Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **14** (2019), No. 3, pp. 331-383
DOI: 10.21915/BIMAS.2019303

ASYMPTOTICS OF TORUS EQUIVARIANT SZEGŐ KERNEL ON A COMPACT CR MANIFOLD

WEI-CHUAN SHEN

Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No.1, Sec.4, Roosevelt Road, Taipei 10617, TAIWAN.
E-mail: shen1993@gate.sinica.edu.tw

Abstract

For a compact CR manifold $(X, T^{1,0}X)$ of dimension $2n+1$, $n \geq 2$, admitting a $S^1 \times T^d$ action, if the lattice point $(-p_1, \ldots, -p_d) \in \mathbb{Z}^d$ is a regular value of the associate CR moment map μ, then we establish the asymptotic expansion of the torus equivariant Szegő kernel $\Pi^{(0)}_{m,mp_1,\ldots,mp_d}(x,y)$ as $m \to +\infty$ under certain assumptions of the positivity of Levi form and the torus action on $Y := \mu^{-1}(-p_1, \ldots, -p_d)$.

1. Introduction and Statement of The Main Results

Let $(X, T^{1,0}X)$ be a Cauchy–Riemann (CR for short) manifold of dimension $2n+1$, and $\Box^{(q)}_b$ be Kohn Laplacian for $(0,q)$-forms on X. The orthogonal projection $\Pi^{(q)} : L^2_{(0,q)}(X) \to \ker \Box^{(q)}_b$ is called the Szegő projection, and we call its distributional kernel $\Pi^{(q)}(x,y)$ the Szegő kernel. The study of the Szegő kernel is a classical subject in several complex variables and CR geometry. For example, when X is the boundary of a strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, which implies that X is a strongly pseudoconvex CR manifold, Boutet de Monvel and Sjöstrand [4] proved that when $q = 0$, $\Pi^{(0)}(x,y)$ is a Fourier integral operator with complex valued phase function. This kind of description of kernel function has profound impact in many aspects, such as spectral theory for Toeplitz operator, geometric quantization and Kähler geometry [3, 10, 19, 20, 22, 24].

Received October 15, 2018.
AMS Subject Classification: Primary: 32V20; Secondary: 35S30, 58J40.
Key words and phrases: Equivariant Szegő kernel asymptotics, Analysis on CR manifolds.

In some recent progress [8, 16, 17], people start to consider CR manifolds with Lie group action G. The study of G-equivariant Szegő kernels is closely related to the problems of equivariant CR embedding and geometric quantization on CR manifolds. The goal of this paper is especially to understand the asymptotic behavior of torus equivariant Szegő kernel. Let us briefly explain out motivation. Within the manifolds drastically studied on this topic, Sasakian manifold, which is a compact, strongly pseudoconvex and torsion free CR manifold, stands for the odd-dimensional counter part in Kähler geometry and serves as a significant example. We say a CR manifold $(X, T^{1,0}X)$ is a quasi-regular Sasakian manifold if it admits a CR and transversal circle action. If X is quasi-regular, in [12] they showed that $\dim H^0_{b,m}(X) \approx m^n$ and the Szegő kernel for $H^0_{b,m}(X)$ admits a full asymptotic expansion as $m \to +\infty$, where $H^0_{b,m}(X)$ is the space of m-th Fourier component with respect to the circle action of the global CR functions on X. We say a CR manifold $(X, T^{1,0}X)$ is an irregular Sasakian manifold if it endows with a CR transversal $\mathbb{R}$-action, which does not come from any circle action. Suppose now X is irregular and T be the fundamental vector field of the $\mathbb{R}$-action. Take a $\mathbb{R}$-invariant L^2-inner product on X and consider the weak maximal extension of T on L^2 functions, then in [11] it was shown that T is a self-adjoint operator, and the spectrum of T, denoted by $\mathrm{Spec}(T)$, is a countable subset in $\mathbb{R}$. Moreover, all the spectrum of T are eigenvalues. On irregular Sasakian manifolds, it is important to understand the space $H^0_{b,\alpha}(X) := \{u \in \mathscr{C}^\infty(X) : \overline{\partial}_b u = 0, Tu = i\alpha u\}$, where $\overline{\partial}_b$ denotes the tangential Cauchy–Riemann operator on X. Different from the quasi-regular situation, in general it is very difficult to see which $\alpha \in \mathrm{Spec}(T)$ makes $\dim H^0_{b,\alpha}(X) > 0$. It is revealed in [13] that if we sum over α between 0 and k then the weigted Szegő kernel for the space $\bigoplus_{\substack{\alpha\in\mathrm{Spec}(T)\\ 0\le\alpha\le k}} H^0_{b,\alpha}(X)$ admits an asymptotic expansion in k as $k \to +\infty$. Accordingly, there are many $\alpha \in \mathrm{Spec}(T)$ such that $H^0_{b,\alpha}(X)$ is non-trivial, and it is natural to fix such α and consider the Szegő kernel for the space $H^0_{b,m\alpha}(X)$ as $m \to +\infty$. This is the motivation of our work. In fact, in [11, Section 3] they pointed out that for an irregular Sasakian manifold X, the $\mathbb{R}$-action on X comes from a CR torus action denoted by T^{d+1}, the Reeb vector field T can also be induced by T^{d+1} and the spectrum $\mathrm{Spec}(T) = \{\mu_0 p_0 + \mu_1 p_1 + \cdots + \mu_d p_d : (p_0, p_1, \ldots, p_d) \in \mathbb{Z}^{d+1}\}$, where $\mu_0, \ldots \mu_d$ are real numbers linearly independent over $\mathbb{Q}$. Hence the problem above is equivalent to the study of the Szegő kernel for $H^0_{b,mp_0,mp_1,\ldots,mp_d}(X)$

as $m \to +\infty$ for some lattice point $(p_0, p_1, \dots, p_d)$. For simplicity, we take $p_0 = 1$ in this article. Write $T^{d+1} = S^1 \times T^d$. Suppose that the S^1-action is CR and transversal and the T^d-action is CR. We prove that if the lattice point $(-p_1, \dots, -p_d)$ is a regular value of the associate CR moment map μ, see (1.1), then there is a full asymptotic expansion of the torus equivariant Szegő kernel as $m \to +\infty$ under certain assumptions of the positivity of Levi form and the torus action on $Y := \mu^{-1}(-p_1, \dots, -p_d)$. In particular, for $\alpha := \mu_0 + \mu_1 p_1 + \cdots + \mu_d p_d \in \operatorname{Spec}(T)$, the space $H^0_{b,m\alpha}(X)$ is non-trivial as $m \to +\infty$.

We now briefly introduce some notations and the main results. Let T_0 be the induced vector field of the S^1-action, and $T_1, \dots, T_d$ be the ones for T^d-action. In other words, $T_0 u(x) := \frac{\partial}{\partial\theta}\big|_{\theta=0} u(e^{i\theta} \circ x)$ and $T_j u(x) := \frac{\partial}{\partial\theta_j}\Big|_{\theta_j=0} u\left((1, \dots, e^{i\theta_j}, \dots, 1) \circ x\right)$, for $j = 1, \dots, d$ and $u \in \mathscr{C}^\infty(X)$. We define T_0 and T_j act on $(0,q)$-forms via Lie derivatives $\mathcal{L}_{T_0}$ and $\mathcal{L}_{T_j}$, $j = 1, \dots, d$, respectively. Choose T_0 to be the Reeb vector field on X, see (2.1), and ω_0 to be its dual one form, see (2.2). Fix a lattice point $(p_1, \dots, p_d) \in \mathbb{Z}^d$ and any $m \in \mathbb{N} := \{1, 2, 3, \dots\}$, we consider the space of equivariant smooth $(0,q)$-forms

$$\Omega^{(0,q)}_{m,mp_1,\dots,mp_d}(X) := \{u \in \Omega^{(0,q)}(X) : -iT_0 u = mu, -iT_j u = mp_j u, j = 1, \dots, d\}.$$

Since the group action is assumed to be CR, we can take the $\overline{\partial}_b$-subcomplex $\left(\overline{\partial}_b, \Omega^{(0,\bullet)}_{m,mp_1,\dots,mp_d}(X)\right)$ and define the corresponding Kohn–Rossi cohomology $H^q_{b,m,mp_1,\dots,mp_d}(X)$. Here $\overline{\partial}_b$ we mean the tangential Cauchy–Riemann operator on X with respect to T_0, see (2.3). Let $\langle\cdot|\cdot\rangle$ be a torus invariant Hermitian metric on $\mathbb{C}TX$ and let $(\cdot|\cdot)$ be the torus invariant L^2-inner product on $\Omega^{(0,q)}(X)$ induced by $\langle\cdot|\cdot\rangle$. Let $L^2_{(0,q),m,mp_1,\dots,mp_d}(X)$ be the completion of $\Omega^{(0,q)}_{m,mp_1,\dots,mp_d}(X)$ with respect to the given torus invariant L^2-inner product $(\cdot|\cdot)$ and take the Gaffney extension of Kohn Laplacian $\Box^{(q)}_b$ to the L^2-space, see (2.8), then we have the Hodge theorem such that

$$\begin{aligned} H^q_{b,m,mp_1,\dots,mp_d}(X) &\cong \mathscr{H}^q_{b,m,mp_1,\dots,mp_d}(X) \\ &:= L^2_{(0,q),m,mp_1,\dots,mp_d}(X) \cap \ker \Box^{(q)}_b \subset \Omega^{(0,q)}_{m,mp_1,\dots,mp_d}(X), \end{aligned}$$

and that $H^q_{b,m,mp_1,\dots,mp_d}(X)$ is finite dimensional for each $q = 0, \dots, n$, though the Kohn Laplacian $\Box^{(q)}_b$ is not elliptic, and in general it may not be hypoelliptic, neither. We use the notation $\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,y)$ for the torus equivariant Szegő kernel, which is the distribution kernel of the orthogonal projection $\Pi^{(q)}_{m,mp_1,\ldots,mp_d} : L^2_{(0,q)}(X) \to \mathcal{H}^q_{b,m,mp_1,\ldots,mp_d}(X)$ with respect to $(\cdot|\cdot)$.

Because the group action here is CR, we can check that the one form ω_0 is torus invariant. We hence consider the torus invariant CR moment map

$$\mu : X \to \mathbb{R}^d,\ \mu(x) := \Big(\langle \omega_0(x), T_1(x)\rangle, \ldots, \langle \omega_0(x), T_d(x)\rangle\Big), \tag{1.1}$$

where we identify $(T_eT^d)^* \cong (\mathbb{R}^d)^* \cong \mathbb{R}^d$. In this work, we need

Assumption 1.1. *The given lattice point $(-p_1,\ldots,-p_d) \in \mathbb{Z}^d$ is a regular value of μ.*

Assumption 1.2. *The torus action $S^1 \times T^d$ is free near Y.*

Assumption 1.3. *The induced Levi form $\mathcal{L}$ is positive near the set $Y := \mu^{-1}(-p_1,\ldots,-p_d)$*

The main result in this work is as follows:

Theorem 1.1. *Let $(X, T^{1,0}X)$ be a compact connected CR manifold with $2n+1$, $n \geq 2$, admitting a $S^1 \times T^d$ action, where the S^1-part is CR and transversal and the T^d-part is only required to be CR. For the lattice point $(p_1,\ldots,p_d) \in \mathbb{Z}^d$ and T^d-invariant CR moment map μ satisfying Assumptions* 1.1*,* 1.2 *and* 1.3*, then we have the following full asymptotic expansion for the torus equivariant Szegő kernel: On one hand, let Ω be an open set containing Y, then*

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $(X \setminus \Omega) \times (X \setminus \Omega)$ if $q \in \{0,\ldots,n\}$. On the other hand, for each $p \in Y$, we can find a neighborhood denoted by D_p, such that

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $D_p \times D_p$ if $q \in \{1,\ldots,n\}$. Finally, on $D_p \times D_p$,

$$\Pi^{(0)}_{m,mp_1,\ldots,mp_d}(x,y) \equiv e^{imf(x,y)} b(x,y,m) \bmod O(m^{-\infty}).$$

Here, the phase function $f \in \mathscr{C}^\infty(D_p \times D_p)$ *satisfies* $\mathrm{Im} f \geq 0$, $f(x,x) = 0$ *for all* $x \in Y \cap D_p$ *and* $d_x f(x,x) = -\omega_0(x)$, $d_y f(x,x) = \omega_0(x)$ *for all* $x \in Y \cap D_p$*; also, the symbol satisfies*

$$b(x,y,m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_p \times D_p)$$
$$b(x,y,m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j} b_j(x,y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_p \times D_p)$$

where $b_j(x,y) \in \mathscr{C}^\infty(D_p \times D_p)$, $j = 0, 1, 2, \ldots$ *and* $b_0(x,x) > 0$ *for all* $x \in Y \cap D_p$.

We refer the semi-classical notations such as $O(m^{-\infty})$, $A = B + O(m^{-\infty})$, $S^k_{\mathrm{loc}}(1; D_p \times D_p)$, and asymptotic sums $\sim$ in S^k_{loc} to Section 2.3.

