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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO FORWARD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

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Abstract

In this paper, we study the existence and the uniqueness of solution of coupled G-forward-backward stochastic differential equations (G-FBDSEs in short). Our systems are described by coupled multi-dimensional G-FBDSEs. We construct a mapping for which the fixed point is the solution of our G-FBSDE, where we prove that this mapping is a contraction. In this paper we do not require the monotonicity condition to prove the existence.

1. Introduction

BSDEs were first introduced by Bismut in 1973. Later, the general non-linear BSDEs in the framework of the Brownian motion were first set forth introduced by Pardoux and Peng in [21], and since then, the theory of BSDEs has developed very quickly, see El-Karoui, Peng and Quenez [15]; Bahlali et al. [2], and there have been an interesting applications in stochastic optimal control and finance (see for example [2]).

Associated with the BSDEs theory, the field of fully coupled FBSDEs has also developed very quickly since the work of Antonelli [1]; we refer to,

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Cvitanic and Ma [7], Delarue [8], Ma and Yong [20], Pardoux and Tang [22], Peng and Wu [23], and Zhang [36], etc.

Recently and since the introduction of the g-expectation based on BSDE by Peng [24], this theory has quickly developed and has been of interest to many authors, among whom we mention [6, 11, 12, 25, 31]. After that, an abstract sublinear or G-expectation space with a process called the G-Brownian motion was introduced by Peng, (see [26, 27, 28, 29]) and by Denis and Martini [9] who suggested a structure based on the quasi-sure analysis from the abstract potential theory to construct a similar structure using a tight family of possibly mutually singular probability. The main difference between the small g-framework and the big G-framework is that in the g-framework we can't let the uncertainty in the diffusion, whereas in the G-framework both the drift and the diffusion can have uncertainty. In recent years, the framework of G-expectation has found increasing applications in the domain of finance and economics; for example, Epstein and Ji [10, 13] study the asset pricing with ambiguity preferences, and Beissner [3] has studied the equilibrium theory with ambiguous volatility, in addition to many others see e.g [4, 19, 32, 34, 35].

The application of FBSDE in the standard case has been the interest of many authors; we refer to [16, 23, 20].

In this paper, we study the existence and the uniqueness of the following FBSDE system in the G-framework:

$$\begin{cases} dX_s = b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s)dB_s + h_{ij}(s, X_s, Y_s)d\langle B^i, B^j\rangle_s, \\ dY_s = -f(s, X_s, Y_s, Z_s, M_s)ds - g_{ij}(s, X_s, Y_s)d\langle B^i, B^j\rangle_s \\ \qquad\quad +Z_s dB_s + dM_s,\ s \in [t, T], \\ X_t = x,\ Y_T = \Phi(X_T), M_t = 0, \end{cases} \tag{1}$$

(X, Y, Z, M) is the solution of our FBSDE where X, Y, Z are square integrable adapted processes and M is a decreasing G-martingale, and the initial value $x \in \mathbb{R}^d$ is a given vector, B is a l-dimensional G-Brownian motion, $\langle B\rangle$ is the quadratic variation of the process $(B_s)_{s\geq 0}$.

The coefficients are given by:

$$\begin{aligned}
b &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \to \mathbb{R}^d, \\
\sigma &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{d\times l}, \\
h_{ij} &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d, \\
f &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^m \to \mathbb{R}^n, \\
g_{ij} &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n, \\
\Phi &: \mathbb{R}^d \to \mathbb{R}^n.
\end{aligned}$$

For simplicity we take $t = 0$. We prove the existence of the solution in one dimension, and yet, the result is still valid in a multi-dimension case.

In this paper with additional conditions we prove existence and uniqueness without the monotonicity condition that Wang et. al. [33] suppose. In addition to that here we allow that the solution of the forward equation X can have any dimension, not necessarily one-dimensional like the case of [33], where they needed it for the comparison reasons.

In many applications of stochastic optimal control, solving the corresponding HJB equation with probabilistic methods is required, especially in high dimensional problems see e.g. [14]. In these cases where we do not know the exact value of the volatility, but only a range of it, like the case of finance, the corresponding HJB equation in a fully non-linear G-PDE equation, and in case of high dimension we can't solve this end by the usual methods like the finite difference, so a probabilistic representation is required, and when the control enter the diffusion see e.g. [5, 17], this will produce a fully coupled G-FBSDE, and this which motivate our work, also another application is when we apply the stochastic maximum principle we end with a G-FBSDE.

The paper is organized as follows: In the next section, we give some preliminaries and existing results on the G-fretwork that we will use in our article; we also set the hypotheses that ensure the existence of the solution of the G-FBSDE (1). In section 3, we give our main result and its proof. The proofs of some technical lemmas are recorded in the Appendix.

2. Preliminaries and Hypothesis

In this section, we present some notations of spaces and existing results on the G-expectation that we will use in our article.

2.1. Notations

In this section, we present the following spaces notations that we use in our article:

- $C_{b.lip}(\mathbb{R}^d)$ is the space of bounded and Lipschitz continuous functions on $\mathbb{R}^d$;
- $L_{ip}(\Omega_T) := \left\{\varphi(B_{t_1},\dots,B_{t_n}) : n\geq 1, t_1,\dots,t_n\in[0,T], \varphi\in C_{b.lip}(\mathbb{R}^{d\times n})\right\}$;
- $L^p_G(\Omega_T)$ is the completion of $L_{ip}(\Omega_T)$ under the norm $\parallel \eta \parallel_{p,G}= \left\{\hat{\mathbb{E}}\left(|\eta|^p\right)\right\}^{\frac{1}{p}}$;
- $M^0_G(0,T) := \left\{\eta_t = \sum\limits_{i=0}^{N-1}\xi_i 1_{\{t_i,t_{i+1}\}} : 0 = t_0 < \cdots < t_N = T, \xi\in L_{ip}(\Omega_{t_i})\right\}$;
- $M^{p,0}_G(0,T) := \left\{\eta_t = \sum\limits_{i=0}^{N-1}\xi_i 1_{\{t_i,t_{i+1}\}} : 0 = t_0 < \cdots < t_N = T, \xi_i\in L^p_G(\Omega_{t_i})\right\}$;
- $\bar{M}^p_G(0,T)$ is the completion of $M^{p,0}_G(0,T)$ under the norm $\parallel \eta \parallel_{\bar{M}^p_G}= \left\{\int_0^T \hat{\mathbb{E}}\Big(|\eta_s|^p ds\Big)\right\}^{\frac{1}{p}}$;
- $M^p_G(0,T)$ is the completion of $M^0_G(0,T)$ under the norm $\parallel \eta \parallel_{M^p_G}= \left\{\hat{\mathbb{E}}\Big(\int_0^T |\eta_s|^p ds\Big)\right\}^{\frac{1}{p}}$;
- $H^p_G(0,T) :=$ the completion of $M^0_G(0,T)$ under the norm $\parallel \eta \parallel_{\mathbb{H}^p}= \left\{\hat{\mathbb{E}}\Big[\Big(\int_0^T |\eta_s|^2 ds\Big)^{\frac{p}{2}}\Big]\right\}^{\frac{1}{p}}$;
- $S^0_G(0,T) := \left\{h(B_{t_1\wedge t},\dots,B_{t_n\wedge t}) : t_1,\dots,t_n\in[0,T], h\in C_{b.lip}(\mathbb{R}^{n+1})\right\}$;
- $S^p_G(0,T)$ is the completion of $S^0_G(0,T)$ under the norm $\parallel \eta \parallel_{S^p_G}= \left\{\hat{\mathbb{E}}\Big(\sup\limits_{s\in[0,T]} |\eta_s|^p\Big)\right\}^{\frac{1}{p}}$;
- $\mathbb{L}^p_G(\Omega_T)$ is the space of decreasing G-martingales with $K_0 = 0$ and $K_T \in L^p_G(\Omega_T)$;
- $\mathfrak{S}^p_G(0,T)$ is the collection of processes (Y,Z,K) such that $Y\in S^p_G(0,T)$, $Z\in H^p_G(0,T)$ K is a decreasing G-martingale with $K_0 = 0$ and $K_T\in\mathbb{L}^p_G(\Omega_T)$;
- $H^{q,\alpha}_{G,T} := \bar{M}^q_G([0,T])\times\mathfrak{S}^\alpha_G(0,T)$.

2.2. Preliminaries

This section aims at recalling some introductory concepts in the G-framework. For more details we refer the readers to e.g. [29].

S. Peng in 2006 has introduced the functional $\hat{\mathbb{E}}(.) : \mathcal{L}_{ip}(\Omega) \to \mathbb{R}$, named **G-expectation** which defines a sublinear expectation on $L_{ip}(\Omega)$, and which means that it satisfies the following properties:

(1) *Monotonicity*: $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y) \quad \text{if} \quad X \geq Y.$

(2) *Constant preserving:* $\hat{\mathbb{E}}(l) = l \quad \text{for} \quad l \in \mathbb{R}.$

(3) *Sub-additivity:* For each $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}(X+Y) \leq \hat{\mathbb{E}}(X) + \hat{\mathbb{E}}(Y)$.

(4) *Positive homogeneity:* $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X) \quad \text{for} \quad \lambda \geq 0.$

The corresponding canonical process $(B_t)_{t\geq 0}$ of the functional $\hat{\mathbb{E}}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ is called a **G-Brownian motion** which is characterized by the following:

Definition 1. A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a *G-Brownian motion* if the following properties are satisfied:

(i) $B_0(\omega) = 0$,

(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(\{0\}, s\Sigma)$[1] is distributed and is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \ldots \leq t_n \leq t$,

where Σ is a non-negative $d \times d$ symmetric matrices.

A very interesting property of the G-Brownian motion is that its quadratic variation $\langle B \rangle$ is as the G-Brownian motion B itself; it holds that the increment $\langle B \rangle_{t+s} - \langle B \rangle_t$ is independent from $(\langle B \rangle_{t_1}, \langle B \rangle_{t_2}, \ldots, \langle B \rangle_{t_n})$, $0 \leq t_1 \leq \ldots \leq t_n \leq t$, and $\langle B \rangle_{t+s} - \langle B \rangle_t \overset{d}{=} \langle B \rangle_s$.

We also need the following BDG type of inequalities in the G-framework

[1] $N(\{0\}, s\Sigma)$ is called the G-normal distribution see[29]

Proposition 1 ([29]). *Let $\beta \in M_G^p(0,T)$ with $p \geq 2$. Then we have $\int_0^T \beta_t dB_t \in L_G^p(\Omega_T)$ and*

$$\hat{\mathbb{E}}\Big(\Big|\int_0^T \beta_t dB_t\Big|^p\Big) \leq C_p \hat{\mathbb{E}}\Big(\Big|\int_0^T \beta_t^2 d\langle B\rangle_t\Big|^{\frac{p}{2}}\Big). \tag{2}$$

Proposition 2 ([18]). *For each $\eta \in H_G^\alpha(0,T)$ with $\alpha \geq 1$ and $p \in (0,\alpha]$, we have*

$$\underline{l}^p c_p \hat{\mathbb{E}}\Big(\Big[\int_0^T \eta_s^2 ds\Big]^{\frac{p}{2}}\Big) \leq \hat{\mathbb{E}}\Big(\sup_{t\in[0,T]}\Big|\int_0^t \eta_s dB_s\Big|^p\Big) \leq \bar{l}^p C_p \hat{\mathbb{E}}\Big(\Big[\int_0^T \eta_s^2 ds\Big]^{\frac{p}{2}}\Big),$$

where, $0 < c_p < C_p < \infty$ are constants.

