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THE ALGEBRAIC SPLITTING OF $bu \wedge BSO(2n)$

I-MING TSAI[*,a], TSUNG-HSUAN WU[*,b] AND DONG YUNG YAN[c]

[1]Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan.

[a]E-mail: imingtsai@mx.nthu.edu.tw

[b]E-mail: thwu@math.nthu.edu.tw

[c]E-mail: dyyan@math.nthu.edu.tw

Abstract

We show that the mod 2 cohomology of $bu \wedge BSO(2n)$ is isomorphic to a direct sum of E-modules, $E = \mathbb{Z}/2\langle Q_0, Q_1\rangle$, $n \geq 2$. This would give the algebraic splitting of the complex connective K-theory of $BSO(2n)$.

1. Motivation

The stable splitting of complex connective K-theory started with E. Ossa's computation of $bu \wedge \Sigma^{-2}B\mathbb{Z}/2 \wedge B\mathbb{Z}/2$(see [8]), with Bruner and Greenlees giving more results on $bu \wedge BG$ in [5], where BG is the classifying space of some finite group G. For infinite Lie groups, $bu \wedge BO(n)$ has a splitting derived by Wilson and Yan in [17], and from their results Tsung Hsuan Wu proved that $bu \wedge BSO(2n+1)$ also has a stable splitting for the classifying space of odd dimensional special orthogonal groups[15]. The method he used in his paper only applies to the odd case, whereas leaving the even case unresolved. Based on these results I show that there is indeed an explicit algebraic splitting of $bu \wedge BSO(2n)$, and hopefully in the future use it to find the topological splitting of $bu \wedge BSO(2n)$ to complete the question. First we introduce some of the background material that is essential for my results.

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* Corresponding authors.

2. Introduction

We operate our cohomology in $\mathbb{Z}/2$ coefficients, so for example $\tilde{H}^*(X)$ would always stand for $\tilde{H}^*(X,\mathbb{Z}/2)$. A will denote the mod 2 Steenrod Algebra, bu is the spectrum of the complex connective K-theory, with $H^*(bu) \cong A//A(Q_0,Q_1) \cong A \otimes_E \mathbb{Z}/2$, with $E = \mathbb{Z}/2\langle Q_0, Q_1\rangle$ the exterior algebra over Q_0 and Q_1, $Q_0 = Sq^1$, $Q_1 = Sq^3 + Sq^2Sq^1$ are the Milnor primitives. $H\mathbb{Z}/2$ is the $\mathbb{Z}/2$ Eilenberg -MacLane spectrum. If Y is a spectrum, then $\Sigma^m Y$ is the suspended spectrum with $(\Sigma^m Y)_n = Y_{m+n}$. The tensor product $\otimes$ will stand for $\otimes_{Z/2}$, so all the spectra and its homotopy equivalences in this paper are 2-localised. It is a standard fact that $H^*(BO(n)) \cong \mathbb{Z}/2[\omega_1, \omega_2, \ldots, \omega_n]$, $\omega_i \in H^i(BO(n))$ is the i-th Stiefel-Whitney class. For $H^*(BSO(n))$, one uses the 2-fold covering $h_n : BSO(n) \to BO(n)$ to show it is equivalent to $\mathbb{Z}/2[\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_n}]$, where $h_n^*(\omega_i) = \widehat{\omega}_i, 2 \le i \le n$. First we state the main result of this paper.

Theorem 1. *For every $n \ge 2$, the cohomology ring $\tilde{H}^*(BSO(2n))$ as an E-module, is isomorphic to $B_{2n} \oplus D_{2n} \oplus M_{2n}$, where M_{2n} is a free E-module with a basis T_{2n}. D_{2n} is a trivial E-module with trivial generators $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n}}^{2m_n}$, $\sum_{i=1}^n m_i > 0$, $m_i \ge 0$. B_{2n} is an E-module with generators of the form $\langle t_j\widehat{\omega_{2n}}^{2k+1} | t_j \in S_1^*, k \ge 0\rangle$ with $S_1^* \subsetneq T_{2n-1}$, where T_{2n-1} is the basis of the free E-module of $\tilde{H}^*(BSO(2n-1))$, and $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n}}^{2m_n+1}$, $\sum_{i=1}^n m_i \ge 0$, $m_i \ge 0$. The generators of B_{2n} are subject to the relations*

$$Q_0Q_1(t_j\widehat{\omega_{2n}}^{2k+1}) = Q_0Q_1(\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n}}^{2m_n+1}) = 0.$$

Remark 1. For $n = 1$, $\tilde{H}^*(BSO(2))$ is just the cohomology ring $\mathbb{Z}/2[\widehat{\omega}_2]$, which is of course trivial under the E-actions since $Q_i(\widehat{\omega_2}) = 0$.