Note that Theorem 1.1 holds on a class of manifolds slightly more general than Sasakian manifolds, for we only assume the Levi form is positive near the submanifold Y instead of being positive on the whole X. Also, from Theorem 1.1, we can conclude:

Corollary 1.1. *Let* $(X, T^{1,0}X)$ *be an irregular Sasakian manifold and* T *be the fundamental vector field induced by the prescribed* $\mathbb{R}$*-action on* X*. It is known that* $\mathbb{R}$*-action comes from a* $S^1 \times T^d$*-action. Assume* $T = \mu_0 T_0 + \mu_1 T_1 + \cdots + \mu_d T_d$*, where* T_0 *is the vector field induced by* S^1*-action and* $T_1, \ldots, T_d$ *are the vector fields induced by* T^d*-action and* $\mu_0, \mu_1, \ldots, \mu_d$ *are real numbers linearly independent over* $\mathbb{Q}$*. If the lattice point* $(p_1, \ldots, p_d) \in \mathbb{Z}^d$ *satisfies Assumptions* 1.1, 1.2*, for* $\alpha := \mu_0 + \mu_1 p_1 + \cdots + \mu_d p_d \in \mathrm{Spec}(T)$, $H^0_{b,m\alpha}(X)$ *is non-trivial as* $m \to \infty$*. Here,* $H^0_{b,m\alpha}(X) := \{u \in \mathscr{C}^\infty(X) : \overline{\partial}_b u = 0,\ Tu = im\alpha u\}$.

We now illustrate the strategy for the proof of the theorem. Since X is NOT assumed to be strongly pseudoconvex in this paper, to establish the asymptotics of torus equivariant Szegő kernel, we do not study Szegő projector $\Pi^{(q)}_{m,mp_1,\ldots,mp_d}$ directly; instead. we need to consider a number $\lambda > 0$ and examine the Szegő projector for lower energy forms $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}$, which is the orthogonal projection from $L^2_{(0,q)}(X)$ to $L^2_{(0,q),m,mp_1,\ldots,mp_d}(X) \cap E((-\infty,\lambda])$. Here, $E((-\infty,\lambda])$ is the spectral projector and E is the spectral measure for the self-adjoint operator $\Box^{(q)}_b$ under Gaffney extension 2.8, respectively. We need to establish the content in Theorem 1.1 in the following version:

Theorem 1.2 (=Theorem 3.1+3.2). *With the same notations and assumptions used in Theorem* 1.1 *and any fixed* $\lambda > 0$, *on one hand, for any open set* Ω *containing* Y,

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $(X \setminus \Omega) \times (X \setminus \Omega)$ *if* $q \in \{0, \ldots, n\}$. *On the other hand, for each* $p \in Y$, *we can find a neighborhood denoted by* D_p, *such that*

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $D_p \times D_p$ *if* $q \in \{1, \ldots, n\}$. *Finally, on* $D_p \times D_p$,

$$\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) \equiv e^{imf(x,y)}b(x,y,m) \bmod O(m^{-\infty})$$

for the same phase function $f(x,y) \in \mathscr{C}^\infty(D_p \times D_p)$ *and symbol* $b(x,y,m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_p \times D_p)$ *in Theorem* 1.1.

We will see in the beginning of the Section 3 that combining the spectral property for Kohn Laplacian and Theorem 1.2 for the case $q = 1$, there is:

Theorem 1.3 (=Theorem 3.3). *With the same notations and assumptions used in Theorem* 1.1, *then for any* $\lambda > 0$, *as* $m \to +\infty$,

$$\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d} = \Pi^{(0)}_{m,mp_1,\ldots,mp_d}.$$

Thus, from Theorem 1.3, Theorem 1.2 actually implies Theorem 1.1.

2. Set Up and Notation

In this section, we recall some basic language in CR geometry, definition and properties of torus equivariant Szegő kernel, and tools in semi-classical analysis and microlocal analysis.

2.1. Cauchy–Riemann manifold and Kohn Laplacian

We follow the presentation in [2, Chapter 4] and [15, Chapter 2]. Let X be a smooth orientable manifold of real dimension $2n+1$, $n \geq 1$, we say X is a Cauchy–Riemann manifold (CR manifold for short) if there is a subbundle $T^{1,0}X \subset \mathbb{C}TX$, such that

(1) $\dim_{\mathbb{C}} T_p^{1,0}X = n$ for any $p \in X$.

(2) $T_p^{1,0}X \cap T_p^{0,1}X = \{0\}$ for any $p \in X$, where $T_p^{0,1}X := \overline{T_p^{1,0}X}$.

(3) For $V_1, V_2 \in \mathscr{C}^\infty(X, T^{1,0}X)$, then $[V_1, V_2] \in \mathscr{C}^\infty(X, T^{1,0}X)$, where $[\cdot,\cdot]$ stands for the Lie bracket between vector fields.

For such subbundle $T^{1,0}X$, we call it a CR structure of the CR manifold X. Fix a Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$. For dimension reason and the assumption that X is orientable, we can always take a non-vanishing real global vector field T (Reeb vector field) such that for all $x \in X$, we have the orthogonal decomposition

$$T_x^{1,0}X \oplus T_x^{0,1}X \oplus \mathbb{C}T(x) = \mathbb{C}T_xX \tag{2.1}$$

and $\langle T|T\rangle = 1$ on X. Denote $\langle\cdot,\cdot\rangle$ to be the paring by duality between vector fields and differential forms, and let $\Gamma : \mathbb{C}T_xX \to \mathbb{C}T_x^*X$ be the anti-linear map given by $\langle u|v\rangle = \langle u, \Gamma v\rangle$ for $u, v \in \mathbb{C}T_xX$, then we can take the induced Hermitian metric on $\mathbb{C}T^*X$ by $\langle u|v\rangle := \langle\Gamma^{-1}v|\Gamma^{-1}u\rangle$ for $u, v \in \mathbb{C}T_x^*X$. Put

$$T^{*1,0}X := \Gamma(T^{1,0}X) = \left(T^{0,1}X \oplus \mathbb{C}T\right)^{\perp} \subset \mathbb{C}T^*X,\ T^{*0,1}X := \overline{T^{*1,0}X}$$

and

$$\omega_0 := -\Gamma(T), \tag{2.2}$$

which is a globally defined non-vanishing 1-form satisfying

$$T_x^{*1,0}X \oplus T_x^{*0,1}X \oplus \mathbb{C}\omega_0(x) = \mathbb{C}T_x^*X,\ \langle\omega_0, T^{1,0}X \oplus T^{0,1}X\rangle = 0 \text{ and } \langle\omega_0, T\rangle = -1.$$

We define the Levi form, which is a globally defined $(1,1)$-form, by

$$\mathcal{L}_x(u,\overline{v}) := \frac{1}{2i}\left\langle\omega_0(x), \left[\mathring{u}, \overline{\mathring{v}}\right](x)\right\rangle,$$

where $\mathring{u}, \mathring{v} \in \mathscr{C}^\infty(X, T^{1,0}X)$ such that $\mathring{u}(x) = u \in T_x^{1,0}X$ and $\mathring{v}(x) = v \in T_x^{1,0}X$. Note that by Cartan's formula we can also express the Levi form by

$$\mathcal{L}_x(u,\overline{v}) = \frac{-1}{2i}\left\langle d\omega_0(x), u \wedge \overline{v}\right\rangle,\ \ u, v \in T_x^{1,0}X.$$

In other words,

$$\mathcal{L}_x := \frac{-1}{2i} d\omega_0(x)\Big|_{T_x^{1,0}X}.$$

Take the Hermitian metric on $\Lambda^r \mathbb{C}T^*X$ by

$$\langle u_1 \wedge \cdots \wedge u_r | v_1 \wedge \cdots v_r \rangle = \det\left(\left(\langle u_j | u_k \rangle\right)_{j,k=1}^r\right),$$

where $u_j, v_k \in \mathbb{C}T^*X$, $j,k = 1, \ldots, r$, and the orthogonal projection

$$\pi^{(0,q)} : \Lambda^q \mathbb{C}T^*X \to T^{*0,q}X := \Lambda^q(T^{*0,1}X)$$

with respect to this Hermitian metric. The tangential Cauchy–Riemann operator is defined to be

$$\overline{\partial}_b := \pi^{(0,q+1)} \circ d : \mathscr{C}^\infty(X, T^{*0,q}X) \to \mathscr{C}^\infty(X, T^{*0,q+1}X). \tag{2.3}$$

By Cartan's formula, we can check that

$$\overline{\partial}_b^2 = 0.$$

Take the L^2-inner product $(\cdot|\cdot)$ on $\mathscr{C}^\infty(X, T^{*0,q}X)$ induced by $\langle \cdot|\cdot \rangle$ via

$$(f|g) := \int_X \langle f|g \rangle dV_X, \ f, g \in \mathscr{C}^\infty(X, T^{*0,q}X),$$

where dV_X is the volume form with expression

$$dV_X(x) = \sqrt{\det\left(\left(\left\langle \frac{\partial}{\partial x_j} \middle| \frac{\partial}{\partial x_k} \right\rangle\right)_{j,k=1}^n\right)} dx_1 \wedge \cdots \wedge dx_{2n+1}$$

in local coordinates $(x_1, \ldots, x_{2n+1})$, and we write $\overline{\partial}_b^*$ to denote the formal adjoint of $\overline{\partial}_b$ with respect to the L^2-inner product $(\cdot|\cdot)$. Denote $\Omega^{(0,q)}(X) := \mathscr{C}^\infty(X, T^{*0,q}X)$, then the Kohn Laplacian is the operator

$$\Box_b^{(q)} := \overline{\partial}_b^* \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b^* : \Omega^{(0,q)}(X) \to \Omega^{(0,q)}(X).$$

We have the following Bochner–Kodaira–Kohn formula for $\Box_b^{(q)}$, see [15, Proposition 2.3]:

Theorem 2.1. *Let $p \in X$, $\{e_j(x)\}_{j=1}^n$ be an orthonormal frame of $T_x^{*0,1}X$ varying smoothly with x in a neighborhood of p, and $\{L_j(x)\}_{j=1}^n$ be the dual frame of $T_x^{0,1}X$. Then we have*

$$\Box_b^{(q)} = \sum_{j=1}^n L_j^* L_j + \sum_{j,k=1}^n (e_j \wedge e_k^{\wedge,*})[L_j, L_k^*] + \epsilon(L) + \epsilon(\overline{L}) + \textit{zero order terms.}$$

Here, $\epsilon(L) := \sum_{j=1}^n a_j L_j$, $\epsilon(\overline{L}) := \sum_{j=1}^n b_j \overline{L}_j$ with smooth coefficeints a_j and b_j for $j,k = 1,\ldots,n$. Also, for each $j,k = 1,\ldots,n$, L_j^* is the formal adjoint of the differential operator L_j, and $e_k^{\wedge,*}$ is the adjoint of $e_k\wedge$ given by $\langle e_k \wedge u|v\rangle = \langle u|e_k^{\wedge,*}v\rangle$ for $u \in T^{*0,q}X$ and $v \in T^{*0,q+1}X$.

2.2. Torus equivariant Szegő kernel

From now on, we assume that X admits a torus action in the form of $S^1 \times T^d$. Consider the vector fields

$$T_0 u(x) := \frac{\partial}{\partial\theta}\bigg|_{\theta=0} u(e^{i\theta} \circ x)$$

and

$$T_j u(x) := \frac{\partial}{\partial\theta_j}\bigg|_{\theta_j=0} u\left((1,\ldots,e^{i\theta_j},\ldots,1) \circ x\right),$$

where $u \in \mathscr{C}^\infty(X)$, $x \in X$, $j = 1,\ldots,d$. Note that T_0 and $T_1,\ldots T_d$ are the induced vector fields of the circle action and torus action, respectively. We also assume that the circle action here is CR and transversal, i.e. T_0 satisfies

$$\left[T_0, \mathscr{C}^\infty(X, T^{1,0}X)\right] \subset \mathscr{C}^\infty(X, T^{1,0}X) \tag{2.4}$$

and

$$\mathbb{C}T_0(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX \text{ for all } x \in X; \tag{2.5}$$

the torus action here is also assumed to be CR, i.e. for all $j = 1,\ldots,d$, T_j has the property

$$\left[T_j, \mathscr{C}^\infty(X, T^{1,0}X)\right] \subset \mathscr{C}^\infty(X, T^{1,0}X). \tag{2.6}$$

We choose T_0 to be our Reeb vector field of X, i.e. $T := T_0$. Accordingly, the cooresponding dual one form ω_0 is torus invariant, because $(e^{i\theta_1},\ldots,e^{i\theta_d}) \circ e^{i\theta} = e^{i\theta} \circ (e^{i\theta_1},\ldots,e^{i\theta_d})$. Benefit from the CR and transversal circle action, we have the BRT coordinates patch from Baouendi–Rothschild–Trèves [1, Proposition I.1]:

Theorem 2.2. *Assume X is a CR manifold admitting a CR and transversal circle action. Fix a point $p \in X$, then there exists $\epsilon > 0$, $\delta > 0$ and an open neighborhood $D := \{(z,\theta) : |z| < \epsilon,\ |\theta| < \delta\}$ and local coordinates $(x_1, x_2, \ldots, x_{2n-1}, x_{2n}, x_{2n+1}) = (z_1, \ldots, z_n, \theta)$ near p, where $z_j := x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, $\theta := x_{2n+1}$, such that $(z(p), \theta(p)) = (0, 0)$ and the fundamental vector field induced by circle action is $T_0 = \frac{\partial}{\partial\theta}$. Also, we can find a real valued function $\phi(z) = \sum_{j=1}^{n} \lambda_j |z_j|^2 + O(|z|^3) \in \mathscr{C}^\infty(D, \mathbb{R})$, where $\{\lambda_j\}_{j=1}^{n}$ are eigenvalues of the Levi form of X at p, such that $\left\{Z_j := \frac{\partial}{\partial z_j} + i\frac{\partial\phi(z)}{\partial z_j}\frac{\partial}{\partial\theta}\right\}_{j=1}^{n}$ forms a basis of $T_x^{1,0}X$ for all $x \in D$. Moreover, we can take $\delta = \pi$ when the action at the point p is free.*

By (2.4) and (2.6), we can check that $\overline{\partial}_b T_j = T_j \overline{\partial}_b$ for all $j = 0, 1, \ldots, d$. For any given lattice point $(p_1, \ldots, p_d) \in \mathbb{Z}^d$, we put

$$\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X) := \{u \in \Omega^{(0,q)}(X) : -iT_0 u = mu, -iT_j u = mp_j u, j = 1, \ldots, d\}.$$