Lemma 1 ([30]). *Let $p \geq 1, \eta \in M_G^p([0,T])$ and $0 \leq s \leq t \leq T$. Then*

$$\hat{\mathbb{E}}\Big(\sup_{s\leq u\leq t}\Big|\int_s^u \eta_r d\langle B\rangle_r\Big|^p\Big) \leq \Big(\frac{\underline{l}+\bar{l}}{4}\Big)^p (t-s)^{p-1}\hat{\mathbb{E}}\Big(\int_s^t |\eta_u|^p du\Big). \tag{3}$$

Lemma 2. *For $\theta \in S_G^2$, we have:*

$$\hat{\mathbb{E}}\Big(\int_0^T |\theta_s|^2 d\langle B\rangle_s\Big) \leq T\bar{l}\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\theta_s|^2\Big),$$

Furthermore, for $\eta \in H_G^2(0,T)$, we have that $(\int_0^t \eta_s\theta_s dB_s)_{t\in[0,T]}$ is an uniformly integrable martingale, equal to 0 at time $t = 0$, so,

$$\hat{\mathbb{E}}\Big(\int_t^T \eta_s\theta_s dB\Big) = 0$$

Let be the following well-known inequalities:

Lemma 3. *For $r > 0$ and $1 < q, p < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|a+b|^r \leq \max\{1, 2^{r-1}\}(|a|^r + |b|^r) \quad \textit{for} \quad a, b \in \mathbb{R} \tag{4}$$

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{5}$$

By this we can get

Proposition 3 ([26]). *For each $X, Y \in \mathcal{H}$, we have*

$$\mathbb{E}\left(|X+Y|^r\right) \leq 2^{r-1}\left(\mathbb{E}(|X|^r)+\mathbb{E}(|Y|^r)\right) \tag{6}$$

$$\mathbb{E}(XY) \leq \left(\mathbb{E}(|X|^p)^{\frac{1}{p}}+\mathbb{E}(|Y|^q)^{\frac{1}{q}}\right) \tag{7}$$

$$\left(\mathbb{E}(|X+Y|^p)\right)^{\frac{1}{p}} \leq \left(\mathbb{E}(|X|^p)\right)^{\frac{1}{p}}+\left(\mathbb{E}(|Y|^p)\right)^{\frac{1}{p}}, \tag{8}$$

where $r \geq 1$ and $1 < p, q < \infty$, with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, for $1 \leq p < p'$, we have $\left(\mathbb{E}(|X|^p)\right)^{\frac{1}{p}} \leq \left(\mathbb{E}(|X|^{p'})\right)^{\frac{1}{p'}}$.

We suppose the following hypothesis (H) which ensures the existence and uniqueness of a solution of the fully coupled G-FBSDE (1).

Hypothesis(H)

(H1) For fixed $x \in \mathbb{R}^d, y \in \mathbb{R}^n, z \in \mathbb{R}^{n\times l}$ suppose that $b(.,x,y,z),\ \sigma(.,x,y),\ h_{ij}(.,x,y) \in M_G^2([0,T])$, also for fixed x,y,z,m, we have $f(.,x,y,z,m), g_{ij}(.,x,y) \in M_G^2([0,T]), \Phi \in L_G^p(\Omega_T)$

(H2) We suppose also that:

$|b(s,x,y,z)-b(s,x',y',z')|^2 \leq k(|x-x'|^2+|y-y'|^2+|z-z'|^2)$,

$|h_{i,j}(s,x,y)-h_{i,j}(s,x',y')|^2 \leq k(|x-x'|^2+|y-y'|^2)$,

$|\sigma(s,x,y)-\sigma(s,x',y')|^2 \leq k_1|x-x'|^2+k_2|y-y'|^2$,

$|f(s,x,y,z,m)-f(s,x',y',z',m')|^2 \leq k(|x-x'|^2+|y-y'|^2+|z-z'|^2+|m-m'|^2)$,

$|g_{i,j}(s,x,y)-g_{i,j}(s,x',y')|^2 \leq k(|x-x'|^2+|y-y'|^2)$,

$|\Phi(x)-\Phi(x')|^2 \leq k|x-x'|^2$.

3. Main Results

In this section, we present our main result which is about the existence of a unique solution of our G-FBSDE (1) given in the following theorem.

Theorem 1. *Under our assumption (H), there exist a constant $C_k > 0$ dependant on the Lipschitz coefficients k, k_1, k_2, such that for all $0 < T \leq C_k$, the G-FBSDE (1) has a unique solution $(X,Y,Z,K) \in H_{G,T}^{2,2}$.*

Remark 1. This result can't be extended to the case where σ depend to Z with $|\sigma(s,x,y,z)-\sigma(s,x',y',z')|^2 \leq k_1|x-x'|^2+k_2|y-y'|^2+k_3|z-z'|^2$, indeed, the system:

$$\begin{cases} dX_s = Z_s dB_s, \\ dY_s = Z_s dB_s + dM_s, \ s\in[t,T], \\ X_t = x, \ Y_T = X_T, M_t = 0, \end{cases} \tag{9}$$

has an infinity number of solutions, because for any $Z \in H_G^p(0,T)$ and any decreasing G-martingales M such that $M_t = 0$ and $M_T \in \mathbb{L}_G^p(\Omega_T)$; the tuple (X,Y,Z,M) with $X_u := x + \int_t^u Z_s dB_s$ and $Y_u := x - M_T$ is a solution of the G-FBSDE (9)