Next we give the previously related theorems:

Theorem 2 (Theorem 1.1 of [17]). *As an E-module, $\tilde{H}^*(BO(n))$ is isomorphic to $D_1^* \oplus D_2^* \oplus M$, where M is a free E-module, D_1^* is a trivial E-module with E-generators $\omega_2{}^{2m_1}\omega_4{}^{2m_2}\cdots\omega_{2k}{}^{2m_k}$ such that $\sum_{i=1}^k m_i > 0$, $2k \le n$, D_2^* is an E-module free over the exterior algebra on Q_0 with E-generators*

${\omega_1}^{2j+1}{\omega_2}^{2m_1}{\omega_4}^{2m_2}\cdots{\omega_{2l}}^{2m_l}$ *such that* $\sum_{i=1}^{l} m_i \geq 0$, $j \geq 0$, $2l \leq n-1$. *Furthermore, the generators satisfy the relations*

$$Q_0({\omega_1}^{2j+1}{\omega_2}^{2m_1}{\omega_4}^{2m_2}\cdots{\omega_{2l}}^{2m_l}) = Q_1({\omega_1}^{2j-1}{\omega_2}^{2m_1}{\omega_4}^{2m_2}\cdots{\omega_{2l}}^{2m_l}).$$

The finding of the E-module structure of $H^*(BO(n))$ has its roots in Wilson's earlier paper[14], when he was trying to determine the complex cobordism of $BO(n)$, which gave $H^*(BO(n))$ as a sum of E_j-modules with generators, $E_j = \mathbb{Z}/2\langle Q_0, Q_1, \ldots, Q_{j-1}\rangle$ is the exterior algebra of $Q_i, 0 \leq i \leq j \leq n$.

Theorem 3 (Theorem 1.2 of [17])**.** *For* $\forall n \geq 1$, *there is a stable splitting*

$$bu \wedge BO(n) \simeq \Big[\bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{Z}/2\Big] \vee \Big[\bigvee_{\beta} \Sigma^{\beta} bu\Big] \vee \Big[\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge \mathbb{R}P^{\infty}\Big],$$

where $\alpha = degd_j$, d_j *are the free* E*-generators of* M. β *and their degrees corresponds to trivial generators of* D_1^*. γ *and their degrees corresponds to the monomials* ${\omega_2}^{2m_1}{\omega_4}^{2m_2}\cdots{\omega_{2l}}^{2m_l}$ *described in the theorem above.*

Once they have computed the exact E-module structure of $H^*(BO(n))$, they went on to build a stable map by analyzing each monomial generator to see if there is a topological map that realises them. Details will be omitted here. The next 2 theorems are attributed to [15]:

Theorem 4 (Theorem A of [15])**.** *For each* $n \geq 1$, $\tilde{H}^*(BSO(2n+1))$ *is isomorphic to* $D_{2n+1} \oplus M_{2n+1}$ *as an* E*-module, where* D_{2n+1} *is a trivial* E*-module with generators* $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots, \widehat{\omega_{2n}}^{2m_n}$, $\sum_{i=1}^{n} m_i > 0$, $m_i \geq 0$, *and* M_{2n+1} *is a free* E*-module with a basis of* E*-generators* $T_{2n+1} = \langle t_i | i \in \Lambda_{2n+1}\rangle$.

Theorem 5 (Theorem B of [15])**.** *For each* $n \geq 1$, *there is a stable splitting*

$$bu \wedge BSO(2n+1) \simeq \Big[\bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{Z}/2\Big] \vee \Big[\bigvee_{\beta} \Sigma^{\beta} bu\Big],$$

where $\alpha = degt_j$ *are the degrees of the generators of* M_{2n+1}, *and* β *are the degrees of the trivial generators of* D_{2n+1}.

The proof of the splitting of $bu \wedge BSO(2n+1)$ is partially based on the result of the splitting of $bu \wedge BO(n)$, first one uses the Adams spectral sequence [1] to calculate the $E_2^{1,*}$ term of $\tilde{bu}_*(BO(n))$, which is in fact the group $Ext_E^{1,*}(\tilde{H}^*(BO(n)), \mathbb{Z}/2) \cong Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BO(n)))$, where $E_* = \mathbb{Z}/2\langle \xi_1, \xi_2 \rangle$ is the exterior algebra over the Milnor generators ξ_1 and ξ_2 of the dual Steenrod algebra $A_* = \mathbb{Z}/2[\xi_1, \xi_2, \ldots]$. The E-module structure has already been determined in [17], so calculating the Ext groups from the bar and cobar resolutions is just standard routine. Wu's paper [15] states that there is an epimorphism

$$Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BO(2n))) \rightarrow Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BSO(2n+1))),$$

so he was able to use this property to track the generators in

$$Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BSO(2n+1))) \cong Ext_E^{1,*}(\tilde{H}^*(BSO(2n+1)), \mathbb{Z}/2).$$

With $Ext_E^{1,*}(\tilde{H}^*(BSO(2n+1)), \mathbb{Z}/2)$ determined, the E-module structure of $\tilde{H}^*(BSO(2n+1))$ can then come to light, and with it the topological splitting of $bu \wedge BSO(2n+1)$. When investigating the even case, the homomorphism of the Ext groups

$$Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BO(2n-1))) \rightarrow Ext_{E_*}^{1,*}(\mathbb{Z}/2, \tilde{H}_*(BSO(2n)))$$

is not surjective because there is no stable Becker-Gottlieb transfer of the kind $BSO(2n) \rightarrow BO(2n-1)$ (see [4]), so other approaches have to be made.

The paper will use results from the previous papers to determine the E-module structure of $\tilde{H}^*(BSO(2n))$ directly, which is actually related to E-module structure of $\tilde{H}^*(BSO(2n-1))$. First we introduce some material needed for our work.