For $\overline{\partial}_b : \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X) \to \Omega^{(0,q+1)}_{m,mp_1,\ldots,mp_d}(X)$, we can take the $\overline{\partial}_b$-subcomplex $\left(\overline{\partial}_b, \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)\right)$ and the corresponding Kohn–Rossi cohomology

$$H^q_{b,m,mp_1,\ldots,mp_d}(X) := \frac{\ker \overline{\partial}_b : \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X) \to \Omega^{(0,q+1)}_{m,mp_1,\ldots,mp_d}(X)}{\operatorname{Im} \overline{\partial}_b : \Omega^{(0,q-1)}_{m,mp_1,\ldots,mp_d}(X) \to \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)}$$

From now on, we pick a $S^1 \times T^d$-invariant Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$. Take the restriction $\overline{\partial}_{b,m,mp_1,\ldots,mp_d} := \overline{\partial}_b |_{\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)}$, then we can check that the formal adjoint of $\overline{\partial}_{b,m,mp_1,\ldots,mp_d}$ satisfies

$$\overline{\partial}^*_{b,m,mp_1,\ldots,mp_d} = \overline{\partial}^*_b |_{\Omega^{(0,q+1)}_{m,mp_1,\ldots,mp_d}(X)} : \Omega^{(0,q+1)}_{m,mp_1,\ldots,mp_d}(X) \to \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X).$$

So we can consider the Fourier component of Kohn Laplacian by

$$\square^{(q)}_{b,m,mp_1,\ldots,mp_d} := \square^{(q)}_b \Big|_{\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)} : \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X) \to \Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X).$$

We pause here for a while to handle some issue on extending Kohn Laplacian to L^2-space as a self-adjoint operator. Let $L^2_{(0,q)}(X)$ to be the completion of $\Omega^{(0,q)}(X)$ with respect to the torus invariant L^2-inner product $(\cdot|\cdot)$ induced by $\langle\cdot|\cdot\rangle$. Denote $\|u\|_X^2 := (u|u)$. Define the weak maximal extension of $\Box_b^{(q)}$ by

$$\begin{aligned}\mathrm{Dom}(\Box_{b,\max}^{(q)}) &:= \{u \in L^2_{(0,q)}(X) : \Box_b^{(q)} u \in L^2_{(0,q)}(X) \text{ in the distribution sense}\},\\ \Box_{b,\max}^{(q)} u &= \Box_b^{(q)} u \text{ in distribution sense, for all } u \in \mathrm{Dom}(\Box_{b,\max}^{(q)})\end{aligned} \tag{2.7}$$

as in [22, Section 3.1]. For such extension, $\Box_b^{(q)}$ may not be a self-adjoint operator, because it is non-elliptic, and it could also be non-hypoelliptic. So in general we have to consider the Gaffney extension as in [22, Proposition 3.1.2], and this extension can make $\Box_b^{(q)}$ to be self-adoint. Precisely, take the maximal extension $\overline{\partial}_b := \overline{\partial}_{b,\max}$ and the Hilbert adjoint of $\overline{\partial}_b$ on the L^2-space by

$$\overline{\partial}^*_{b,H} : \mathrm{Dom}(\overline{\partial}^*_{b,H}) \subset L^2_{(0,q+1)}(X) \to L^2_{(0,q)}(X),$$

where the domain is given by

$$\begin{aligned}\mathrm{Dom}(\overline{\partial}^*_{b,H}) := \{v \in L^2_{(0,q)}(X) : \text{for all } u \in \mathrm{Dom}(\overline{\partial}_b), \text{ the operator } u \mapsto (\overline{\partial}_b u|v)\\ \text{is bounded linear}\}.\end{aligned}$$

By Riesz representation theorem, for all $v \in \mathrm{Dom}(\overline{\partial}^*_{b,H})$, there is a $w \in L^2_{(0,q)}(X)$ such that $(\overline{\partial}_b u|v) = (u|w)$ for all $u \in \mathrm{Dom}(\overline{\partial}_b)$, and $\overline{\partial}^*_{b,H} v := w$. Then the Gaffney extension is given by

$$\begin{aligned}\mathrm{Dom}(\Box_b^{(q)}) &:= \{u \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}^*_{b,H}) : \overline{\partial}_b u \in \mathrm{Dom}\overline{\partial}^*_{b,H}, \overline{\partial}^*_{b,H} u \in \mathrm{Dom}\overline{\partial}_b\},\\ \Box_b^{(q)} u &= (\overline{\partial}^*_{b,H}\overline{\partial}_b + \overline{\partial}_b\overline{\partial}^*_{b,H})u, \text{ for all } u \in \mathrm{Dom}(\Box_b^{(q)}).\end{aligned} \tag{2.8}$$

Let $\Omega_m^{(0,q)}(X) := \{u \in \Omega^{(0,q)}(X) : -iT_0 u = mu\}$ and $L^2_{(0,q),m}(X)$ be the completion of $\Omega_m^{(0,q)}(X)$ with respect to $(\cdot|\cdot)$. We need the following:

Proposition 2.1. *The Gaffney extension and the weak maximal extension for $\Box_b^{(q)}$ coincides on $L^2_{(0,q),m}(X)$, and hence on $L^2_{(0,q),m,mp_1,\ldots,mp_d}(X)$.*

Proof. On one hand, by $\overline{\partial}^*_{b,H}|_{\Omega^{(0,q)}(X)} = \overline{\partial}^*_b$, for all $v \in \Omega^{(0,q)}(X)$,

$$(\Box_b^{(q)} u|v) = (u|\Box_b^{(q)} v) = (u|\Box_{b,\max}^{(q)} v) =: (\Box_{b,\max}^{(q)} u|v), \text{ for all } u \in \mathrm{Dom}(\Box_b^{(q)}),$$

so $\Box^{(q)}_{b,\max}u = \Box^{(q)}_b u \in L^2_{(0,q)}(X)$ for $u \in \mathrm{Dom}(\Box^{(q)}_b)$. This implies that $\mathrm{Dom}(\Box^{(q)}_b) \subset \mathrm{Dom}(\Box^{(q)}_{b,max})$. On the other hand, though Theorem 2.1 suggests that $\Box^{(q)}_b$ is not an elliptic operator (since the principal symbol $\sigma_{\Box^{(q)}_b}(x, \lambda\omega_0(x)) = 0$, for all $\lambda \in \mathbb{R} \setminus \{0\}$), the operator $\Box^{(q)}_b - T_0^2$ is elliptic. After applying the elliptic regularity for $\Box^{(q)}_b - T_0^2$ on the space $L^2_{(0,q),m}(X)$, we can check that

$$\mathrm{Dom}(\Box^{(q)}_{b,\max}) \cap L^2_{(0,q),m}(X) = H^2_{(0,q),m}(X).$$

Here, we let $H^s_{(0,q)}(X)$ to be the Sobolev space of order s for $(0, q)$ forms on X with respect to a Sobolev norm $\|\cdot\|_s$ induced by the invariant L^2-inner product $(\cdot|\cdot)$, and $H^s_{(0,q),m}(X) := H^s_{(0,q)}(X) \cap L^2_{(0,q),m}(X)$. Accordingly, $\mathrm{Dom}(\Box^{(q)}_{b,\max}) \cap L^2_{(0,q),m}(X) = H^2_{(0,q),m}(X) \subset \mathrm{Dom}(\overline{\partial}_b)$; also, by Friedrichs lemma [22, Lemma 3.1.3], for $v \in H^2_{(0,q),m}(X)$, we can find a sequence $\{v_j\}_{j=1}^\infty \subset \Omega^{(0,q)}(X)$ such that $v_j \to v$ in $L^2_{(0,q)}(X)$ and $\overline{\partial}^*_b v_j \to \overline{\partial}^*_b v$ in $L^2_{(0,q-1)}(X)$ as $j \to \infty$. So for all $u \in \mathrm{Dom}(\overline{\partial}_b)$, there is a constant $C_1 := \|\overline{\partial}^*_b v\|_X < \infty$ such that

$$\begin{aligned} |(\overline{\partial}_b u|v)| &= \lim_{j\to\infty} |(\overline{\partial}_b u|v_j)| = \lim_{j\to\infty} |(u|\overline{\partial}^*_b v_j)| \\ &\leq \lim_{j\to\infty} \|u\|_X \|\overline{\partial}^*_b v_j\|_X = \|\overline{\partial}^*_b v\|_X \|u\|_X \leq C_1 \|u\|_X. \end{aligned}$$

Thus,

$$\mathrm{Dom}(\Box^{(q)}_{b,\max}) \cap L^2_{(0,q),m}(X) = H^2_{(0,q),m}(X) \subset \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}^*_{b,H}).$$

Next, we check that if $v \in \mathrm{Dom}(\Box^{(q)}_{b,\max}) \cap L^2_{(0,q),m}(X) = H^2_{(0,q),m}(X)$, then there is a constant $C_2 > 0$ such that

$$|(\overline{\partial}_b u|\overline{\partial}_b v)| \leq C_2 \|u\|_X \text{ for all } u \in \mathrm{Dom}(\overline{\partial}_b).$$

If this is true, then $\overline{\partial}_b : \mathrm{Dom}(\Box^{(q)}_{b,\max}) \to \mathrm{Dom}(\overline{\partial}^*_{b,H})$. Now, let $w := \overline{\partial}_b v \in H^1_{(0,q+1),m}(X)$, by Friedrichs lemma again, we can take a sequence $\{w_j\}_{j=1}^\infty \subset \Omega^{(0,q+1)}(X)$ such that $w_j \to w$ in $L^2_{(0,q)}(X)$ and $\overline{\partial}^*_b w_j \to \overline{\partial}^*_b w$ in $L^2_{(0,q-1)}(X)$. Then for a constant $C_2 := \|\overline{\partial}^*_b w\|_X < \infty$,

$$|(\overline{\partial}_b u|\overline{\partial}_b v)| = \lim_{j\to\infty} |(\overline{\partial}_b u|w_j)| = \lim_{j\to\infty} |(u|\overline{\partial}^*_b w_j)|$$

$$\leq \lim_{j\to\infty} \|u\|_X \|\overline{\partial}_b^* w_j\|_X = \|\overline{\partial}_b^* w\|_X \|u\|_X \leq C_2 \|u\|_X.$$

Similarly, we have $\overline{\partial}_{b,H}^* : \mathrm{Dom}(\Box_{b,\max}^{(q)}) \to \mathrm{Dom}(\overline{\partial}_b)$, so we can conclude

$$\mathrm{Dom}(\Box_{b,\max}^{(q)}) \cap L^2_{(0,q),m}(X) \subset \mathrm{Dom}(\Box_b^{(q)}) \cap L^2_{(0,q),m}(X). \qquad \Box$$

With the proposition, from now on, we take the extension

$$\begin{aligned}\Box^{(q)}_{b,m,mp_1,\ldots,mp_d} \; : \; & \mathrm{Dom}(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}) \subset L^2_{(0,q),m,mp_1,\ldots,mp_d}(X) \\ & \to L^2_{(0,q),m,mp_1,\ldots,mp_d}(X)\end{aligned}$$

where

$$\begin{aligned}\mathrm{Dom}\left(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}\right) := & \left\{u \in L^2_{(0,q)}(X) : \Box_b^{(q)} u \in L^2_{(0,q)}(X)\right\} \\ & \cap L^2_{(0,q),m,mp_1,\ldots,mp_d}(X)\end{aligned}$$

and $\Box^{(q)}_{b,m,mp_1,\ldots,mp_d} u := \Box_b^{(q)} u$ in the distribution sense for all $u \in \mathrm{Dom}\left(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}\right)$, to extend $\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}$. We have some standard spectral properties for $\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}$:

Theorem 2.3. *For*

$$\Box^{(q)}_{b,m,mp_1,\ldots,mp_d} : \mathrm{Dom}\left(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}\right) \to L^2_{(0,q)m,mp_1,\ldots,mp_d}(X),$$

we have

(1) $\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}$ *is a non-negative and a self-adjoint operator.*

(2) *The spectrum* $\mathrm{Spec}(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d})$ *consists only of eigenvalues, and it is a countable and discrete subset in* $[0,\infty)$.

(3) *For each* $\mu \in \mathrm{Spec}(\Box^{q}_{b,m,mp_1,\ldots,mp_d})$, *the space of eigenforms*

$$\mathcal{H}^q_{b,\mu,m,mp_1,\ldots,mp_d}(X) := \{u \in \mathrm{Dom}(\Box^{(q)}_{b,m,mp_1,\ldots,mp_d}) : \Box_b^{(q)} u = \mu u\}$$

is a finite dimensional subspace of $\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)$.

(4) $H^q_{b,m,mp_1,\ldots,mp_d}(X) \cong \mathcal{H}^q_{b,m,mp_1,\ldots,mp_d}(X) := \mathcal{H}^q_{b,\mu=0,m,mp_1,\ldots,mp_d}(X)$.