Proof. The proof of Theorem 1 is based on the fixed point theorem, and for this, we define the following map: For $(X,Y,Z,M),(U,V,W,R) \in M_G^2(0,T)\times \mathfrak{S}_G^2(0,T) := H_{G,T}^{2,2}$ we define $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{M})$(resp. $(\tilde{U},\tilde{V},\tilde{W},\tilde{R})$) as the image of (X,Y,Z,M) (resp.(U,V,W,R)) by the map $\digamma$ where:
$\mathfrak{S}_G^2(0,T) := S_G^2(0,T)\times H_G^2(0,T)\times \mathbb{L}_G^2(\Omega_T)$, and

$$\digamma : H_{G,T}^{2,2} \to H_{G,T}^{2,2}, \ (X,Y,Z,M) \to \digamma(X,Y,Z,M) := (\tilde{X},\tilde{Y},\tilde{Z},\tilde{M}) \tag{10}$$

where $(\tilde{X},\tilde{Y},\tilde{Z},\tilde{M})$ are defined by:

for $t\in[0,T]$,

$$\tilde{X}_t = x + \int_0^t b(s,\tilde{X}_s,Y_s,Z_s)ds + \int_0^t \sigma(s,\tilde{X}_s,Y_s)dB_s + \int_0^t h(s,\tilde{X}_s,Y_s)d\langle B\rangle_s, \tag{11}$$

and,

$$\tilde{Y}_t = \Phi(X_T) + \int_t^T f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s)ds + \int_t^T g(s,X_s,\tilde{Y}_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - \int_t^T d\tilde{M}_s. \tag{12}$$

Remark 2.

(1) The space $H_{G,T}^{2,2}$ is a Banach space as a product of Banach spaces $M_G^2(0,T)$, $S_G^2(0,T)$, $H_G^2(0,T)$, and $L_G^2(\Omega_T)$.

(2) The map $\mathcal{F}$ is well defined. Indeed, because Y and Z are given respectively in $S^2_G(0,T)$, $H^2_G(0,T)$, and the coefficients b, σ and h hold the conditions (H), then $\tilde{X}$ in equation (11) exists (see e.g. [29]) as the solution of this equation and belongs to $M^2_G([0,T])$, and so we plug-in $\tilde{X}$ in the G-BSDE equation (12); then we have also $\tilde{Y}_s, \tilde{Z}_s, \tilde{M}_s$ which exist (see e.g. [18]) as the solution of the BSDE (12) and belong to $\mathfrak{S}^2_G(0,T) = S^2_G(0,T) \times H^2_G(0,T) \times \mathbb{L}^2_G(\Omega_T)$ for fixed $(\tilde{X}, M) \in M^2_G(0,T) \times \mathbb{L}^2_G(\Omega_T)$.

The idea of the proof is to show that the map $\mathcal{F}$ is a contraction, and for this, let the following notations: $\bar{x}_s = \tilde{X}_s - \tilde{U}_s, \bar{z} = \tilde{Z}_s - \tilde{W}_sand\bar{y}_s = \tilde{Y}_s - \tilde{V}_s, x_s = X_s - U_s, y_s = Y_s - V_s, w_s = Z_s - W_s, \bar{m}_t = \tilde{M}_t - \tilde{R}_t, m_t = M_t - R_t$.

So,

$$\bar{x}_t = \int_0^t (b(s, \tilde{X}_s, Y_s, Z_s) - b(s, \tilde{U}_s, V_s, W_s))ds + \int_0^t (\sigma(s, \tilde{X}_s, Y_s) - \sigma(s, \tilde{U}_s, V_s))dB_s + \int_0^t (h(s, \tilde{X}_s, Y_s) - h(s, \tilde{U}_s, V_s))d\langle B\rangle_s \tag{13}$$

and,

$$\bar{y}_t = \tilde{Y}_t - \tilde{V}_t = \bar{y}_T + \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s))ds + \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s))d\langle B\rangle_s - \int_t^T (\tilde{Z}_s - \tilde{W}_s)dB_s - \int_t^T d(\tilde{M}_t - \tilde{R}_t). \tag{14}$$

The proof that the map $\mathcal{F}$ is a contraction mapping is based on the following lemmas:

Lemma 4. *For a given $\beta > 0$, there exist positive constants C_1, C_2 depending only on $k, k_1, k_2, \bar{l}, \underline{l}, T, \beta$ s.t.:*

$$\int_0^T e^{-2\beta t} \hat{\mathbb{E}}(|\bar{x}_t|^2)dt \le C_1 \hat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |y_t|^2\Big) + C_2 \hat{\mathbb{E}}\Big(\int_0^T |z_s|^2 ds\Big). \tag{15}$$

Proof. For the proof see Appendix A. □

Lemma 5. *For a given $\beta > 0$, there exist positive constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$*

depending only on $k, k_1, k_2, \bar{l}, \underline{l}, T, \beta$ *s.t.:*

$$\hat{\mathbb{E}}(|\bar{x}_T|^2) \leq \tilde{C}_1 \hat{\mathbb{E}}(\sup_{t\in[0,T]} |y_t|^2) + \tilde{C}_2 \hat{\mathbb{E}}\Big(\int_0^T |z_s|^2 ds\Big) + \tilde{C}_3 \int_0^T e^{-2\beta t} \hat{\mathbb{E}}(|\bar{x}_t|^2) dt. \quad (16)$$

Proof. For the proof see Appendix B. □

Lemma 6. *There exist a positive constants* C_3, C_4, C_5, C_6*, depending only on* $T, k, \bar{l}, \underline{l}, \beta$ *s.t.:*

$$\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |\bar{y}_s|^2\Big) \leq C_3 \hat{\mathbb{E}}\left(|m_T|^2\right) + C_4 \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2) ds + C_5 \hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) + C_6 \hat{\mathbb{E}}(|\bar{x}_T|^2). \quad (17)$$

Proof. For the proof see Appendix C. □

Lemma 7. *There exist a positive constants* C_7, C_8, C_9, C_{10}*, depending only on* $T, k, \bar{l}, \underline{l}, \beta$ *s.t.:*

$$\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \leq C_7 \hat{\mathbb{E}}(\sup_{s\in[0,T]} |\bar{y}_s|^2) + C_8 \hat{\mathbb{E}}\left(|m_T|^2 ds\right) + C_9 \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2 ds) + C_{10} \hat{\mathbb{E}}(|\bar{x}_T|^2). \quad (18)$$