3. Basic Notions

Before we prove our results some basic machinery must be quoted. As a cohomology operation, the Steenrod Squares Sq^i satisfy the Cartan relation $Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$, $x, y \in H^*(X)$ for a space or spectrum X [12]. As elements of A, the multiplication satifies the Adem relations: $Sq^aSq^b = \sum_{j=0}^{[a/2]} \binom{b-i-j}{a-2j} Sq^{a+b-j}Sq^j$, $0 < a < 2b$. The Milnor primitives

Q_n [10] can be defined inductively as $Q_{n+1} = Q_n Sq^{2^{n+1}} + Sq^{2^{n+1}} Q_n, Q_0 = Sq^1, n \geq 0$. Using these relations we can see that $Q_1 = Sq^3 + Sq^2 Sq^1$, higher primitives won't be needed since we are dealing with $E = \mathbb{Z}/2\langle Q_0, Q_1\rangle$. Note that the Q_n has the special property $Q_n(xy) = Q_n(x)y + xQ_n(y)$, which is quite helpful in our calculations. To use calculate the values of the Stiefel-Whitney classes with E-actions applied to it, one more formula must be mentioned.

Proposition 1 (Wu formula [16])**.** *For the Stiefel-Whitney classes ω_m,*

$$Sq^k(\omega_m) = \sum_{t=0}^{k} \binom{m-k+t-1}{t} \omega_{k-t}\omega_{m+t}, m \geq k.$$

Now we can start the calculations. For $H^*(BSO(n))$, if $2m \leq n$, then $Q_0(\widehat{\omega}_{2m}) = \widehat{\omega}_{2m+1}$, and $Q_1(\widehat{\omega}_{2m}) = \widehat{\omega}_3\widehat{\omega}_{2m} + \widehat{\omega}_{2m+3}$. While for $2m+1 \leq n$, $Q_0(\widehat{\omega}_{2m+1}) = 0$, $Q_1(\widehat{\omega}_{2m+1}) = \widehat{\omega}_3\widehat{\omega}_{2m+1}$. $Q_0Q_1(\widehat{\omega_k})$ can be deduced from these equations. If $k > n$, then we let $\widehat{\omega}_k = 0$. So in $H^*(BSO(2n))$, $Q_0(\widehat{\omega}_{2n}) = 0$, $Q_1(\widehat{\omega}_{2n}) = \widehat{\omega}_3\widehat{\omega}_{2n}$, and $Q_1(\widehat{\omega}_{2n-2}) = \widehat{\omega}_3\widehat{\omega}_{2n-2}$. We now know ow Q_i acts on $\widehat{\omega_k}$, so finding out the value for a monomial composed of $\widehat{\omega_k}$'s is just pure calculation, using the derivation property given above. For a general monomial $\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega_{2n}}^{m_{2n}}$, we give its explicit form for all the E-actions:

$$\begin{aligned}
&Q_0(\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega}_{2n}^{m_{2n}})\\
&=\sum_{k=1}^{n-1} m_{2k}\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega_{2k}}^{m_{2k}-1}\widehat{\omega_{2k+1}}^{m_{2k+1}+1}\cdots\widehat{\omega_{2n}}^{m_{2n}}
\end{aligned}$$

$$\begin{aligned}
Q_1(\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega}_{2n}^{m_{2n}}) &= \Big(\sum_{i=2}^{2n} m_i\Big)\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3+1}\cdots\widehat{\omega}_{2n}^{m_{2n}}\\
&+\sum_{k=1}^{n-2} m_{2k}\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega_{2k}}^{m_{2k}-1}\cdots\widehat{\omega_{2k+3}}^{m_{2k+3}+1}\cdots\widehat{\omega_{2n}}^{m_{2n}}
\end{aligned}$$

$$\begin{aligned}
&Q_0Q_1(\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3}\cdots\widehat{\omega}_{2n}^{m_{2n}})\\
&=(\sum_{i=2}^{2n} m_i)\sum_{j=1}^{n-1} m_{2j}\widehat{\omega_2}^{m_2}\widehat{\omega_3}^{m_3+1}\cdots\widehat{\omega}_{2j}^{m_{2j}-1}\widehat{\omega}_{2j+1}^{m_{2j+1}+1}\cdots\widehat{\omega}_{2n}^{m_{2n}}\\
&\quad+\sum_{i=1}^{n-2}\sum_{j=1}^{n-1} m_{2i}m_{2j}^{(2i)}\widehat{\omega_2}^{m_2^{(2i)}}\cdots\widehat{\omega}_{2j}^{m_{2j}^{(2i)}-1}\widehat{\omega}_{2j+1}^{m_{2j+1}^{(2i)}+1}\cdots\widehat{\omega}_{2n}^{m_{2n}^{(2i)}}
\end{aligned}$$

where $(m_2^{(2i)}, m_3^{(2i)}, \ldots, m_{2n}^{(2i)}) = (m_2, \ldots, m_{2i} - 1, \ldots, m_{2i+3} + 1, \ldots, m_{2n})$. Keep in mind all the coefficients are in mod 2, and if one tries to calculate kerQ_0, kerQ_1 and kerQ_0Q_1, set all the coefficients of the Q_i-polynomial to zero and solve the system of equations. For the sake of simplicity, we denote $\widehat{\omega_2}^{s_2}\widehat{\omega_3}^{s_3}\cdots\widehat{\omega}_{2n}^{s_{2n}} = W(s_2, s_3, \ldots, s_{2n})$ if necessary. Finally let's introduce some concepts related to the E-modules:

Definition 1. E is the exterior algebra of the generators Q_0, Q_1, then an E-module M is a $\mathbb{Z}/2$-module equipped with the module homomorphism $\varphi : E \otimes M \to M$, defined by Q_i-actions on the elements of M, $\varphi(Q_i \otimes m) = Q_i(m)$. A generating set S is a subset of M such that every element in M is an E-linear combination of the elements in S, and the multiplication is defined as $e \cdot m = \varphi(e \otimes m) = e(m)$. The elements of S are called E-generators. An E-submodule N is a $\mathbb{Z}/2$-submodule of M which is closed under the E-actions.

