other hand,

$$\int |u|\rho_\delta d\mathbf{m}(w,\theta) \leq \Big\{\int |u|^2 d\mathbf{m}(w,\theta)\Big\}^{\frac{1}{2}}\Big\{\int \rho_\delta^2 d\mathbf{m}(w,\theta)\Big\}^{\frac{1}{2}} < \infty, \tag{88}$$

and since $\chi_\epsilon K\mathcal{F}\rho_\delta u \in L^1_{\eta\cdot\varphi}(\mathbb{H})$, by Proposition 9 that

$$\mathcal{F}^{-1}(\chi_\epsilon K\mathcal{F}\rho_\delta u) = \int u\rho_\delta \int \tilde{S}\chi_\epsilon d\mathbf{m}(\eta) d\mathbf{m}(w,\theta) = \int u\rho_\delta S_\epsilon. \tag{89}$$

Since S_ϵ is bounded, (87) and (89) show (85). This proves the result. □

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