Proof. For the proof see Appendix D. □

Lemma 8. *There exist a positive constants* $C_{11}, C_{12}, C_{13}, C_{14}, C_{15}$*, depending only on* $T, k, \bar{l}, \underline{l}, \beta$ *s.t.:*

$$\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq C_{11} \hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |m_s|^2\Big) + C_{12} \hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |\bar{y}_s|^2\Big) + C_{13} \hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) + C_{14} \int_0^T e^{-2\beta s} \hat{\mathbb{E}}(|x_s|^2) ds + C_{15} \hat{\mathbb{E}}(|\bar{x}_T|^2). \quad (19)$$

Proof. For the proof see Appendix E. □

Now we return to the proof of our main results. From Lemmas 4 to 8, for a given $\beta > 0$, there exists a constant $C = C_{k,k_1,k_2,\bar{l},\underline{l},\beta} > 0$, such that

$\forall T; 0 < T \leq C$, and there exist constants $\omega_1, \omega_2, \omega_3, \omega_4 \in (0,1)$ depending only on $k, T, \beta, \bar{l}, \underline{l}$ s.t.:

$$\int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|\bar{x}_s|^2)ds + \hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) + \hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) + \hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{m}_s|^2\Big)$$
$$\leq \omega_1 \int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|x_s|^2)ds + \omega_2\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|y_s|^2\Big)$$
$$+ \omega_3\hat{\mathbb{E}}\Big(\int_0^T |z_s|^2 ds\Big) + \omega_4\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|m_s|^2\Big). \tag{20}$$

It is noted that the following two norms are equivalent on

$$\bar{M}_G^p(0,T), \int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt \sim \int_0^T \hat{\mathbb{E}}(|\bar{x}_t|^2)dt$$

and so, the map $\mathcal{F}$ is a contracting mapping from the Banach space $H_{G,T}^{2,2}$ to itself, which ensures the existence of a unique fixed point $(X,Y,Z,M) \in H_{G,T}^{2,2}$ which is (from the definition of $\mathcal{F}$) the solution of the FBSDE (1) and this proves our main theorem. □

Appendices

Appendix A. Proof of Lemma 4

We have from (13),

$$\bar{x}_t = \int_0^t (b(s,\tilde{X}_s,Y_s,Z_s) - b(s,\tilde{U}_s,V_s,W_s))ds + \int_0^t (\sigma(s,\tilde{X}_s,Y_s)$$
$$- \sigma(s,\tilde{U}_s,V_s))dB_s + \int_0^t (h(s,\tilde{X}_s,Y_s) - h(s,\tilde{U}_s,V_s))d\langle B\rangle_s$$

By Young's inequality and simple calculations, we have for $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\hat{\mathbb{E}}(|\bar{x}_t|^2) \leq \Big(\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2}\Big)\hat{\mathbb{E}}(|\bar{x}_t|^2)$$
$$+ \frac{t}{2\varepsilon_1}\hat{\mathbb{E}}\Big(\int_0^t |(b(s,\tilde{X}_s,Y_s,Z_s) - b(s,\tilde{U}_s,V_s,W_s))|^2 ds\Big)$$
$$+ \frac{1}{2\varepsilon_2}\hat{\mathbb{E}}\Big(\Big|\int_0^t (\sigma(s,\tilde{X}_s,Y_s) - \sigma(s,\tilde{U}_s,V_s))dB_s\Big|^2\Big)$$
$$+ \frac{1}{2\varepsilon_3}\hat{\mathbb{E}}\Big(\Big|\int_0^t (h(s,\tilde{X}_s,Y_s) - h(s,\tilde{U}_s,V_s))d\langle B\rangle_s\Big|^2\Big).$$

Using Lemma 1 and Lipschitz conditions

$$\hat{\mathbb{E}}(|\bar{x}_t|^2)\leq\Big(\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}\Big)\,\hat{\mathbb{E}}(|\bar{x}_t|^2)+\frac{kt}{2\varepsilon_1}\hat{\mathbb{E}}\Big(\int_0^t(|\bar{x}_s|^2+|y_s|^2+|z_s|^2)ds\Big)$$
$$+\frac{C_2\bar{l}^2}{2\varepsilon_2}\hat{\mathbb{E}}\Big(\int_0^t(k_1|\bar{x}_s|^2+k_2|y_s|^2)ds\Big)+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\hat{\mathbb{E}}\Big(\int_0^t(|\bar{x}_s|^2+|y_s|^2)ds\Big)$$
$$\Big(1-(\frac{\varepsilon_1}{2}+\frac{k_1\varepsilon_2}{2}+\frac{\varepsilon_3}{2})\Big)\,\hat{\mathbb{E}}(|\bar{x}_t|^2)$$
$$\leq\Big(\frac{k_1T}{2\varepsilon_1}+\frac{kC_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\Big)\int_0^t\hat{\mathbb{E}}(|\bar{x}_s|^2)ds$$
$$+T\Big(\frac{kT}{2\varepsilon_1}+\frac{k\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\Big)\,\hat{\mathbb{E}}(\sup_{s\in[0,T]}|y_s|^2)+\frac{kT}{2\varepsilon_1}\hat{\mathbb{E}}\Big(\int_0^T|z_s|^2ds\Big)$$

We multiply both sides of the inequality by $e^{-2\beta t}$ and integrate them on $[0,T]$, and simple calculations give

$$\Big(1-(\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2})\Big)\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt$$
$$\leq\frac{1}{2\beta}\Big(\frac{kT}{2\varepsilon_1}+\frac{k_1C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\Big)\int_0^T e^{-2\beta s}\hat{\mathbb{E}}(|\bar{x}_s|^2)ds$$
$$+\frac{T}{2\beta}\Big(\frac{kT}{2\varepsilon_1}+\frac{k_2C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\Big)(1-e^{-2\beta T})\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|y_s|^2\Big)$$
$$+\frac{kT}{4\beta\varepsilon_1}(1-e^{-2\beta T})\hat{\mathbb{E}}\Big(\int_0^T|z_s|^2ds\Big).$$