Definition 2. A free E-module M is an E-module with a basis of E-generators $\{r_i | i \in I\}$, such that every element a in M can be represented uniquely by a linear combination of r_i with coefficients in E. r_i are the free E-generators of M.

Definition 3. A trivial E-module D is an E-module generated by a set of E-generators $\{g_j | j \in J\}$, such that for every g_j, $e \cdot g_j = 0, \forall e \in E\backslash\{1\}$. g_j are the trivial E-generators of D.

Note that there are generators that are neither free or trivial, so they should satisfy some E-relations, which will be explained later. With all the materials introduced, we can move on to show our results.

4. Algebraic Splitting of the Spectra

We now enter the main problem. As described in the previous chapter, if one has to know the behaviour of the stable splitting of $bu \wedge BSO(2n)$, first we must find out what the E-module structure looks like. Recall that $H^*(BSO(n)) \cong \mathbb{Z}/2[\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_n}]$, since this is a polynomial ring and the Stiefel Whitney classes are algebraically independent, then it can be viewed as $\mathbb{Z}/2[\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_n}] = \mathbb{Z}/2[\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_{n-1}}][\widehat{\omega_n}]$, then the cohomology of $BSO(2n-1)$ and $BSO(2n)$ are related as polynomial rings

$$H^*(BSO(2n)) \cong \bigoplus_{k\geq 0} H^*(BSO(2n-1))\widehat{\omega_{2n}}^k.$$

Since $H^0 \cong \tilde{H}^0 \oplus \mathbb{Z}/2$, so $\tilde{H}^*(BSO(2n))$ can be viewed as $\tilde{H}^*(BSO(2n-1)) \oplus \bigoplus_{k\geq 1} H^*(BSO(2n-1)\widehat{\omega}_{2n}^k$. The generators of the E-modules D_{2n-1} and M_{2n-1} have been described, but the monomial generators in $\tilde{H}^*(BSO(2n-1))$ might not behave the same way when applied to the E-action in $\tilde{H}^*(BSO(2n))$, owing to the fact that it has an extra variable $\widehat{\omega_{2n}}$. Luckily Proposition 1 tells us that all the E-actions of $\widehat{\omega}_i, 2 \leq i \leq 2n-1$ in $\tilde{H}^*(BSO(2n-1))$ remain the same as in $\tilde{H}^*(BSO(2n))$, so it's an E-submodule of $\tilde{H}^*(BSO(2n))$. We only need to worry about $\widehat{\omega_{2n}}$. $Q_0(\widehat{\omega_{2n}}) = 0$, $Q_1(\widehat{\omega_{2n}}) = \widehat{\omega_3}\widehat{\omega_{2n}}$, so the E-action of all elements in $\tilde{H}^*(BSO(2n))$ can be determined. Now for any monomial $W(s_2, s_3, \ldots, s_{2n})$,

$$\begin{aligned} Q_0 W(s_2, s_3, \ldots, s_{2n}) &= (Q_0 W(s_2, s_3, \ldots, s_{2n-1}))\widehat{\omega_{2n}}^{s_{2n}}, \\ Q_1 W(s_2, s_3, \ldots, s_{2n}) & \\ =&(Q_1 W(s_2, s_3, \ldots, s_{2n-1}) + s_{2n} W(s_2, s_3+1, \ldots, s_{2n-1}))\widehat{\omega_{2n}}^{s_{2n}}, \end{aligned}$$

so from this one can actually see that every E-action on the monomial $W(s_2, s_3, \ldots, s_{2n})$ preserves the degree of $\widehat{\omega_{2n}}$, and no $\widehat{\omega}_i$ can map to $\widehat{\omega_{2n}}$ through the E-actions, hence we've just proved that every $H^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ is in fact an E-submodule of $\tilde{H}^*(BSO(2n))$:

Lemma 1. *For every $k \geq 1, H^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ is an E-submodule of $\tilde{H}^*(BSO(2n))$, and $\tilde{H}^*(BSO(2n-1))$ is an E-submodule of $\tilde{H}^*(BSO(2n))$.*

So from above we can see that $\tilde{H}^*(BSO(2n))$ is actually an infinite direct sum of E-submodules $H^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ and $\tilde{H}^*(BSO(2n-1))$, what is left is the need to determine the exact module structure of each one. From Theorem 4, $\tilde{H}^*(BSO(2n-1)) \cong D_{2n-1} \oplus M_{2n-1}$, so for this reason we choose to divide $H^*(BSO(2n-1))\widehat{\omega_{2n}}^k \cong \tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k \oplus \mathbb{Z}/2\widehat{\omega_{2n}}^k$ into a sum of two modules, with $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ the elements can be represented as E-generators related to D_{2n-1}, M_{2n-1}. While for the elements $\widehat{\omega_{2n}}^k$, from calculations above it is straightforward that when k is even $\widehat{\omega_{2n}}^k$ is trivial, for k odd it satisfies the relation $Q_0Q_1 = 0$. We divide $\tilde{H}^*(BSO(2n-$