Proof. The argument is almost the same as the case [5, Section 3] when only circle action is involved, and for the modification to torus action we refer to the proof in [11, Section 4]. $\Box$

Let $n_q := \dim \mathcal{H}^q_{b,m,mp_1,\ldots,mp_d} < \infty$, and the torus equivariant Szegő kernel function

$$\mathrm{Tr}\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x) := \sum_{j=1}^{n_q} |f^q_j(x)|^2_h := \sum_{j=1}^{n_q} \left\langle f^q_j(x) \middle| f^q_j(x) \right\rangle \tag{2.9}$$

where $\{f^q_j\}_{j=1}^{n_q}$ is an orthonormal basis for $\mathcal{H}^q_{b,m,mp_1,\ldots,mp_d}$. For an open set $D \subset X$, take $\{e_j(x)\}_{j=1}^n$ varying smoothly for $x \in D$ such that $\{e_j(x)\}_{j=1}^n$ is an orthonormal basis for $T^{0,1}_x X$ at every $x \in D$. For a strictly increasing index set $J = \{j_1, \ldots, j_q\}$ with $|J| = q$, write $e^J := e_{j_1} \wedge \cdots \wedge e_{j_q}$ and for any $u \in \Omega^{(0,q)}(X)$, write $u(x) = \sum'_{|J|=q} u_J(x) e^J(x)$, where $\Sigma'_{|J|=q}$ we mean the summation only over a strictly increasing index set. Then we can find the torus equivariant Szegő kernel function is the peak function similar in [18, Lemma 2.1], i.e.

$$\mathrm{Tr}\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x) = \sum_{|J|=q}{}' \sup\left\{ |u_J(x)|^2_h : u \in \mathcal{H}^q_{b,m,mp_1,\ldots,mp_d}(X), \|u\|^2_X = 1 \right\}.$$

Also, for all $q = 0, \ldots, n$, the torus equivariant Szegő kernel

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,y) \in \mathscr{C}^\infty\left(X \times X, \mathrm{Hom}(T^{*0,q}X, T^{*0,q}X) \right)$$

is the distribution kernel of the orthogonal projection

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d} : L^2_{(0,q)}(X) \to \mathcal{H}^{(q)}_{m,mp_1,\ldots,mp_d}(X)$$

with respect to $(\cdot|\cdot)$. By Theorem 2.3, the projection is a smoothing operator, and we can check that locally on D, we have the expression

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,y) = \sum_{|I|=q}{}' \sum_{|J|=q}{}' \Pi^{(q)}_{m,mp_1,\ldots,mp_d,I,J}(x,y) e^I(x) \otimes \left(e^J(y) \right)^* \tag{2.10}$$

in the sense that

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d} u(x) = \sum_{|I|=q}{}' \sum_{|J|=q}{}' \left(\int_D \Pi^{(q)}_{m,mp_1,\ldots,mp_d,I,J}(x,y) u_J(y) dV_X(y) \right) e^I(x) \tag{2.11}$$

for $u = \sum'_{|J|=q} u_J e^J \in \Omega^{(0,q)}(X)$. We can check that for all strictly increasing index set I, J, $|I| = |J| = q$,

$$\sum_{|J|=q}{}' \left\langle \Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x,x)e^J(x)|e^J(x) \right\rangle = \mathrm{Tr}\Pi^{(q)}_{m,mp_1,\ldots,mp_d}(x), \tag{2.12}$$

Here, $\Pi^{(q)}_{m,mp_1,\ldots,mp_d,I,J}(x,y) \in \mathscr{C}^\infty(D \times D)$ for all strictly increasing index set I and J, $|I| = |J| = q$. Moreover, we can also check that for all strictly invrasing index set I and J, $|I| = |J| = q$,

$$\Pi^{(q)}_{m,mp_1,\ldots,mp_d,I,J}(x,y) = \sum_{j=1}^{n_q} f^q_{j,I}(x)\overline{f^q_{j,J}}(y), \tag{2.13}$$

where $f^q_j = \sum'_{|K|=q} f^q_{j,K} e^K$, $j = 1, \ldots, n_q$, is an orthonormal basis for $\mathcal{H}^q_{b,m,mp_1,\ldots,mp_d}(X)$.

2.3. Notations in semi-classical analysis and microlocal analysis

We here present some convention in semi-classical analysis and microlocal analysis [7, 9, 23] to describe and calculate the asymptotic behavior of the torus equivariant Szegő kernel. Let U be an open set in $\mathbb{R}^{n_1}$ and let V be an open set in $\mathbb{R}^{n_2}$. Let E and F be vector bundles over U and V, respectively. Let $\mathscr{C}^\infty_0(V,F)$ and $\mathscr{C}^\infty(U,E)$ be the space of smooth sections of F over V with compact support in V and the space of smooth sections of E over U, respectively; $\mathscr{D}'(U,E)$ and $\mathscr{E}'(V,F)$ be the space of distributional sections of E over U and the space of distributional sections of F over V with compact support in V, respectively. We say an m-dependent continuous linear operator

$$A_m : \mathscr{C}^\infty_0(V,F) \to \mathscr{D}'(U,E)$$

is m-negligible if

(1) for all m large enough A_m is a smoothing operator, which is equivalent to ([14, Section 5.2])

$$A_m : \mathscr{E}'(V,F) \to \mathscr{C}^\infty(U,E) \text{ is continuous}$$

or its Schwartz kernel $A_m(x,y)$ is smooth, i.e.

$$A_m(x,y) \in \mathscr{C}^\infty(U \times V, E \boxtimes F^*),$$

where $E \boxtimes F^*$ is the vector bundle over $U \times V$ with fiber $\mathrm{Hom}(F_y, E_x)$ at each $(x,y) \in U \times V$.

(2) for any compact set K in $U \times V$, multi-index $\alpha, \beta \in \mathbb{N}_0^N$ and $N \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N} := \{1,2,3,\ldots\}$, there exists a constant $C_{K,\alpha,\beta,N} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta A_m(x,y)| \leq C_{K,\alpha,\beta,N} m^{-N} \text{ for all } (x,y) \in K,\ m \text{ large enough.}$$

From now on, we also use the notation $A_m(x,y) = O(m^{-\infty})$ or $A_m = O(m^{-\infty})$ on $U \times V$ for an m-negligible continuous linear operator A_m. Moreover, for two continuous linear operators $A_m, B_m : \mathscr{C}_0^\infty(V,F) \to \mathscr{D}'(U,E)$, we write

$$A_m(x,y) \equiv B_m(x,y) \bmod O(m^{-\infty}) \text{ on } U \times V$$

or

$$A_m \equiv B_m \bmod O(m^{-\infty}) \text{ on } U \times V$$

if

$$(A_m - B_m)(x,y) = O(m^{-\infty}) \text{ on } U \times V.$$

Let W be an open set in $\mathbb{R}^N$, we define the space

$$S(1;W) := \{a \in \mathscr{C}^\infty(W) : \sup_{x \in W} |\partial_x^\alpha a(x)| < \infty \text{ for all } \alpha \in \mathbb{N}_0^N\}.$$

Consider the space $S^0_{\mathrm{loc}}(1;W)$ containing all smooth functions $a(x,m)$ with real parameter m such that for all multi-index $\alpha \in \mathbb{N}_0^N$, any cut-off function $\chi \in \mathscr{C}_0^\infty(W)$, we have

$$\sup_{\substack{m \in \mathbb{R} \\ m \geq 1}} \sup_{x \in W} |\partial_x^\alpha(\chi(x) a(x,m))| < \infty.$$

For general $k \in \mathbb{R}$, we can also consider

$$S^k_{\mathrm{loc}}(1;W) := \{a(x,m) : m^{-k} a(x,m) \in S^0_{\mathrm{loc}}(1;W)\}.$$

In other words, $S^k_{\mathrm{loc}}(1;W)$ takes all the smooth function $a(x,m)$ with parameter $m\in\mathbb{R}$ satisfying the estimate: for all $x\in W$, multi-index $\alpha\in\mathbb{N}_0^n$ and $\chi\in\mathscr{C}_0^\infty(W)$ with $\mathrm{supp}(\chi)\subset K$, K is a compact subset of W, then there is a constant $C_{K,\alpha}>0$ independent of m such that

$$|\partial_x^\alpha(\chi(x)a(x,m))|\leq C_{K,\alpha}m^k,\ \text{for all } m>0.$$

For a sequence of $a_j\in S^{k_j}_{\mathrm{loc}}(1;W)$ with k_j decreasing, $k_j\to-\infty$, and $a\in S^{k_0}_{\mathrm{loc}}(1;W)$. We say

$$a(x,m)\sim\sum_{j=0}^\infty a_j(x,m) \text{ in } S^{k_0}_{\mathrm{loc}}(1;W)$$

if for all $l\in\mathbb{N}_0$, we have

$$a-\sum_{j=0}^l a_j\in S^l_{\mathrm{loc}}(1;W).$$

In fact, for all sequence a_j above, there always exists an element a as the asymptotic sum, which is unique up to the elements in $S^{-\infty}_{\mathrm{loc}}(1;W):=\cap_k S^k_{\mathrm{loc}}(1;W)$. The above discussion can be found in [7], and all the notations introduced above and can be generalized to the case on paracompact manifolds.

Let M be an open set in $\mathbb{R}^n$, $\dot{\mathbb{R}}^n:=\mathbb{R}^n\setminus\{0\}$, $\mathbb{N}:=\{1,2,3,\ldots\}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. Let ρ,δ be real numbers such that $0\leq\delta<\rho\leq 1$.

Definition 2.1.

(1) The symbol space with order m in type (ρ,δ) on $M\times\mathbb{R}^N$ is denoted by $S^m_{\rho,\delta}(M\times\mathbb{R}^N)$, which is the space of all $a\in\mathscr{C}^\infty(M\times\mathbb{R}^N)$ satisfying: for all compact subset K in M, and all multi-indices $\alpha\in\mathbb{N}_0^n$, $\beta\in\mathbb{N}_0^N$, there exists a constant $C_{K,\alpha,\beta}>0$ such that

$$\sup_{(x,\theta)\in K\times\mathbb{R}^N}|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)|\leq C_{K,\alpha,\beta}(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}.$$

(2) A function $\phi(x,\theta)\in\mathscr{C}^\infty(M\times\dot{\mathbb{R}}^N)$ is called a phase function if it satisfies: $\mathrm{Im}(\phi)\geq 0$, $\phi(x,\lambda\theta)=\lambda\phi(x,\theta)$ for all $\lambda>0$ and every $(x,\theta)\in M\times\dot{\mathbb{R}}^N$, and $d\phi:=\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}dx_i+\sum_{j=1}^N\frac{\partial\phi}{\partial\theta_j}d\theta_j\neq 0$ everywhere.

Let $a_j \in S^{m_j}_{\rho,\delta}(M \times \mathbb{R}^N),\ j \in \mathbb{N}_0$ and $m_j \searrow -\infty$ as $j \to \infty$, then there exists $a \in S^{m_0}_{\rho,\delta}$ unique modulo $S^{-\infty}(M \times \mathbb{R}^N)$ such that

$$a - \sum_{0 \le j \le k} a_j \in S^{m_k}_{\rho,\delta} \text{ for all } k \in \mathbb{N}_0.$$

We call such a the asymptotic sum of a_j, denoted by $a \sim \sum_{j=0}^{\infty} a_j$. The space of classical symbols $S^m_{\mathrm{cl}}(M \times \mathbb{R}^N)$ collects all $a(x,\theta) \in S^m_{1,0}(M \times \mathbb{R}^N)$ such that $a \sim \sum_{j=0}^{\infty} a_{m-j}(x,\theta)$, where the function $a_{m-j} \in \mathscr{C}^\infty(M \times \mathbb{R}^N)$ is positively homogeneous of degree $m-j$ in the variable $\theta \neq 0$. Let the symbol $a \in S^m_{\mathrm{cl}}(M \times \mathbb{R}^N)$ and ϕ be a phase function on $X \times \dot{\mathbb{R}}^N$. If there is $k \in \mathbb{N}_0$ such that $m + k < -N$, then the oscillatory integral

$$I(a,\phi) := \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta) d\theta$$

converges absolutely, and is a function in $\mathscr{C}^k(M)$. Moreover,

Proposition 2.2. *For any m and $a \in S^m_{\mathrm{cl}}(M \times \mathbb{R}^N)$, there is a unique way of defining $I(a,\phi) \in \mathscr{D}'(M)$ such that*

(1) *$I(a,\phi) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta) d\theta$ when $m < -N$.*

(2) *The map $a \mapsto I(a,\phi)$ is continuous.*

For open sets $U \subset \mathbb{R}^{n_1},\ V \subset \mathbb{R}^{n_2}$, by the Schwartz kernel theorem [14, Theorem 5.2.1], there exists a bijection between $K_A \in \mathscr{D}'(U \times V)$ and a continuous linear map $A : \mathscr{C}^\infty_0(V) \to \mathscr{D}'(U)$ by the correspondence

$$\langle Au, v \rangle_U = \langle K_A, v \otimes u \rangle_{U \times V}$$

for all $u \in \mathscr{C}^\infty_0(V),\ v \in \mathscr{C}^\infty_0(U)$, where $\langle \cdot, \cdot \rangle$ means the pairing by duality and the tensor product $v \otimes u$ is defined $(v \otimes u)(x,y) := v(x)u(y) \in \mathscr{C}^\infty_0(U \times V)$. We call K_A the distribution kernel of A. Let ϕ be a phase function on $(U \times V) \times \dot{\mathbb{R}}^N$, and $a \in S^m_{\mathrm{cl}}\big((U \times V) \times \mathbb{R}^N\big)$. A continuous linear operator $A : \mathscr{C}^\infty_0(V) \to \mathscr{D}'(U)$ is called a Fourier integral operator if its distribution kernel is an oscillatory integral of the form

$$K_A(x,y) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta) d\theta.$$

We formally write

$$Au(x) = \iint e^{i\phi(x,y,\theta)} a(x,y,\theta)u(y)dyd\theta, \text{ for all } u \in \mathscr{C}_0^\infty(V).$$

In particular, when $U = V$ and $\phi(x,y,\theta) = \langle x-y,\theta\rangle$, A is said to be a pseudodifferential operator. We say a Fourier integral operator is properly supported if the projections $\pi_1 : \mathrm{supp}(K_A) \to U$ and $\pi_2 : \mathrm{supp}(K_B) \to V$ are proper maps. The discussion here can be found in [9], and all the notations introduced above can be generalized to the case on manifolds.