Let

$$\varepsilon=\Bigg(1-\Big(\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{2}+\frac{\varepsilon_3}{2}+\frac{1}{2\beta}\big(\frac{kT}{2\varepsilon_1}+\frac{k_1C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\big)\Big)\Bigg)$$

for big β and small $\varepsilon_1,\varepsilon_2,\varepsilon_3$, we have that $\varepsilon>0$, and then

$$\int_0^T e^{-2\beta t}\hat{\mathbb{E}}(|\bar{x}_t|^2)dt$$
$$\leq\frac{T}{2\varepsilon\beta}\Big(\frac{kT}{2\varepsilon_1}+\frac{k_2C_2\bar{l}^2}{2\varepsilon_2}+\frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3}\Big)(1-e^{-2\beta T})\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|y_s|^2\Big)$$
$$+\frac{k}{4\varepsilon\beta\varepsilon_1}(1-e^{-2\beta T})\hat{\mathbb{E}}\Big(\int_0^T|z_s|^2ds\Big).$$

Appendix B. Proof of Lemma 5

We have

$$\bar{x}_T = \int_0^T (b(s,\tilde{X}_s,Y_s,Z_s) - b(s,\tilde{U}_s,V_s,W_s))ds + \int_0^T (\sigma(s,\tilde{X}_s,Y_s) - \sigma(s,\tilde{U}_s,V_s))dB_s + \int_0^T (h(s,\tilde{X}_s,Y_s) - h(s,\tilde{U}_s,V_s))d\langle B\rangle_s$$

By using the same technique as the previous lemma, we have that

$$\hat{\mathbb{E}}(|\bar{x}_T|^2) \le \frac{C}{\bar{\varepsilon}'}\Big(\frac{kT}{2\varepsilon_1'} + \frac{k_1\bar{l}^2}{2\varepsilon_2'} + \frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3'}\Big)\int_0^T \hat{\mathbb{E}}(|\bar{x}_t|^2)e^{-2\beta s}ds + \frac{T}{\bar{\varepsilon}'}\Big(\frac{kT}{2\varepsilon_1'} + \frac{k_2\bar{l}^2}{2\varepsilon_2'} + \frac{kT(\bar{l}+\underline{l})^2}{32\varepsilon_3'}\Big)\hat{\mathbb{E}}\Big(\sup_{t\in[0,T]}|y_t|^2\Big) + \frac{kT}{2\bar{\varepsilon}'\varepsilon_1'}\hat{\mathbb{E}}\Big(\int_0^T |z_s|^2 ds\Big)$$

for $\bar{\varepsilon}' = \Big(1 - (\frac{\varepsilon_1'}{2} + \frac{\varepsilon_2'}{2} + \frac{\varepsilon_3'}{2})\Big)$, for strictly positive, and small enough $\varepsilon_1', \varepsilon_2', \varepsilon_3'$.

Appendix C. Proof of Lemma 6

We have

$$\bar{y}_t - \bar{y}_T = \int_t^T (f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s) - f(s,U_s,\tilde{V}_s,\tilde{W}_s,R_s))ds + \int_t^T (g(s,X_s,\tilde{Y}_s) - g(s,U_s,\tilde{V}_s))d\langle B\rangle_s - \int_t^T (\tilde{Z}_s - \tilde{W}_s)dB_s - \int_t^T d(\tilde{M}_t - \tilde{R}_t).$$

We apply Itô's formula on $|\bar{y}_t|^2$

$$|\bar{y}_t|^2 = -\int_t^T 2\bar{y}_t\bar{z}_s dB_s + |\varphi(\tilde{X_T}) - \varphi(\tilde{U_T})|^2 + \int_t^T 2\bar{y}_t(f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s) - f(s,U_s,\tilde{V}_s,\tilde{W}_s,R_s))ds + \int_t^T 2\bar{y}_t(g(s,X_s,\tilde{Y}_s) - g(s,U_s,\tilde{V}_s)d\langle B\rangle_s - \int_t^T |\bar{Z}_s|^2 d\langle B\rangle_s - \int_t^T 2\bar{y}_t d(\tilde{M}_t - \tilde{R}_t)$$

$$
\begin{aligned}
|\bar{y}_t|^2+\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s \leq & -\int_t^T 2\bar{y}_s\bar{z}_s dB_s + k|\bar{x_T}|^2 \\
& + k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\
& + k\int_t^T 2\bar{y}_s(x_s+\bar{y}_s)d\langle B\rangle_s - \int_t^T 2\bar{y}_s d\bar{m}_s.
\end{aligned}
$$

Let

$$
J_t = \int_0^T 2|\bar{y}_s|d\bar{m}_s + \int_0^T 2\bar{y}_s\bar{z}_s dB_s
$$

$$
\begin{aligned}
\sup_{s\in[0,T]}|\bar{y}_s|^2 \leq & \, 2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T x_s ds + 2k\sup_{s\in[0,T]}|\bar{y}_s|^2\int_0^T ds \\
& + 2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T |\bar{z}_s|ds + 2k\sup_{s\in[0,T]}|\bar{y}_s|\int_0^T |m_s|ds \\
& + k\int_0^T (\frac{1}{\varsigma_1}|x_s|^2 + \varsigma_1|\bar{y}_s|^2)d\langle B\rangle_s \\
& + k\int_t^T |\bar{y}_s|^2 d\langle B\rangle_s + k|\bar{x}_T|^2 + J_T - J_t.
\end{aligned}
$$