$1))\widehat{\omega_{2n}}^k$ into two cases: the first case is when k is even. For any $e \in E$, $e\cdot(W(s_2, s_3, \ldots, s_{2n-1})\widehat{\omega_{2n}}^{2r}) = (e\cdot W(s_2, s_3, \ldots, s_{2n-1}))\widehat{\omega_{2n}}^{2r}$ by direct computation. So $\widehat{\omega_{2n}}^{2r}$ can be looked as a coefficient independent of E multiplied to the monomial $W(s_2, s_3, \ldots, s_{2n-1}) \in \tilde{H}^*(BSO(2n-1))$, from this we can tell that the E-generators $t_i \in M_{2n-1}$, $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n-2}}^{2m_{n-1}} \in D_{2n-1}$ of $\tilde{H}^*(BSO(2n-1))$ when multiplied by $\widehat{\omega_{2n}}^{2r}$ they still retain their property in $\tilde{H}^*(BSO(2n))$: $t_i\widehat{\omega_{2n}}^{2r}$ is a free E-generator and $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n-2}}^{2m_{n-1}}\widehat{\omega_{2n}}^{2r}$ is a trivial E-generator. We summarize this into the following lemma.

Lemma 2. *The direct sum of E-modules $\bigoplus_{r\geq 0}\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^{2r}$ can be further splitted into direct sums of free E-submodules $M_{2n-1}\widehat{\omega_{2n}}^{2r}$ with free E-generators $t_j\widehat{\omega_{2n}}^{2r}$, $\forall t_j \in M_{2n-1}$, and a direct sum of E-trivial submodules $D_{2n-1}\widehat{\omega_{2n}}^{2r}$, with trivial generators $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n-2}}^{2m_{n-1}}\widehat{\omega_{2n}}^{2r}$, $\forall\ \widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n-2}}^{2m_{n-1}} \in D_{2n-1}$.*

Proof. The verification of trivial generators is easy check since the $\widehat{\omega_{2j}}$'s all have even indices and they cannot be generated by other monomials for they are even dimensional Stifel-Whitney classes. For the monomials $t_i\widehat{\omega_{2n}}^{2r}$, suppose there exist coefficients $e_i^{(r)} \in E$ such that $\sum_{j,r} e_i^{(r)}\cdot t_i\widehat{\omega_{2n}}^{2r} = 0$, the E action on them turns out to act on just t_i and multiplied by $\widehat{\omega_{2n}}^{2r}$, which gives us

$$\sum_{i,r} e_i^{(r)}\cdot(t_i\widehat{\omega}_{2n}^{2r}) = \sum_{i,r}(e_i^{(r)}\cdot t_i)\widehat{\omega_{2n}}^{2r} = 0.$$

We rearrange the relation into a polynomial of $\widehat{\omega_{2n}}^{2r}$ with coefficients in $\tilde{H}^*(BSO(2n-1))$, hence they are also a polynomial sum of t_i with E-coefficients. The relation states that the sum is zero, by standard fact on Stiefel-Whitney classes of $BSO(2n)$, they satisfy no polynomial relations whatsoever, which leaves us only the possibility that every coefficient of $\widehat{\omega_{2n}}^{2r}$ must be zero. This means for every fixed r, $\sum_i e_i^{(r)}\cdot t_i = 0$. But t_i are free E-generators in $\tilde{H}^*(BSO(2n-1))$ and $\tilde{H}^*(BSO(2n))$, hence $e_i^{(r)}=0$ for all i and r, showing that $t_i\widehat{\omega_{2n}}^{2r}$ is a generator and free at the same time. $\square$

This concludes the algebraic splitting of the submodules $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ with k even, now we have to deal with the case for k odd, which is a bit more complicated. As stated before when the Q_i-actions applied to the monomials, in particular for $i=1$ we find that

$$\begin{aligned} &Q_1(W(s_2,s_3,\ldots,s_{2n-1})\widehat{\omega_{2n}}^{2r+1}) \\ &=Q_1(W(s_2,s_3,\ldots,s_{2n-1}))\widehat{\omega_{2n}}^{2r+1}+W(s_2,s_3+1,\ldots,s_{2n-1})\widehat{\omega_{2n}}^{2r+1}, \end{aligned}$$

this is the main complication. Again replicating the methods above, any monomial element $W(s_2,s_3,\ldots,s_{2n-1})\in\tilde{H}^*(BSO(2n-1))$ is representable as a finite sum of generators in D_{2n-1}, M_{2n-1} with E-coefficients, when multiplied by $\widehat{\omega_{2n}}^{2r+1}$ it can be rearranged as the following:

$$W(s_2,s_3,\ldots,s_{2n-1})\widehat{\omega_{2n}}^{2r+1}=\Big(\sum_{i,r}e_i^{(r)}\cdot t_i+\sum_{J,r}e_J^{(r)}\cdot d_J\Big)\widehat{\omega_{2n}}^{2r+1},$$