In the rest part of this section, we collect the essential tool in Melin–Sjöstrand theory on Fourier integral operators with complex phase [23]. For $z = x+iy \in \mathbb{C}$, we write $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$. Let $W \subset \mathbb{C}^n$ be an open set, we say a $f \in \mathscr{C}^\infty(W)$ is almost analytic if for any compact subset $K \subset W$ and any $N \in \mathbb{N}_0$, there is a constant $C_N > 0$ such that

$$\left|\frac{\partial f}{\partial \bar{z}}(z)\right| \leq C_N |\mathrm{Im} z|^N \text{ for all } z \in K.$$

We say two almost analytic functions f_1 and f_2 are equivalent, or $f_1 \sim f_2$, if for any compact subset $K \subset W$ and any $N \in \mathbb{N}_{\nvdash}$, there is a constant $C_N > 0$ such that

$$|(f_1 - f_2)(z)| \leq C_N |\mathrm{Im} z|^N \text{ for all } z \in K.$$

For $W_{\mathbb{R}} := W \cap \mathbb{R}^n$, then for $f \in \mathscr{C}^\infty(W_{\mathbb{R}})$, f always admits an almost analytic extension up to equivalence. One way is again via the Borel construction, for example, see [6, Section 2.2]. The following proposition is about the critical point in the sense of Melin–Sjöstrand:

Proposition 2.3. *Assume $f(x,w)$ is a smooth complex-valued function in a neighborhood of $(0,0) \in \mathbb{R}^{n+m}$ and that $\mathrm{Im} f \geq 0$, $\mathrm{Im} f(0,0) = 0$, $f'_x(0,0) = 0$, $\det f''_{xx}(0,0) \neq 0$. Let $\tilde{f}(z,w)$ be an almost analytic extension of f to a complex neighboehood of $(0,0)$, where $z = x+iy$ and $w \in \mathbb{C}^m$. By implicit function theorem, we denote $Z(w)$ to be the solution of*

$$\frac{\partial \tilde{f}}{\partial z}(Z(w),w) = 0$$

in a neighborhood of of $0 \in \mathbb{C}^m$*. Then when* w *is real, for every* $N \in \mathbb{N}$*, there is a constant* $C_N > 0$ *such that for all* $w \in \mathbb{R}^m$ *near* 0*,*

$$\left|\frac{\partial}{\partial w}\left(\tilde{f}(Z(w), w)\right) - \frac{\partial}{\partial w}\left(\tilde{f}(z, w)\Big|_{z=Z(w)}\right)\right| \le C_N |\mathrm{Im} Z(w)|^N.$$

Moreover, there are constants C_1*,* $C_2 > 0$ *such that*

$$\mathrm{Im}\tilde{f}(Z(w), w) \ge C_1 |\mathrm{Im} Z(w)|^2, \ w \in \mathbb{R}^m, \ w \text{ near } 0$$

and

$$\mathrm{Im}\tilde{f}(Z(w), w) \ge C_2 \inf_{x \in \Omega} \left(\mathrm{Im} f(x, w) + |d_x f(x, w)|^2\right), \ w \in \mathbb{R}^m, \ w \text{ near } 0$$

where Ω *is an open set near the origin in* $\mathbb{R}^n$*. We call* $\tilde{f}(Z(w), w)$ *the corresponding critical value.*

We end this part by the Melin–Sjöstrand complex stationary phase formula [23, Theorem 2.3]:

Theorem 2.4. *Let* $f(x, w)$ *be as in the Proposition* 2.3*. Then there are neighborhood* U *and* V *of the origin of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and differential operators* $C_{f,j}$ *in* x *of order less equals to* $2j$ *with smooth coefficient of* $w \in V$ *such that*

$$\left|\int e^{itf(x,w)} u(x, w) dx - \left(\det\left(\frac{t\tilde{f}''_{zz}(Z(w), w)}{2\pi i}\right)\right)^{-\frac{1}{2}} e^{it\tilde{f}}(Z(w), w) \sum_{j=0}^{N-1} (C_{f,j}\tilde{u})(Z(w), w)) t^{-j}\right| \tag{2.14}$$

is bounded by $C_N t^{-N-\frac{n}{2}}$*, where* C_N *is a positive constant,* $t \ge 1$ *and* $u \in \mathscr{C}_0^\infty(U \times V)$*. Here, the function*

$$\left(\det\left(\frac{t\tilde{f}''_{zz}(Z(w), w)}{2\pi i}\right)\right)^{-\frac{1}{2}}$$

is the branch of the square root of the

$$\left(\det\left(\frac{t\tilde{f}''_{zz}(Z(w), w)}{2\pi i}\right)\right)^{-1}$$

which is continuously deformed into 1 *under the homotopy*

$$s \in [0,1] \mapsto -i(1-s)\tilde{f}''_{zz}(Z(w),w) + sI \in \mathrm{GL}(n,\mathbb{C}).$$

We note that all the discussion above can be generalize to the case on manifolds.

3. Asymptotics for Lower Energy Torus Equivariant Szegő Kernel

In this section we study the asymptotics for lower energy torus equivariant Szegő kernel. First, we fix a number $\lambda > 0$. Denote

$$\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X) = \bigoplus_{0\leq\mu\leq\lambda} \mathcal{H}^q_{b,\mu,m,mp_1,\ldots,mp_d}(X)$$

where the space $\mathcal{H}^q_{b,\mu,m,mp_1,\ldots,mp_d}(X)$ is defined as in Theorem 2.3. Apply Theorem 2.3, it is clear that $\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$ is still a finite dimensional subspace of $\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X)$. Consider the spectral projector

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d} : L^2_{(0,q)}(X) \to \mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X).$$

Denote $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y)$ to be the distribution kernel of the spectral projector $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}$. Let $N_q := \dim_{\mathbb{C}} \mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X) < \infty$ and $\{f^q_{\leq\lambda,j}\}_{j=1}^{N_q}$ be an orthonormal basis for the space $\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$. Define the torus equivariant Szegő kernel function on lower energy forms by

$$\mathrm{Tr}\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x) := \sum_{j=1}^{N_q} |f^q_{\leq\lambda,j}(x)|^2_h := \sum_{j=1}^{N_q} \left\langle f^q_{\leq\lambda,j}(x) \middle| f^q_{\leq\lambda,j}(x) \right\rangle \tag{3.1}$$

where $\{f^q_{\leq\lambda,j}\}_{j=1}^{N_q}$ is an orthonormal basis for $\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$. We note that the relation (2.9) and (2.12) in Section 2.1 also holds here: for an open set $D \subset X$, we take $\{e_j(x)\}_{j=1}^n$ varying smoothly in $x \in D$ such that $\{e_j(x)\}_{j=1}^n$ form an orthonormal basis of $T^{0,1}_x X$ for every $x \in D$. For a strictly increasing index set $J = \{j_1, \ldots, j_q\}$ with $|J| = q$, if we write

$e^J := e_{j_1} \wedge \cdots \wedge e_{j_q}$, then

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) = {\sum_{|I|=q}}' {\sum_{|J|=q}}' \Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d,I,J}(x,y)e^I(x) \otimes \left(e^J(y)\right)^* \tag{3.2}$$

in the sense that

$$\begin{aligned}&\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}u(x)\\ &\quad= {\sum_{|I|=q}}' {\sum_{|J|=q}}' \left(\int_D \Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d,I,J}(x,y)u_J(y)dV_X(y)\right) e^I(x).\end{aligned} \tag{3.3}$$

for $u = \sum'_{|J|=q} u_J e^J \in \Omega^{(0,q)}(X)$. We can check that

$${\sum_{|J|=q}}' \left\langle \Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,x)e^J(x)|e^J(x)\right\rangle = \mathrm{Tr}\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x). \tag{3.4}$$

Here, $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d,I,J}(x,y) \in \mathscr{C}^\infty(D\times D)$ for all strictly increasing index set I, J, $|I| = |J| = q$. Moreover, we can also check that for all strictly increasing index set I, J, $|I| = |J| = q$,

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d,I,J}(x,y) = \sum_{j=1}^{N_q} f^q_{\leq\lambda,j,I}(x)\overline{f^q_{\leq\lambda,j,J}}(y), \tag{3.5}$$

where $f^q_{\leq\lambda,j} = \sum'_{|K|=q} f^q_{\leq\lambda,j,K}e^K$, $j = 1,\ldots,N_q$, is an orthonormal basis for $\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$.

We divide our discussion of Szegő kernel on space lower energy forms into the cases of the one away from Y and the one near Y, where $Y := \mu^{-1}(-p_1,\ldots,-p_d)$ define by the torus invariant CR moment map

$$\mu : X \to \mathbb{R}^d,\ \mu(x) := \Big(\langle\omega_0(x),T_1(x)\rangle,\ldots,\langle\omega_0(x),T_d(x)\rangle\Big),$$

satisfying Assumption 1.1, 1.2 and 1.3. On one hand, for the case away from Y, we can estimate the bound $\sup_{\substack{x\in D\\ D\cap Y=\emptyset}} |u(x)|$ for the functions $u \in \mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$ with $\|u\|^2_X := (u|u) = 1$ by Bochner formula of $\Box^{(q)}_b$ and some standard PDEs argument. Combine with the relation (3.5), we can show that:

Theorem 3.1. *For any open set Ω containing Y,*

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $(X \setminus \Omega) \times (X \setminus \Omega)$ if $q \in \{0,\ldots,n\}$.

On the other hand, for the case near Y, if we also assume the Levi form is positive near Y, then we can apply the Boutet–Sjöstrand type theorem [21, Theorem 4.1], i.e. in local picture $\Pi^{(q)}_{\leq\lambda}(x,y)$ is in the form of a complex phase Fourier integral operator, and we can study the asymptotic behavior of $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y)$ using integration by parts and Melin–Sjöstrand complex stationary phase formula. Precisely, we have

Theorem 3.2. *For each $p \in Y$, we can find a neighborhood denoted by D_p, such that*

$$\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) = O(m^{-\infty})$$

on $D_p \times D_p$ if $q \in \{1,\ldots,n\}$. Finally, on $D_p \times D_p$,

$$\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) \equiv e^{imf(x,y)}b(x,y,m) \bmod O(m^{-\infty}).$$

Here, the phase function $f \in \mathscr{C}^\infty(D_p \times D_p)$ satisfies $\mathrm{Im} f \geq 0$, $f(x,x) = 0$ for all $x \in Y \cap D_p$ and $d_x f(x,x) = -\omega_0(x)$, $d_y f(x,x) = \omega_0(x)$ for all $x \in Y \cap D_p$; also, the symbol satisfies

$$\begin{aligned} &b(x,y,m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_p \times D_p) \\ &b(x,y,m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j} b_j(x,y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_p \times D_p) \end{aligned}$$

where $b_j(x,y) \in \mathscr{C}^\infty(D_p \times D_p)$, $j = 0,1,2,\ldots$ and $b_0(x,x) > 0$ for all $x \in Y \cap D_p$.

Finally, we show that the asymptotic behavior of kernel on lower energy forms Theorem 3.1, Theorem 3.2 actually coincide with the one for genuine kernel Theorem 1.1 as $m \to +\infty$. The precise statement is as followed:

Theorem 3.3. *Under the same assumption in Theorem 1.1, then for any $\lambda > 0$, as $m \to +\infty$*

$$\Pi^{(0)}_{m,mp_1,\ldots,mp_d} = \Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}.$$

Proof. Decompose the space

$$\mathcal{H}^0_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)=\mathcal{H}^0_{b,m,mp_1,\ldots,mp_d}(X)\oplus\mathrm{Span}_{0<\mu\leq\lambda}\mathcal{H}^0_{b,\mu,m,mp_1,\ldots,mp_d}(X).$$

where

$$\mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X):=\bigoplus_{0\leq\mu\leq\lambda}\{u\in\Omega^{(0,q)}_{m,mp_1,\ldots,mp_d}(X):\Box^{(q)}_b u=\mu u\}.$$

Apply Theorem 3.2 and Theorem 3.3 for the case $q=1$, then by the compactness of X we know that

$$\dim_{\mathbb{C}}\mathcal{H}^1_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)=\int_X \mathrm{Tr}\Pi^{(1)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x)dV_X(x)\leq C_N m^{-N}.$$

After fixing an $N\in\mathbb{N}_{\not{\,}}$, we know that as $m\to+\infty$

$$\mathcal{H}^1_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)=\{0\}.$$

Since the group action is required to be CR, $T_j\overline{\partial}_b=\overline{\partial}_bT_j$ for all $j=0,1,\ldots,d$. Combine this with $\Box^{(q+1)}_b\overline{\partial}_b=\overline{\partial}_b\Box^{(q)}_b$, we can find that for any $u\in\mathcal{H}^0_{b,\mu,m,mp_1,\ldots,mp_d}(X)$, $0<\mu\leq\lambda$,

$$\overline{\partial}_b u\in\mathcal{H}^1_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X).$$

for m large enough. However, this means that for some $0<\mu\leq\lambda$, for m large enough

$$\mu u=\Box^{(0)}_b u=\overline{\partial}^*_b\left(\overline{\partial}_b u\right)=0,$$

i.e. $\mathrm{Span}_{0<\mu\leq\lambda}\mathcal{H}^0_{b,\mu,m,mp_1,\ldots,mp_d}(X)=\{0\}$ as $m\to+\infty$. Therefore, as $m\to+\infty$

$$\mathcal{H}^0_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)=\mathcal{H}^0_{b,m,mp_1,\ldots,mp_d}(X).$$

Thus, for any $\lambda>0$, as $m\to+\infty$

$$\Pi^{(0)}_{m,mp_1,\ldots,mp_d}=\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}.\qquad\Box$$