By Lemma (3.4) in [18] J_t is a G-martingale. By using Young's and the BDG inequalities, with simple calculations

$$
\begin{aligned}
\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) \leq & \big(\varsigma_3 k + 2kT + \varsigma_4 k + \varsigma_5 k + C_2\bar{l}T\,(k+k\varsigma_1)\big)\,\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) \\
& + \Big(\frac{kT}{\varsigma_3}+\frac{k(\bar{l}+\underline{l})^2T}{16\varsigma_1}\Big)\int_0^T \hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma_4}\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \\
& + k\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\varsigma_5}\hat{\mathbb{E}}(|m_T|^2)
\end{aligned}
$$

$$
\varsigma = 1 - \big(\varsigma_3 k + 2kT + \varsigma_4 k + \varsigma_5 k + C_2\bar{l}T\,(k+k\varsigma_1)\,\big)
$$

$$
\begin{aligned}
\varsigma\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) \leq & \Big(\frac{kT}{\varsigma_3}+\frac{k(\bar{l}+\underline{l})^2T}{16\varsigma_1}\Big)\int_0^T \hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma_4}\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \\
& + k\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\varsigma_5}\hat{\mathbb{E}}(|m_T|^2). \\
\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) \leq & \frac{1}{\varsigma}\Big(\frac{kT}{\varsigma_3}+\frac{k(\bar{l}+\underline{l})^2T}{16\varsigma_1}\Big)\int_0^T \hat{\mathbb{E}}(|x_s|^2)ds + \frac{kT}{\varsigma\varsigma_4}\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \\
& + \frac{k}{\varsigma}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\varsigma\varsigma_5}\hat{\mathbb{E}}(|m_T|^2).
\end{aligned}
$$

Appendix D. Proof of Lemma 7

$$\begin{aligned}\bar{y}_t - \bar{y}_T = &\int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s))ds \\ &+ \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s))d\langle B\rangle_s \\ &- \int_t^T (\tilde{Z}_s - \tilde{W}_s)dB_s - \int_t^T d(\tilde{M}_t - \tilde{R}_t).\end{aligned}$$

We apply Itô's formula on $|\bar{y}_t|^2$

$$\begin{aligned}|\bar{y}_t|^2 = &- \int_t^T 2\bar{y}_t\bar{z}_s dB_s + |\varphi(\tilde{X_T}) - \varphi(\tilde{U_T})|^2 \\ &+ \int_t^T 2\bar{y}_t(f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s))ds \\ &+ \int_t^T 2\bar{y}_t(g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)d\langle B\rangle_s \\ &- \int_t^T |\bar{Z}_s|^2 d\langle B\rangle_s - \int_t^T 2\bar{y}_t d(\tilde{M}_t - \tilde{R}_t)\end{aligned}$$

$$\begin{aligned}\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s \leq &- \int_t^T 2\bar{y}_s\bar{z}_s dB_s + k|\bar{x_T}|^2 \\ &+ k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\ &+ k\int_t^T 2\bar{y}_s(x_s + \bar{y}_s)d\langle B\rangle_s - \int_t^T 2\bar{y}_s d\bar{m}_s\end{aligned}$$

$$\begin{aligned}\hat{\mathbb{E}}\Big(\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s\Big) \leq &\hat{\mathbb{E}}\Big(k|\bar{x_T}|^2 + k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\ &+ k\int_t^T 2\bar{y}_s(x_s + \bar{y}_s)d\langle B\rangle_s - \Big(\int_t^T 2\bar{y}_s d\bar{m}_s + \int_t^T 2\bar{y}_s\bar{z}_s dB_s\Big)\Big)\end{aligned}$$

$$\begin{aligned}\hat{\mathbb{E}}\Big(\int_t^T |\bar{z}_s|^2 d\langle B\rangle_s\Big) \leq &\hat{\mathbb{E}}(k|\bar{x_T}|^2 + k\int_t^T 2(x_s\bar{y}_s + |\bar{y}_s|^2 + \bar{y}_s\bar{z}_s + \bar{y}_s m_s)ds \\ &+ k\int_t^T 2\bar{y}_s(x_s + \bar{y}_s)d\langle B\rangle_s) \\ &+ \hat{\mathbb{E}}\Big(-\Big(\int_t^T 2\bar{y}_s d\bar{m}_s + \int_t^T 2\bar{y}_s\bar{z}_s dB_s\Big)\Big)\end{aligned}$$

With some simple calculations, we have for some strictly positive $\varsigma_1', \varsigma_3', \varsigma_4'$, ς_5'

$$\begin{aligned}\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 d\langle B\rangle_s\Big) \le \hat{\mathbb{E}}\Big(&\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k + \sup_{s\in[0,T]} |\bar{y}_s|^2 \\ &+ \frac{kT'}{\varsigma_3}\int_0^T |x_s|^2 ds + \frac{kT}{\varsigma_4'}\int_0^T |\bar{z}_s|^2 ds + k|\bar{x}_T|^2 + \frac{kT^2}{\varsigma_5'}|m_T|^2 \\ &+ \big(k + k\varsigma_1'\big)\int_0^T |\bar{y}_s|^2 d\langle B\rangle_s + \frac{k}{\varsigma_1'}\int_0^T |x_s|^2 d\langle B\rangle_s\Big),\end{aligned}$$