where d_J abbreviates for the trivial generators $\widehat{\omega_2}^{2m_1}\widehat{\omega_4}^{2m_2}\cdots\widehat{\omega_{2n-2}}^{2m_{n-1}}$. This form has yet to be presented in a sum of E-generators with E-coefficients. So $(e_i^{(r)}\cdot t_i)\widehat{\omega_{2n}}^{2r+1}$ and $(e_J^{(k)}\cdot d_J)\widehat{\omega_{2n}}^{2r+1}$ has to be rearranged to a sum of monomials with E-coefficients. For Q_0 we have $Q_0(t_i)\widehat{\omega_{2n}}^{2r+1}=Q_0(t_i\widehat{\omega_{2n}}^{2r+1})$, whilst Q_1 we get $Q_1(t_i)\widehat{\omega_{2n}}^{2r+1}=Q_1(t_i\widehat{\omega_{2n}}^{2r+1})+\widehat{\omega_3}t_i\widehat{\omega_{2n}}^{2r+1}$. On the right side of the relation, we have the anomaly $\widehat{\omega_3}t_i\widehat{\omega_{2n}}^{2r+1}$ which we are not quite sure how to classify it at first glance. But if we look at $\widehat{\omega_3}t_i$ alone we can tell it actually lies in $\tilde{H}^*(BSO(2n-1))$, moreover in M_{2n-1} since D_{2n-1} consists of trivial generators and cannot generate monomials containing odd Stifel-Whitney classes through E-actions. So either $\widehat{\omega_3}t_i=t_j$ is itself another free E-generator, or a E-linear combination of free generators $\sum_j e_j^{(k)}\cdot t_j$. For the first case $Q_1(t_i)\widehat{\omega_{2n}}^{2r+1}=Q_1(t_i\widehat{\omega_{2n}}^{2r+1})+t_j\widehat{\omega_{2n}}^{2r+1}$ is now a formal sum of monomials with E-coefficients, for the second case when we place $\sum_j e_j^{(r)}\cdot t_j$ inside the equation the sum is again multiplied by $\widehat{\omega_{2n}}^{2r+1}$, by decomposing the $e_j^{(r)}$'s we might encounter Q_1 or Q_0Q_1 acting on t_j again, we go back to the previous process to determine the status of $\widehat{\omega_3}t_j$, eventually we will get a finite sum of monomials $t_i\widehat{\omega_{2n}}^{2r+1}$ with coefficients in E (Partially owing to the fact that the Q_i-actions raises the degrees of monomials, so any polynomial must be a finite sum of other monomials in E-coefficients). For $Q_0Q_1(t_i)\widehat{\omega}_{2n}^{2r+1}$, it is just $Q_0Q_1(t_i\widehat{\omega}_{2n}^{2r+1})+Q_0(\widehat{\omega}_3t_i)\widehat{\omega}_{2n}^{2r+1}$, $Q_0(\widehat{\omega}_3t_i)\in M_{2n-1}$ is dealt in the same way above. Summarizing the facts, for $r\geq 0$ every $W(s_2,s_3,\ldots,s_{2n-1})\widehat{\omega_{2n}}^{2r+1}$ can be written as a sum of monomials $t_i\widehat{\omega_{2n}}^{2r+1}, d_J\widehat{\omega_{2n}}^{2r+1}$ with E-coefficients, but we still have to check if they qualify as E-generators. For $d_J\widehat{\omega_{2n}}^{2r+1}$, it is simply a generator since it's a product of even Stiefel-Whitney classes $\widehat{\omega_{2i}}$, which can't be generated by E-actions through other monomials ($Q_i(W) \neq 0$ contains a term which has $\widehat{\omega}_{2p+1}$ as its factor for some p). The monomial $t_i\widehat{\omega_{2n}}^{2r+1}$ is what we need to investigate in the whole problem. Once we have determined its properties we then can claim to know the entire E-module structure of $\tilde{H}^*(BSO(2n))$, concluding our question. We now bring the result on the odd case of $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k$:

Lemma 3. *For $\forall r \geq 0$, we have the E-module isomorphism $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^{2r+1} \cong B_{2n}^{(r)} \oplus M_{2n}^{(r)}$, where $M_{2n}^{(r)}$ is a free E-submodule with a basis of free generators of the form $\langle t_i\widehat{\omega_{2n}}^{2r+1} | t_i \in S_0^*\rangle$ for some set $S_0^* \subsetneq T_{2n-1}$, where T_{2n-1} is the basis for the free E-module M_{2n-1} of $\tilde{H}^*(BSO(2n-1))$. $B_{2n}^{(r)}$ is a E-submodule with a generating set $\langle t_j\widehat{\omega_{2n}}^{2r+1} | t_j \in S_1^*\rangle \cup \langle d_J\widehat{\omega_{2n}}^{2r+1} | d_J \in D_{2n-1}\rangle$ for some set $S_1^* \subsetneq T_{2n-1}$, $S_0^* \cap S_1^* = \varnothing$. In $B_{2n}^{(r)}$, every generator is subject to the E-relation*

$$Q_0Q_1(t_j\widehat{\omega_{2n}}^{2r+1}) = Q_0Q_1(d_J\widehat{\omega_{2n}}^{2r+1}) = 0.$$

Proof. First look at $d_J\widehat{\omega_{2n}}^{2r+1}$ which is fairly easy to verify. Its status as an E-generator has been verified above, so we check the relation it satisfies. $Q_1(d_J\widehat{\omega_{2n}}^{2r+1}) = \widehat{\omega_3}d_J\widehat{\omega_{2n}}^{2r+1}$, now if we apply the Q_0-action to it, the result is zero because $Q_0(\widehat{\omega_3}) = Q_0(d_J\widehat{\omega_{2n}}^{2r+1}) = 0$, and the fact that $Q_0(xy) = Q_0(x)y + xQ_0(y)$. For $t_i\widehat{\omega_{2n}}^{2r+1}$, the entire collection of these monomials with $d_J\widehat{\omega_{2n}}^k$ altogether would of course generate $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^{2r+1}$, but they need to be classified. To do this we select all the $t_i\widehat{\omega_{2n}}^{2r+1}$ from the set $T_{2n}^{(r)} = \langle t_i\widehat{\omega_{2n}}^{2r+1} | t_i \in T_{2n-1}\rangle$ such that they qualify as E-generators, i.e., they themselves are not generated by other $t_l\widehat{\omega_{2n}}^{2r+1}$'s, and will also generate $M_{2n-1}\widehat{\omega_{2n}}^{2r+1}$. Such selection exists, for we can rearrange $T_{2n}^{(r)}$ into an infinite pairwise disjoint union of sets $W_{2n,\eta}^{(r)} = \langle t_i\widehat{\omega_{2n}}^{2r+1} | deg(t_i) = \eta\rangle$. $W_{2n,\eta}^{(r)}$ is obviously finite, so we use induction on η. Take the union $\bigcup_{u\leq\eta} W_{2n,u}^{(r)}$, this set spans to some finite E-module, what we do is pick mechanically all the suitable $t_i\widehat{\omega_{2n}}^{2r+1} \in \bigcup_{u\leq\eta} W_{2n,u}^{(r)}$ such that it is not generated by elements inside this E-module other than itself, and at the same time check that the span of these generators is of equal to the span of $\bigcup_{u\leq\eta} W_{2n,u}^{(r)}$. When there are no generators left to pick, move on to search generators of $\bigcup_{u\leq\eta+1} W_{2n,u}^{(r)}$ by looking at the newly included set $W_{2n,\eta+1}^{(r)}$. When $\eta \to \infty$, the span of the selected generators will be the same as the span of the set $T_{2n}^{(r)}$. In this way we will eventually obtain the generating set of $M_{2n-1}\widehat{\omega_{2n}}^{2r+1}$ as required. Call that set $U_{2n}^{(r)}$. The free E-submodule of $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^{2r+1}$ must of course have generators of the form $t_i\widehat{\omega_{2n}}^{2r+1}$ because $d_J\widehat{\omega_{2n}}^{2r+1}$ is relational, so every free generator would lie in $U_{2n}^{(r)}$. We collect all these free generators so they form a basis $V_{2n}^{(r)} = \langle t_j\widehat{\omega_{2n}}^{2r+1} | t_j \in S_0^*\rangle$ for the free submodule $M_{2n}^{(r)}$, with $S_0^* \subsetneq T_{2n-1}$. Now for the set $U_{2n}^{(r)} \backslash V_{2n}^{(r)} = \langle t_j\widehat{\omega_{2n}}^{2r+1} | t_j \in S_1^*\rangle$, the remaining generators $t_j\widehat{\omega_{2n}}^{2r+1}$ should all satisfy some E-relation. Note that S_0^*, S_1^* can select to be the same for every r. Here $U_{2n}^{(r)} \backslash V_{2n}^{(r)}$ and $\langle d_J\widehat{\omega_{2n}}^{2r+1} | d_J \in D_{2n-1}\rangle$ altogether forms the generating set for the module $B_{2n}^{(r)}$. Now pick any generator in $U_{2n}^{(r)} \backslash V_{2n}^{(r)}$, say $t_l\widehat{\omega_{2n}}^{2r+1}$. $(t_l\widehat{\omega_{2n}}^{2r+1})$ is a cyclic E-submodule. Since it's not free, then by definition there should exist some $e_l \in E\backslash\{1\}$ such that $e_l \cdot t_l\widehat{\omega_{2n}}^{2r+1} = 0$. Owing to E as an exterior algebra, we can definitely find some $e_l' \in E$ that $e_l'e_l = Q_0Q_1$, so in the end for every generator $t_l\widehat{\omega_{2n}}^{2r+1}$ not free, the common E-relation they satisfy is $Q_0Q_1(t_l\widehat{\omega_{2n}}^{2r+1}) = 0$. $\square$

Remark 2. The existence of both free and non-free E-generators can be further verified by actually computing $Q_0Q_1(W(s_2, s_3, \dots, s_{2n}))$.
If $W(s_2, s_3, \dots, s_{2n})$ is free then $Q_0Q_1(W)$ cannot be zero. For generators belonging to $B_{2n}^{(r)}$, $Q_0Q_1(W(s_2, s_3, \dots, s_{2n})) = 0$. From chapter 3 the exact solution of $Q_0Q_1(W) = 0$ is solved by setting all its $\mathbb{Z}/2$-coefficients to zero, so $W \in B_{2n}^{(r)}$ should be monomials of the form

$$\widehat{\omega_{2k}}\widehat{\omega_{2n}}^{2r+1} \prod_{i=1}^{n-1} \widehat{\omega_{2i}}^{2s_i} \prod_{j=1}^{n-1} \widehat{\omega_{2j+1}}^{p_j},$$