From Proposition 2

$$\begin{aligned}&\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \\ &\le \frac{1}{\underline{l}^2 c_2}\left(\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k\right)\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) + \frac{kT}{\underline{l}^2 c_2 \varsigma_3'}\int_0^T \hat{\mathbb{E}}(|x_s|^2) ds \\ &\quad + \frac{kT}{\underline{l}^2 c_2 \varsigma_4'}\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) + \frac{k}{\underline{l}^2 c_2}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\underline{l}^2 c_2 \varsigma_5'}\hat{\mathbb{E}}(|m_T|^2) \\ &\quad + \frac{C_2 \bar{l} T}{\underline{l}^2 c_2}\left(k + k\varsigma_1'\right)\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) + \frac{k(\bar{l}+\underline{l})^2 T}{16\varsigma_1'}\int_0^T \hat{\mathbb{E}}(|x_s|^2) ds,\end{aligned}$$

$$\begin{aligned}&\Big(1 - \frac{kT}{\underline{l}^2 c_2 \varsigma_4'}\Big)\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big) \\ &\le \frac{1}{\underline{l}^2 c_2}\Big(\varsigma_3' k + 2kT + \varsigma_4' k + \varsigma_5' k + \frac{C_2 \bar{l} T}{\underline{l}^2 c_2}\left(k + k\varsigma_1'\right)\Big)\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big) \\ &\quad + C\Big(\frac{k(\bar{l}+\underline{l})^2 T}{16\varsigma_1'} + \frac{kT}{\underline{l}^2 c_2 \varsigma_3'}\Big)\int_0^T \hat{\mathbb{E}}(|x_s|^2) e^{-2\beta s} ds \\ &\quad + \frac{k}{\underline{l}^2 c_2}\hat{\mathbb{E}}(|\bar{x}_T|^2) + \frac{kT^2}{\underline{l}^2 c_2 \varsigma_5'}\hat{\mathbb{E}}(|m_T|^2).\end{aligned}$$

Appendix E. Proof of Lemma 8

$$\begin{aligned}\bar{m}_T - \bar{m}_t = &-\bar{y}_t + \bar{y}_T + \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds \\ &+ \int_t^T (g(s, X_s, \tilde{Y}_s) - g(s, U_s, \tilde{V}_s)) d\langle B\rangle_s - \int_t^T (\tilde{Z}_s - \tilde{W}_s) dB_s.\end{aligned}$$

$$\bar{m}_T = \bar{m}_t - \bar{Y}_t + \varphi(\bar{x}_T) + \int_t^T (f(s, X_s, \tilde{Y}_s, \tilde{Z}_s, M_s) - f(s, U_s, \tilde{V}_s, \tilde{W}_s, R_s)) ds$$

$$+\int_t^T (g(s,X_s,\tilde{Y}_s)-g(s,U_s,\tilde{V}_s))d\langle B\rangle_s-\int_t^T \bar{z}_s dB_s.$$

By taking $t=0$, and using that $\bar{m}_t=0$, we have for some $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5$

$$\begin{aligned}|\bar{m}_T|^2 \leq& \frac{\delta_1}{2}|\bar{m}_T|^2+\frac{1}{2\delta_1}\sup_{s\in[0,T]}|\bar{Y}_s|^2+\frac{\delta_2}{2}|\bar{m}_T|^2\\ &+\frac{1}{2\delta_2}\Big|\int_0^T (f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s)-f(s,U_s,\tilde{V}_s,\tilde{W}_s,R_s))ds\Big|^2\\ &+\frac{\delta_3}{2}|\bar{m}_T|^2+\frac{1}{2\delta_3}\Big|\int_0^T (g(s,X_s,\tilde{Y}_s)-g(s,U_s,\tilde{V}_s))d\langle B\rangle_s\Big|^2\\ &+\frac{\delta_4}{2}|\bar{m}_T|^2+\frac{1}{2\delta_4}\Big|\int_0^T \bar{z}_s dB_s\Big|^2+\frac{k\delta_5}{2}|\bar{m}_T|^2+\frac{k}{2\delta_5}|\bar{x}_T|^2.\end{aligned}$$

$$\begin{aligned}\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq& \frac{1}{2}(\delta_1+\delta_2+\delta_3+\delta_4+k\delta_5)\,\hat{\mathbb{E}}(|\bar{m}_T|^2)+\frac{1}{2\delta_1}\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big)\\ &+\frac{T}{2\delta_2}\hat{\mathbb{E}}\Big(\int_0^T |(f(s,X_s,\tilde{Y}_s,\tilde{Z}_s,M_s)-f(s,U_s,\tilde{V}_s,\tilde{W}_s,R_s))|^2 ds\Big)\\ &+\frac{1}{2\delta_3}\hat{\mathbb{E}}\Big(\sup_{t\in[0,T]}\Big|\int_0^T (g(s,X_s,\tilde{Y}_s)-g(s,U_s,\tilde{V}_s))d\langle B\rangle_s\Big|^2\Big)\\ &+\frac{1}{2\delta_4}\hat{\mathbb{E}}\Big(\sup_{t\in[0,T]}\Big|\int_0^T \bar{z}_s dB_s\Big|^2\Big)+\frac{k}{2\delta_5}\hat{\mathbb{E}}(|\bar{x}_T|^2).\end{aligned}$$

From Proposition2, we have for $\delta=1-\frac{1}{2}(\delta_1+\delta_2+\delta_3+\delta_4+k\delta_5)$, and we chose δ_i for $i=1,2,3,4,5$ small enough such that $\delta>0$ and

$$\begin{aligned}\hat{\mathbb{E}}(|\bar{m}_T|^2) \leq& \frac{C}{\delta}\Big(\frac{Tk(\underline{l}+\bar{l})^2}{32\delta_3}+\frac{Tk}{2\delta_2}\Big)\hat{\mathbb{E}}\Big(\int_0^T |x_s|^2 e^{-2\beta s}ds\Big)\\ &+\frac{1}{\delta}\Big(\frac{T^2k(\underline{l}+\bar{l})^2}{32\delta_3}+\frac{T^2k}{2\delta_2}+\frac{1}{2\delta_1}\Big)\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|\bar{y}_s|^2\Big)\\ &+\frac{1}{\delta}\Big(\frac{Tk}{2\delta_2}+\frac{\bar{l}^2C_2}{2\delta_4}\Big)\hat{\mathbb{E}}\Big(\int_0^T |\bar{z}_s|^2 ds\Big)+\frac{Tk}{2\delta\delta_2}\hat{\mathbb{E}}(|m_T|^2)+\frac{k}{2\delta\delta_5}\hat{\mathbb{E}}(|\bar{x}_T|^2).\end{aligned}$$

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