with $0 \leq k \leq n-1$, $\sum_{i=1}^{n-1} s_i \geq 0, s_i \geq 0$ and $\sum_{j=1}^{n-1} p_j$ even, $p_j \geq 0$ (For $k = 0$, we let $\widehat{\omega_0} = 1$ for convenience). Then t_j would of course be $\widehat{\omega_{2k}} \prod_{i=1}^{n-1} \widehat{\omega_{2i}}^{2s_i} \prod_{j=1}^{n-1} \widehat{\omega_{2j+1}}^{p_j}$. Finally for the special cases $k = 0$ and $p_j = 0$ for all j, the monomial is just the generator $d_J\widehat{\omega_{2n}}^{2r+1}$, while for $1 \leq k \leq n-1$ and $p_j = 0$ for all j, the monomial $\widehat{\omega_{2k}}\widehat{\omega_{2n}}^{2r+1} \prod_{i=1}^{n-1} \widehat{\omega_{2i}}^{2s_i}$ is easily verified as a relational E-generator since it has no odd Stiefel-Whitney classes, and the same can be said of the generator $t_j = \widehat{\omega}_{2k} \prod_{i=1}^{n-1} \widehat{\omega_{2i}}^{2s_i}$ which is free over E.

Remark 3. From [17], when the author describes their E-module decompostion of $\tilde{H}^*(BO(n))$, for the module D_2^*, he gives the E-relations as $Q_0(W_0) = Q_1(W_1)$, this is in fact equivalent to $Q_0Q_1(W_0) = Q_0Q_1(W_1) = 0$, which is done by just applying the actions Q_0 and Q_1 respectively to the whole relation. He chose to write it in that fashion partly because he can compute the exact monomial, whilst for the free generators they can only show the existence of a basis. Note from Remark 2 we also have the similar relation

$$Q_0\Big(\widehat{\omega_{2k+2}}\widehat{\omega_{2n}}^{2r+1}\prod_{i=1}^{n-1}\widehat{\omega_{2i}}^{2s_i}\prod_{j=1}^{n-1}\widehat{\omega_{2j+1}}^{p_j}\Big)=Q_1\Big(\widehat{\omega_{2k}}\widehat{\omega_{2n}}^{2r+1}\prod_{i=1}^{n-1}\widehat{\omega_{2i}}^{2s_i}\prod_{j=1}^{n-1}\widehat{\omega_{2j+1}}^{p_j}\Big).$$

So if one must describe the generators of $B_{2n}^{(r)}$ in full, more information about $t_i \in M_{2n-1}$ must then be given, which is not a very big concern since we are just trying to determine what $\tilde{H}^*(BSO(2n))$ looks like as an E-module.

***Proof of Theorem* 1.** We already know from Lemma 1 that $\tilde{H}^*(BSO(2n)) \cong \tilde{H}^*(BSO(2n-1)) \oplus \bigoplus_{k\geq 1} H^*(BSO(2n-1))\widehat{\omega}_{2n}^k$, each is an E-submodule of $\tilde{H}^*(BSO(2n))$. By the previous Lemmas 2 and 3, the respective E-module decomposition of each $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ is given, so we sum them up and the module $\bigoplus_{k\geq 1}\mathbb{Z}/2\widehat{\omega_{2n}}^k$ to get the result. The direct sum of all the free submodules in $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^k$ for all $k \geq 0$ is the free E-module M_{2n} of $\tilde{H}^*(BSO(2n))$ with basis T_{2n}, while the direct sum of the modules $\bigoplus_{r\geq 0} D_{2n-1}\widehat{\omega_{2n}}^{2r}$ and $\bigoplus_{r\geq 1}\mathbb{Z}/2\widehat{\omega_{2n}}^{2r}$ is the entire trivial E-module D_{2n}, and the sum of all the submodules $B_{2n}^{(r)}$ in $\tilde{H}^*(BSO(2n-1))\widehat{\omega_{2n}}^{2r+1}$ and $\bigoplus_{r\geq 0}\mathbb{Z}/2\widehat{\omega_{2n}}^{2r+1}$ gives our anomaly E-module B_{2n}, every generator belonging to it satisfies the relation $Q_0Q_1 = 0$. The elements of $B_{2n} \oplus D_{2n} \oplus M_{2n}$ of course belongs to $\tilde{H}^*(BSO(2n))$, and since every element of $\tilde{H}^*(BSO(2n))$ is a polynomial $P(\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_{2n}}) = \sum_{k=0}^{q} P_k(\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_{2n-1}})\widehat{\omega_{2n}}^k$, we sort each $P_k(\widehat{\omega_2}, \widehat{\omega_3}, \ldots, \widehat{\omega_{2n-1}})\widehat{\omega_{2n}}^k$ out as a sum of monomial generators with E-coefficients shown in Lemma 2 and Lemma 3, with each generator belonging to one of the modules B_{2n}, D_{2n}, M_{2n}, and so we are done.

5. Addendum

Although the E-module structure of $bu \wedge BSO(2n)$ has been given, the generators of B_{2n} does not have a suitable spectra representing them, this is because the structures of the generators themselves do not correspond to any known space or spectra. If such a space exists, then we can go on and obtain the stable topological splitting. Hopefully positive results will turn out.

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