3.1. The asymptotic behavior away from Y

In this section we prove Theorem 3.1. Fix a number $\lambda \geq 0$ and a point $p \notin Y$. Since Y is closed, there always exist a neighborhood D_p near p and a $j \in \{1, \dots, d\}$ such that

$$\langle \omega_0(x), T_j(x) \rangle \neq -p_j \text{ for every } x \in D_p.$$

We may assume $j = 1$, i.e. $p_1 + \langle \omega_0(x), T_1(x) \rangle \neq 0$ for all $x \in D_p$. Consider the vector field

$$F := T_1 + \langle \omega_0, T_1 \rangle T_0,$$

which is in $\mathscr{C}^\infty\left(X, T^{1,0}X \oplus T^{0,1}X\right)$ because $\langle \omega_0, F \rangle = 0$. We hence decompose $F = L + \overline{L}$, where the vector field $L \in \mathscr{C}^\infty\left(X, T^{1,0}X\right)$. From now on, we assume $u \in \mathcal{H}^q_{b,\leq\lambda,m,mp_1,\dots,mp_d}(X)$ with $\|u\|_X^2 = 1$. Take a cut-off function $\chi \in \mathscr{C}_0^\infty(D_p)$ with $\chi = 1$ near p. By Fourier inversion formula, we can see the multiplication operator $\chi\cdot$ as a properly supported zero order pseudodifferential operator. Precisely, for any $v \in \mathscr{C}_0^\infty(D_p)$

$$\chi v(x) = \frac{1}{(2\pi)^{2n+1}} \iint e^{i\langle (x-y), \xi \rangle} \chi(x) v(y) dy d\xi.$$

If we also regard F as a first order differential operator, then

$$\begin{aligned} F(\chi u) &= \chi(Fu) + [F, \chi]u \\ &= (im(p_1 + \langle \omega_0, T_1 \rangle)) \chi u + [F, \chi]u, \end{aligned} \tag{3.6}$$

Cause we assume $p_1 + \langle \omega_0, T_1 \rangle \neq 0$ on D_p, from (3.6) there are constants $C, C_0 > 0$ such that

$$\begin{aligned} \|\chi u\|_X^2 &\leq \frac{1}{Cm^2} \left(\|F(\chi u)\|_X^2 + \|[F, \chi]u\|_X^2 \right) \\ &\leq \frac{1}{Cm^2} \left(\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 + C_0 \|u\|_X^2 \right), \end{aligned} \tag{3.7}$$

where $[\cdot, \cdot]$ denotes the commutator between differential operators and we see $[F, \chi]$ as an order 0 properly supported pseudodifferential operator admitting L^2 continuity (See [9, Theorem 3.6 and Theorem 4.5] for example). From now on, we use notations such as $C_0, C_1, C_2, \dots$ or $C', C'', \dots$ to denote positive constants independent of m. We need the following L^2-estimate to control (3.7):

Proposition 3.4. *For a given vector filed $Z \in \mathscr{C}^\infty(X, T^{1,0}X)$, there is a positive constant $C' = C'(Z)$ such that*

$$\|Z\phi\|_X^2 + \|\overline{Z}\phi\|_X^2 \leq C'\left(\left(\Box_b^{(q)}\phi\middle|\phi\right) + |(T_0\phi|\phi)| + \|\phi\|_X^2\right).$$

for every $\phi \in \Omega_0^{(0,q)}(D_p)$.

Proof. First of all, for any $p \in X$, let $\{\overline{e}_j(x)\}_{j=1}^n$ varying smoothly in x near p such that $\{\overline{e}_j(x)\}_{j=1}^n$ be an orthonomal frame of $T_x^{*0,1}X$ for all x near p and $\{\overline{L}_j\}_{j=1}^n$ be its dual frame on $T_x^{0,1}X$, x is near p. Since L_j's are first order differential operators, using integration by parts we can find the formal adjoint $\overline{L}_j^*$'s and L_j^*'s of $\overline{L}_j$'s and L_j are

$$\overline{L}_j^* = -L_j + E_j,\ L_j^* = -\overline{L}_j + \overline{E}_j,$$

respectively, where E_j are some terms of zero order. Now, we rewrite Theorem 2.1 into the form

$$\Box_b^{(q)} = \sum_{j=1}^n \overline{L}_j^*\overline{L}_j + \sum_{j=1}^n a_jL_j + \sum_{j=1}^n b_j\overline{L}_j + cT_0 + d,$$

where a_j, b_j, c_j , c and d are smooth coefficeints, $j = 1, \ldots, n$. Then for $\phi \in \mathscr{C}_0^\infty(D_p)$,

$$\begin{aligned}\left(\Box_b^{(q)}\phi\middle|\phi\right) \geq &\sum_{j=1}^n \|\overline{L}_j\phi\|_X^2 - \sum_{j=1}^n |(a_jL_j\phi|\phi)| - \sum_{j=1}^n |(b_j\overline{L}_j\phi|\phi)| \\ &- |(cT_0\phi|\phi)| - |(d\phi|\phi)|\end{aligned}$$

Note that for any $\epsilon > 0$, we have

$$\begin{aligned}|(a_jL_j\phi|\phi)| &\leq |(L_ja_j\phi|\phi)| + |([a_j, L_j]\phi|\phi)| \\ &\leq |(a_j\phi|L_j^*\phi)| + C_1\|\phi\|_X^2 \\ &\leq |(a_j\phi|\overline{L}_j\phi)| + |(a_j\phi|\overline{E}_j\phi)| + C_1\|\phi\|_X^2 \\ &\leq \frac{C_2}{\epsilon}\|\phi\|_X^2 + \epsilon\|\overline{L}_j\phi\|_X^2 + C_3\|\phi\|_X^2 + C_1\|\phi\|_X^2,\end{aligned}$$

and

$$|(b_j\overline{L}_j\phi|\phi)| \leq C_4\epsilon\|\overline{L}_j\phi\|_X^2 + \frac{1}{\epsilon}\|\phi\|_X^2,$$

and

$$|(cT_0\phi|\phi)| \leq C_5\|T_0\phi\|\cdot\|\phi\|,\ |(d\phi|\phi)| \leq C_6\|\phi\|_X^2.$$

Take ϵ small enough, then we get

$$\|\overline{L}_j\phi\|^2 \leq C_7\left((\Box_b^{(q)}\phi|\phi) + |(T_0\phi|\phi)| + \|\phi\|_X^2\right), \tag{3.8}$$

and sum over j we have

$$\|\overline{Z}\phi\|^2 \leq C_7'\left((\Box_b^{(q)}\phi|\phi) + |(T_0\phi|\phi)| + \|\phi\|_X^2\right). \tag{3.9}$$

So it remains to estimate $\|L_j\phi\|_X^2$ and $\|Z\phi\|_X^2$. Observe that

$$\begin{aligned}
\|L_j\phi\|_X^2 =&|(L_j^*L_j\phi|\phi)| \\
\leq&|(L_jL_j^*\phi|\phi)| + |([L_j^*, L_j]\phi|\phi)| \\
\leq&\|L_j^*\phi\|_X^2 + \left|\left(\left(\sum_{k=1}^n f_kL_k + \sum_{k=1}^n g_k\overline{L}_k) + cT_0\right)\phi\middle|\phi\right)\right| \\
\leq& \left(\|\overline{L}_j\phi\|_X^2 + C_8\|\phi\|_X^2\right) \\
&+ C_9\left(\epsilon\|L\phi\|_X^2 + \frac{1}{\epsilon}\|\phi\|_X^2 + (\|\overline{L}\phi\|_X^2 + \|\phi\|_X^2) + |(T_0\phi|\phi)|\right),
\end{aligned}$$

where $f_k,\ g_k,\ c$ are some smooth coefficients. Choose ϵ small enough, sum over j, and apply (3.8), then we can also get

$$\|Z\phi\|_X^2 \leq C_{10}\left((\Box_b^{(q)}\phi|\phi) + |(T_0\phi|\phi)| + \|\phi\|_X^2\right). \tag{3.10}$$

Hence,

$$\|Z\phi\|_X^2 + \|\overline{Z}\phi\|_X^2 \leq C'\left((\Box_b^{(q)}\phi|\phi) + |(T_0\phi|\phi)| + \|\phi\|_X^2\right). \qquad \square$$

Recall that from (3.7) we have

$$\|\chi u\|_X^2 \leq \frac{1}{Cm^2}\left(\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 + C_0\right) \tag{3.11}$$

and now Proposition 3.4 implies that

$$\|L(\chi u)\|_X^2+\|\overline{L}(\chi u)\|_X^2+C_0\leq C'\Big((\Box_b^{(q)}(\chi u)|\chi u)+(T_0(\chi u)|\chi u)+\|\chi u\|_X^2\Big)+C_0. \tag{3.12}$$

We now estimate all terms in (3.12) one by one. First, we have

$$\begin{aligned}
(\Box_b^{(q)}(\chi u)|\chi u) &= (\chi\Box_b^{(q)}u|\chi u)+([\Box_b^{(q)},\chi]u|\chi u)\\
&\leq \lambda\|\chi u\|_X^2+\left(\Big(\sum_{j=1}^n c_jL_j+\sum_{j=1}^n d_j\overline{L}_j+eT_0\Big)u|\chi u\right)\\
&\leq \lambda\|\chi u\|_X^2+\left|\left(\Big(\sum_{j=1}^n c_jL_j+\sum_{j=1}^n d_j\overline{L}_j+eT_0\Big)u|\chi u\right)\right|\\
&\leq \lambda\|\chi u\|_X^2+\sum_{j=1}^n |(c_jL_ju|\chi u)|+\sum_{j=1}^n \big|\big(d_j\overline{L}_ju|\chi u\big)\big|+|(eT_0u|\chi u)| \qquad (3.13)
\end{aligned}$$

for some smooth coefficeints $c_j,\ d_j,\ e,\ j=1,\ldots,n$. Note that

$$|(eT_0u|\chi u)|=|(eu|m\chi u)|\leq \frac{C_{11}}{\epsilon}\|u\|_X^2+\epsilon m^2\|\chi u\|_X^2; \tag{3.14}$$

and for all $j=1,\ldots,n$,

$$\begin{aligned}
|(c_jL_ju|\chi u)| &= \big|(c_ju|L_j^*(\chi u))\big|+|([c_j,L_j]u|\chi u)|\\
&\leq \left(\frac{C_{12}}{\epsilon}\|u\|_X^2+\epsilon\left(\|\overline{L}_j(\chi u)\|_X^2+C_{13}\|\chi u\|_X^2\right)\right)\\
&\quad+\left(\frac{C_{14}}{\epsilon}\|u\|_X^2+\epsilon\|\chi u\|_X^2\right)\\
&= \epsilon\|\overline{L}_j(\chi u)\|_X^2+(C_{13}\epsilon+\epsilon)\|\chi u\|_X^2+\frac{C_{12}+C_{14}}{\epsilon}\|u\|_X^2, \qquad (3.15)
\end{aligned}$$

and similarly

$$\big|\big(d_j\overline{L}_ju|\chi u\big)\big|\leq \epsilon\|L_j(\chi u)\|_X^2+(C_{16}\epsilon+\epsilon)\|\chi u\|_X^2+\frac{C_{15}+C_{17}}{\epsilon}\|u\|_X^2. \tag{3.16}$$

Second,

$$\begin{aligned}
|(T_0(\chi u)|\chi u)| &\leq |(\chi T_0u|\chi u)|+|([T_0,\chi]u|\chi u)|\\
&\leq m\|\chi u\|_X^2+\|[T_0,\chi]u\|_X\cdot\|\chi u\|_X
\end{aligned}$$

$$\leq\ m\|\chi u\|_X^2 + \left(\frac{C_{18}}{\epsilon}\|u\|_X^2 + \epsilon\|\chi u\|_X^2\right). \tag{3.17}$$

Therefore, $\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2$ is bounded above by

$$C'\left(\left(m+(m^2+C_{19})\epsilon\right)\|\chi u\|_X^2+\epsilon\left(\|L_j(\chi u)\|_X^2+\|\overline{L}_j(\chi u)\|_X^2\right)+C_{20}\epsilon^{-1}\|u\|_X^2\right).$$

for some constant $C' > 0$ independent of m and u. Take ϵ small enough and sum over j, then when m large enough, there is also a constant $C'' > 0$ independent of m and u such that

$$\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 \leq C''\left((m+\epsilon m^2)\|\chi u\|_X^2 + \epsilon^{-1}\|u\|_X^2\right)$$

holds. Back to the estimate (3.11), i.e.

$$Cm^2\|\chi u\|_X^2 \leq \|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 + C_0.$$

Recall that we assume $\|u\|_X^2 = 1$ here, so if we take a suitably small ϵ such that $\epsilon < \frac{C}{2C''}$, then when m large enough we can find

Proposition 3.5. *For a point $p \in Y$ and D_p a neighborhood of p with $D_p \cap Y = \emptyset$, if we fix a number $\lambda \geq 0$, then for each function $\chi \in \mathscr{C}_0^\infty(D_p)$ and q-form $u \in \mathcal{H}^q_{b,\leq\lambda,m,mp_1,\ldots,mp_d}(X)$ with $\|u\|_X^2 = 1$, we can find a constant $C_{0,1} > 0$ independent of m and u such that as $m \to +\infty$*

$$\|\chi u\|_X \leq \frac{1}{C_{0,1}m}.$$

In fact, we can modify the above argument and improve the estimate Proposition 3.5:

Proposition 3.6. *Assume the same p, D_p, u in Proposition 3.5. For each $\chi \in \mathscr{C}_0^\infty(D_p)$ and any $N \in \mathbb{N}_0$, we can find a constant $C_{0,N} > 0$ independent of m and u such that as $m \to +\infty$*

$$\|\chi u\|_X \leq \frac{1}{C_{0,N}m^N}.$$

To see this, we need to look back the estimate we did before, i.e. the one appeared in (3.7), (3.12), (3.13) and (3.17). First, take another cut-off function $\tau \in \mathscr{C}_0^\infty(D_p)$ with $\tau \equiv 1$ near $\mathrm{supp}(\chi)$, observe that (3.7) can be reformulated as

$$\begin{aligned}\|\chi u\|_X^2 &\leq \frac{1}{Cm^2}(\|F(\chi\tau u)\|_X^2 + \|[F,\chi]\tau u\|) \\ &\leq \frac{1}{Cm^2}\left(\|L(\chi\tau u)\|_X^2 + \|\overline{L}(\chi\tau u)\|_X^2 + C_0\|\tau u\|_X^2\right). \qquad (3.18)\end{aligned}$$

Note that (3.14) can be also viewed as

$$|(eT_0u|\chi u)| = |(e\tau u|m\chi u)| \leq \frac{C_0}{\epsilon}\|\tau u\|_X^2 + \epsilon m^2\|\chi u\|_X^2.$$

Similarly, we rewrite (3.15), (3.16) into

$$\begin{aligned}&|(c_jL_ju|\chi u)| = |(c_j\tau L_ju|\chi u)| \\ &\leq \left|(c_j\tau u|L_j^*(\chi u))\right| + |([c_j\tau, L_j]u|\chi u)| \\ &\leq \left(\frac{C_{12}}{\epsilon}\|\tau u\|_X^2 + \epsilon\left(\|\overline{L}_j(\chi u)\|_X^2 + C_{13}\|\chi u\|_X^2\right)\right) + 2C_{14}\|u\|_X\cdot\|\chi u\|_X \\ &= \epsilon\|\overline{L}_j(\chi u)\|_X^2 + \frac{C_{12}}{\epsilon}\|\tau u\|_X^2 + C_{13}\epsilon\|\chi u\|_X^2 + 2C_{14}\|u\|_X\cdot\|\chi u\|_X, \qquad (3.19)\end{aligned}$$

and

$$\begin{aligned}\left|\left(d_j\overline{L}_ju|\chi u\right)\right| \leq \epsilon\|L_j(\chi u)\|_X^2 + \frac{C_{15}}{\epsilon}\|\tau u\|_X^2 + C_{16}\epsilon\|\chi u\|_X^2 \\ +2C_{17}\|u\|_X\cdot\|\chi u\|_X, \qquad (3.20)\end{aligned}$$

respectively. Also, we take (3.17) in the form of

$$|(T_0(\chi u)|\chi u)| \leq m\|\chi u\|_X^2 + 2C_{18}\|u\|_X\cdot\|\chi u\|_X. \qquad (3.21)$$

Therefore, we have a slightly different upper bound

$$\begin{aligned}C'\left(\left(m + (m^2 + C_8)\epsilon\right)\|\chi u\|_X^2 + \epsilon^{-1}\|\tau u\|_X^2 + \epsilon\left(\|L_j(\chi u)\|_X^2 + \|\overline{L}_j(\chi u)\|_X^2\right)\right. \\ \left. +2C_{19}\|u\|_X\cdot\|\chi u\|_X\right).\end{aligned}$$

for $\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2$. Take ϵ small enough and sum over j, then for all large enough m,

$$\begin{aligned}&\|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 \\ &\quad\leq C''\left(\left(m + \epsilon m^2\right)\|\chi u\|_X^2 + \epsilon^{-1}\|\tau u\|_X^2 + 2C_{19}\|u\|_X\cdot\|\chi u\|_X\right).\end{aligned}$$

From (3.18)

$$Cm^2\|\chi u\|_X^2 \le \|L(\chi u)\|_X^2 + \|\overline{L}(\chi u)\|_X^2 + C_0\|\tau u\|_X^2,$$

we can take sutibly small ϵ such that as m large enough

$$\begin{aligned} m^2\|\chi u\|_X^2 &\le C''' \left(\epsilon^{-1}\|\tau u\|_X^2 + \|u\|_X \cdot \|\chi u\|_X + \|\tau u\|_X^2\right) \\ &\le C'''(\frac{1}{\epsilon C_{0,1}(\tau)m^2} + \frac{1}{C_{0,1}(\chi)m} + \frac{1}{C_{0,1}(\tau)m^2}) \\ &\le \frac{1}{C_{0,\frac{3}{2}}m}. \end{aligned} \tag{3.22}$$

So $\|\chi u\|_X \le \frac{1}{C_{0,\frac{3}{2}}m^{\frac{3}{2}}}$, and we can inductively apply (3.22) to get Proposition 3.6.

Finally, for $p \in Y$, we take neighborhoods $O_p \Subset D_p$ of p where $D_p \cap Y = \emptyset$, and pick a bump function $\chi \in \mathscr{C}_0^\infty(D_p)$ with $\chi \equiv 1$ on O_p. Denote $\|\cdot\|_k$ to be a torus invariant Sobolev k-norm induced by $(\cdot|\cdot)$. After applying the Gårding inequality to the $2k$-order strongly elliptic operator $(\Box_b^{(q)} - T_0^2)^k$, we have

$$\|u\|_k^2 \le C_k' \left(\left((\Box_b^{(q)} - T_0^2)^k u \middle| u\right) + \|u\|_X^2\right) = O(m^{2k}).$$

Similarly, with the help of elliptic estimate on $(\Box_b^{(q)} - T_0^2)^k$, for large enough m

$$\begin{aligned} \|\chi u\|_k^2 &\le C_k' \left(\left((\Box_b^{(q)} - T_0^2)^k \chi u \middle| \chi u\right) + \|\chi u\|_X^2\right) \\ &\le C_k' \left(\left(\chi(\Box_b^{(q)} - T_0^2)^k u \middle| \chi u\right) + \left(\left[(\Box_b^{(q)} - T_0^2)^k, \chi\right] u \middle| \chi u\right) + \|\chi u\|_X^2\right) \\ &\le C_k'' \left((\lambda + m^2)^k \|\chi u\|_X^2 + \|u\|_{2k-1}\|\chi u\|_X + \|\chi u\|_X^2\right). \end{aligned} \tag{3.23}$$

By Proposition 3.6, for any $N \in \mathbb{N}_0$, there is a constant $C_{k,N} > 0$ independent of m and u such that

$$\|\chi u\|_k \le C_{k,N} m^{-N}, \tag{3.24}$$

for all m large enough. Combine (3.24) with Sobolev inequality, for all $x \in O_p$, $p \notin Y$, k large enough, then for any $N \in \mathbb{N}$, there is a constant

$C_N > 0$ independent of m and u such that

$$|u(x)|_h^2 = |\chi(x)u(x)|_h^2 \leq C_k''' \|\chi u\|_k^2 \leq C_N m^{-N}, \tag{3.25}$$

for all m large enough. Now, consider a cut-off function $\tau \in \mathscr{C}_0^\infty(D_p)$, $\tau \equiv 1$ on $\mathrm{supp}(\chi)$. For any differential operator $P : \Omega_0^{(0,q)}(D_p) \to \Omega^{(0,q)}(D_p)$ of order l, where $\Omega_0^{(0,q)}(D_p) := \mathscr{C}_0^\infty(D_p, T^{*0,q}X)$. Note that

$$Pu(x) = \chi(x)Pu(x) = P(\chi u)(x) + [\chi, P](\tau u)(x).$$

Thus, similar in (3.25), for every $x \in O_p$, $p \notin Y$, k large enough and any $N \in \mathbb{N}$, there is a constant $C_N > 0$ independent of m and u such that

$$\begin{aligned} |Pu(x)|_h^2 &\leq C_k''' \left(\|P(\chi u)\|_k^2 + \|[\chi, P](\tau u)\|_k^2 \right) \\ &\leq C_k'''' \left(\|\chi u\|_{k+l}^2 + \|\tau u\|_{k+l-1} \right) \leq C_N m^{-N} \end{aligned} \tag{3.26}$$

for all m large enough. From (3.25) and (3.26) and (3.5), Theorem 3.1 holds.

3.2. The full asymptotic expansion near Y

In this section, we prove Theorem 3.2. We first calculate the cirle equivariant Szegő kernel. From now on, we fix a point $p \in Y$ and take a BRT patch D near p as in Proposition 2.2 . Let

$$\Omega_m^{(0,q)}(X) := \left\{ u \in \Omega^{(0,q)}(X) : -iT_0 u := mu \right\}$$

and $L^2_{(0,q),m}(X)$ be the completion of $\Omega_m^{(0,q)}(X)$ with respect to $(\cdot|\cdot)$. With respect to $(\cdot|\cdot)$, denote $Q_m^{(q)}$ to be the orthogonal projection

$$Q_m^{(q)} : L^2_{(0,q)}(X) \to L^2_{(0,q),m}(X).$$

Extend $\Box_b^{(q)}$ by Gaffney extension (2.8), then $\Box_b^{(q)}$ is a self-adjoint operator. We can hence apply generel theory for self-adjoint operator such as [6] to take the spectral projector

$$\Pi_{\leq\lambda}^{(q)} : L^2_{(0,q)}(X) \to \mathcal{H}_{b,\leq\lambda}^q(X) := E((-\infty, \lambda]),$$

for any $\lambda > 0$, where $E((-\infty,\lambda])$ is the spectral projection and E is the spectral measure for $\Box_b^{(q)}$, respectively. Denote the m-th Fourier component of the spectral projector by

$$\Pi^{(q)}_{\leq\lambda,m} : L^2_{(0,q)}(X) \to \mathcal{H}^q_{b,m,\leq\lambda}(X) := \mathcal{H}^q_{b,\leq\lambda}(X) \cap L^2_{(0,q),m}(X).$$

We can check that on $\Omega^{(0,q)}(X)$,

$$\Pi^{(q)}_{\leq\lambda} = Q^{(q)}_m \Pi^{(q)}_{\leq\lambda} = \Pi^{(q)}_{\leq\lambda} Q^{(q)}_m.$$

For a given point $p \in Y$, let $x = (x_1,\ldots,x_{2n},x_{2n+1}) = (\mathring{x},x_{2n+1})$, $y = (y_1,\ldots,y_{2n},y_{2n+1}) = (\mathring{y},y_{2n+1})$ be BRT trivialization as in Theorem 2.2 defined on $D := \tilde{D}\times(-\pi,\pi) \subset X$ near p, where $\tilde{D}$ is an open set of $\mathbb{C}^n$. Note that by theory of Fourier series on the circle, $Q^{(q)}_m u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}u(e^{i\theta}\circ x)d\theta$ for any $u \in \Omega^{(0,q)}(X)$. In particular, under BRT coordinates, for all $u \in \Omega^{(0,q)}_0(D) := \mathscr{C}^\infty_0(D, T^{*0,q}X)$,

$$\begin{aligned}
\Pi^{(q)}_{\leq\lambda,m}u(x) &= Q^{(q)}_m \Pi^{(q)}_{\leq\lambda}u(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\left(\int_D \Pi^{(q)}_{\leq\lambda}(e^{i\theta}\circ x,y)u(y)dV_X(y)\right)d\theta \\
&= \int_D \left(\frac{1}{2\pi}\int \Pi^{(q)}_{\leq\lambda}(e^{i\theta}\circ x,y)e^{-im\theta}d\theta\right)u(y)dV_X(y).
\end{aligned}$$

In other words,

$$\Pi^{(q)}_{\leq\lambda,m}(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Pi^{(q)}_{\leq\lambda}(e^{i\theta}\circ x,y)e^{-im\theta}d\theta. \tag{3.27}$$

Similarly, for a fixed $(p_1,\ldots,p_d) \in \mathbb{Z}^d$, we can find that $\Pi^{(0)}_{m,mp_1\ldots,mp_d}u(x)$ is

$$(2\pi)^{-d}\int_X \left(\int_{T^d} \Pi^{(0)}_{\leq\lambda,m}\left((e^{i\theta_1},\ldots,e^{i\theta_d})\circ x,y\right)e^{-im\sum_{j=1}^d p_j\theta_j}d\theta_1,\ldots d\theta_d\right) u(y)dV_X(y),$$

for all $u \in \Omega^{(0,q)}(X)$.Therefore,

$$\begin{aligned}
&\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) \\
&= \frac{1}{(2\pi)^d}\int_{T^d} \Pi^{(0)}_{\leq\lambda,m}\left((e^{i\theta_1},\ldots,e^{i\theta_d})\circ x,y\right)e^{-im\sum_{j=1}^d p_j\theta_j}d\theta_1,\ldots d\theta_d.
\end{aligned} \tag{3.28}$$

Since we assume the Levi form is positive on Y, we can apply the result in [21] for the case of constant signature $(n_-, n_+) = (0, n)$ near Y. On one hand, from [21, Theorem 4.1], we have:

Proposition 3.7. *For each $q = 1, \ldots, n-1$, $\Pi^{(q)}_{\leq\lambda}$ is a smoothing operator near Y.*

Form Proposition 3.7, using integration by parts with respect to θ in (3.27), beacuse the boundary term vanishes for periodic reason, we can show that on $\Omega \times \Omega$,

$$\Pi^{(q)}_{\leq\lambda,m}(x,y) = O(m^{-\infty}) \text{ for all } q = 1, \ldots, n-1,$$

where Ω is an open set containing Y. In particular, from (3.28) and Theorem 3.1, we have:

Proposition 3.8. *For each $q{=}1, \ldots, n{-}1$, $\Pi^{(q)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y){=}O(m^{-\infty})$ on $X \times X$.*

On the other hand, from the statement and the proof of [21, Theorem 4.1], we know:

Proposition 3.9. *For $q = \{0, n\}$, locally on a coordinates patch $D \subset X$, $\Pi^{(q)}_{\leq\lambda}$ is in the form of complex Fourier integral operator. Precisely, in the sense of oscillatory integral*

$$\Pi^{(0)}_{\leq\lambda}(x,y) = \int_0^\infty e^{i\phi_-(x,y)t} a_-(x,y,t)dt.$$

Moreover, for any small open neighborhood Ω containing Y and all $\chi, \tau \in \mathscr{C}^\infty_0(\Omega)$ such that $\operatorname{supp}(\chi) \cap \operatorname{supp}(\tau) = \emptyset$, then $\chi\Pi^{(0)}_{\leq\lambda}\tau$ is a smoothing operator. Here the phase function locally on D is

$$\phi_-(x,y) = x_{2n+1} - y_{2n+1} + \Phi(\mathring{x}, \mathring{y}),$$

where $\mathring{x} := (x_1, \ldots, x_{2n})$, $\mathring{y} := (y_1, \ldots, y_{2n})$ and $\Phi(\mathring{x}, \mathring{y})$ is a complex-valued function satisfying for some constant $C{>}0$, $\operatorname{Im}\Phi(\mathring{x}, \mathring{y}) \geq C|\mathring{x}{-}\mathring{y}|^2$, $\Phi(\mathring{x}, \mathring{y}) = -\overline{\Phi(\mathring{y}, \mathring{x})}$, and $\Phi(\mathring{x}, \mathring{y}) = 0$ if and only if $\mathring{x} = \mathring{y}$, for all $(x,y) \in D \times D$. And the symbol here satisfies

$$a_-(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+),$$

$$a_-(x,y,t) \sim \sum_{j=0}^{\infty} t^{n-j}(a_-)_j(x,y) \ \textit{in} \ S^n_{\mathrm{cl}}(D\times D\times\mathbb{R}_+),$$

where $(a_-)_j(x,y) \in \mathscr{C}^\infty(D\times D)$, $j=0,1,2,\ldots,$ *and*

$$(a_-)_0(x,x) = \frac{1}{2\pi^{n+1}}|\det\mathcal{L}_x|.$$

Similarly,

$$\Pi^{(n)}_{\le\lambda}(x,y) = \int_0^\infty e^{i\phi_+(x,y)t}a_+(x,y,t)dt,$$

where the phase on D *is*

$$\phi_+(x,y) = -\overline{\phi}_-(x,y) = -x_{2n+1} + y_{2n+1} - \overline{\Phi}(\mathring{x},\mathring{y})$$

and the symbol

$$a_+(x,y,t) \in S^n_{\mathrm{cl}}(D\times D\times\mathbb{R}_+, T^{*0,n}X \boxtimes T^{*0,n}X).$$

Also, for any small open neighborhood Ω *containing* Y *and all* $\chi,\tau\in\mathscr{C}^\infty_0(\Omega)$ *such that* $\mathrm{supp}(\chi)\cap\mathrm{supp}(\tau)=\emptyset$, *then* $\chi\Pi^{(n)}_{\le\lambda}\tau$ *is a smoothing operator.*

To calculate (3.27) via Proposition 3.9, we need to consider the integral under the BRT coordinates patch $D := \tilde{D}\times(-\pi,\pi)$ in Theorem 2.2. First, note that under BRT coordinates, for $x\in D$ and $e^{i\theta}\circ x\in D$, we have $e^{i\theta}\circ x = (x_1,\ldots,x_{2n},x_{2n+1}+\theta)$. Second, to make sure all the calculation are under BRT patch, for $(x,y)\in D\times D$, we have to consider a smaller open set $D_p\subset D$ such that $\overline{D}_p\subset D$, and cut-off functions $\chi_0,\chi_1\in\mathscr{C}^\infty_0(D)$, where $\chi_0\equiv 1$ on D_p and $\chi_1\equiv 1$ on $\mathrm{supp}(\chi)$. Notice that

$$\Pi^{(0)}_{\le\lambda,m}(x,y) = e^{imx_{2n+1}}\Pi^{(0)}_{\le\lambda,m}(\hat{x},y),$$

where $\hat{x} := (x_1,\ldots,x_{2n},0)$. This holds because from (3.5), $\Pi^{(0)}_{\le\lambda,m}(x,y) = \sum_{j=1}^{N_0} f^0_{\le\lambda,j}(x)\overline{f^0_{\le\lambda,j}}(y)$ and $f^0_{\le\lambda,j}(x) = e^{imx_{2n+1}}f^0_{\le\lambda,j}(\hat{x})$ by $T_0 = \frac{\partial}{\partial x_{2n+1}}$, where $\{f^0_{\le\lambda,j}\}_{j=1}^{N_0}$ is an orthonormal basis for $\mathcal{H}^0_{b,\le\lambda,m}(X)$. Now, for $(x,y)\in D_p\times D_p$, we write

$$\Pi^{(0)}_{\le\lambda,m}(x,y) = e^{imx_{2n+1}}\Pi^{(0)}_{\le\lambda,m}(\hat{x},y)$$

$$
\begin{aligned}
&= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi} \Pi^{(0)}_{\leq\lambda}(e^{i\theta}\circ\hat{x},y)e^{-im\theta}d\theta \\
&= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi} \Pi^{(0)}_{\leq\lambda}(e^{i\theta}\circ\hat{x},y)\chi_0(y)e^{-im\theta}d\theta \\
&= I_1 + I_2 \qquad (3.29)
\end{aligned}
$$

where

$$I_1 := \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi} \chi_1(e^{i\theta}\circ\hat{x})\Pi^{(0)}_{\leq\lambda}(e^{i\theta}\circ\hat{x},y)\chi_0(y)e^{-im\theta}d\theta \qquad (3.30)$$

and

$$
\begin{aligned}
I_2 &:= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}(1-\chi_1)(e^{i\theta}\circ\hat{x},y)\Pi^{(0)}_{\leq\lambda}(e^{i\theta}\circ\hat{x},y)\chi_0(y)e^{-im\theta}d\theta \\
&= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\Big((1-\chi_1)\Pi^{(0)}_{\leq\lambda}\chi_0\Big)(e^{i\theta}\circ\hat{x},y)e^{-im\theta}d\theta. \qquad (3.31)
\end{aligned}
$$

In (3.31), since $(1-\chi_1)\Pi^{(0)}_{\leq\lambda}\chi_0$ is a smoothing operator in view of Proposition 3.9, we can apply integration by parts with respect to θ. Because the boundary term vanishes for periodic reason, we can find that $I_2 = O(m^{-\infty})$. As for (3.30), we shall write

$$
\begin{aligned}
I_1 &= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi} \chi_1(e^{i\theta}\circ\hat{x},y)\Pi^{(0)}_{\leq\lambda}(e^{i\theta}\circ\hat{x})\chi_0(y)e^{-im\theta}d\theta \\
&= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} e^{i\phi_-(e^{i\theta}\circ\hat{x},y)t}\chi_1(e^{i\theta}\circ\hat{x})a_-(e^{i\theta}\circ\hat{x},y,t)\chi_0(y)e^{-im\theta}dtd\theta \\
&= \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} e^{i\phi_-((\mathring{x},\theta),y)t-im\theta}\chi_1(\mathring{x},\theta)a_-\big((\mathring{x},\theta),y,t\big)\chi_0(y)dtd\theta \\
&= \frac{me^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} e^{im\psi_-(x,y,t,\theta)}\chi_1(\mathring{x},\theta)a_-\big((\mathring{x},\theta),y,mt\big)\chi_0(y)dtd\theta \qquad (3.32)
\end{aligned}
$$

where

$$\psi_-(x,y,t,\theta) := \Big(\theta - y_{2n+1} + \Phi(\mathring{x},\mathring{y})\Big)t - \theta.$$

Similarly, on $D_P \times D_p$, we write $\Pi^{(n)}_{\leq\lambda,m}(x,y) = I_3 + I_4$, where

$$I_3 := \frac{me^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} e^{im\psi_+(x,y,t,\theta)}\chi_1(\mathring{x},\theta)a_+\big((\mathring{x},\theta),y,mt\big)\chi_0(y)dtd\theta, \qquad (3.33)$$

and

$$I_4 := \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\left((1-\chi_1)\Pi^{(n)}_{\leq\lambda}\chi_0\right)(e^{i\theta}\circ\hat{x},y)e^{-im\theta}d\theta = O(m^{-\infty}). \quad (3.34)$$

Here,

$$\psi_+(x,y,t,\theta) := -\Big(\theta - y_{2n+1} + \overline{\Phi}(\mathring{x},\mathring{y})\Big)t - \theta.$$

We first handle the case for $q = n$:

Proposition 3.10. $\Pi^{(n)}_{\leq\lambda,m}(x,y) = O(m^{-\infty})$ *on* $D_p \times D_p$.

Proof. Consider a cut-off function $\chi_2(\theta) \in \mathscr{C}^\infty_0(\mathbb{R})$, $\chi_2(\theta) \equiv 1$ when $|\theta| \leq \frac{\pi}{4}$ and $\chi_2(\theta) \equiv 0$ when $\frac{\pi}{2} \leq |\theta| < \pi$. Write

$$I_3 = I_5 + I_6$$

where I_5 has the integrand cut off by χ_2 and I_6 is the one cut off by $1-\chi_2$. Since $t \geq 0$, the term $\frac{\partial\psi_+}{\partial\theta} = -t-1 \neq 0$ for all $\theta \in (-\pi,\pi)$, so we can write $e^{im\psi_+} = \frac{\partial}{\partial\theta}\left(\frac{e^{im\psi_+}}{-im(t+1)}\right)$. Take $I_5 = I_5' + I_5''$, where I_5' is the integration taken over $0 \leq t \leq 1$ and I_5'' is the one taken over $t > 1$. By using integration by parts with respect to θ, we can find both $I_5' = O(m^{-\infty})$ and $I_5'' = O(m^{-\infty})$. Thus, $I_5 = O(m^{-\infty})$. As for I_6, for the case $\mathring{x} \neq \mathring{y}$, we have $\mathrm{Im}\psi_+ = \mathrm{Im}\Phi(\mathring{x},\mathring{y}) > 0$, so $\Pi^{(n)}_{\leq\lambda,m}(x,y) = O(m^{-\infty})$ by the elementary inequality that for any $m, N \in \mathbb{N}_{\not{}}$, $m^N e^{-m} \leq C_N$ for some constant $C_N > 0$. For the case $\mathring{x} = \mathring{y}$, $\psi_+ = -(\theta - y_{2n+1})t - \theta$. Notice that we may assume $\theta - y_{2n+1} \neq 0$ on I_6 by taking the open set D_p small enough. Consider a cut-off function $\tau \in \mathscr{C}^\infty_0(\mathbb{R})$, $\tau(t) \equiv 1$ when $|t| \leq 1$ and $\tau(t) \equiv 0$ when $|t| \geq 2$. Set

$$\tau_j(t) := \tau(2^{-j}t) - \tau(2^{1-j}t),\ j \in \mathbb{N},\ \tau_0 := \tau. \quad (3.35)$$

Note that $\sum_{j=0}^{\infty}\tau_j = 1$ and

$$2^{j-1} \leq |t| \leq 2^{j+1} \text{ for } t \in \mathrm{supp}(\tau_j),\ j > 0. \quad (3.36)$$

By the construction of oscillatory integral, for example see [14, Theorem 7.8.2], in this case

$$I_6 = \frac{e^{imx_{2n+1}}}{2\pi}\sum_{j=0}^{\infty}\int_{-\pi}^{\pi}\int_0^{\infty} e^{im\psi_+(x,y,t,\theta)}\tau_j(t)(1-\chi_2(\theta))\chi_1(\mathring{x},\theta)$$

$$\times a_+\big((\mathring{x},\theta),y,mt\big)\chi_0(y)dtd\theta.$$

Decompose $I_6 = I_6' + I_6''$, where

$$I_6' := \frac{e^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} e^{im\psi_+(x,y,t,\theta)}\tau_0(t)(1-\chi_2(\theta))\chi_1(\mathring{x},\theta) \\ a_+\big((\mathring{x},\theta),y,mt\big)\chi_0(y)dtd\theta \quad (3.37)$$

and

$$I_6'' := \frac{e^{imx_{2n+1}}}{2\pi}\sum_{j=1}^{\infty}\int_{-\pi}^{\pi}\int_0^{\infty} e^{im\psi_+(x,y,t,\theta)}\tau_j(t)(1-\chi_2(\theta))\chi_1(\mathring{x},\theta) \\ a_+\big((\mathring{x},\theta),y,mt\big)\chi_0(y)dtd\theta. \quad (3.38)$$

On one hand, in (3.38), because $e^{-im(-(\theta-y_{2n+1})t-\theta)} = \frac{\partial}{\partial t}\left(\frac{e^{-im(-(\theta-y_{2n+1})t-\theta)}}{-im(\theta-y_{2n+1})}\right)$, we can integration by parts with respect to t, and after combining (3.35), (3.36) and $a_+(x,y,t) \in S^n_{\mathrm{cl}}(D\times D\times\mathbb{R}_+, T^{*0,n}X\boxtimes T^{*0,n}X)$, we can find that $I_6'' = O(m^{-\infty})$. On the other hand, in (3.37), we can also integration by parts with respect to t; however, the boundary term appears at $t=0$. Fortunately, thanks to χ_1 has compact support in $(-\pi,\pi)$ and $e^{-im(-(\theta-y_{2n+1})t-\theta)} = \frac{\partial}{\partial\theta}\left(\frac{e^{-im(-(\theta-y_{2n+1})t-\theta)}}{im(t+1)}\right)$, we can again apply integration with respect to θ, and no boundary term will appear. In this way, we can also find

$$I_6' = O(m^{-\infty}). \qquad \square$$

In particular, from (3.28) and Theorem 3.1, we find that:

Proposition 3.11. $\Pi^{(n)}_{\le\lambda,m,mp_1,\dots,mp_d}(x,y) = O(m^{-\infty})$ *on* $X\times X$.

Next, for the case $q=0$, take the point $p\in Y$ which we set in the beginning of this section, and we can find that at the point

$$(x,y,t,\theta) = (p,p,1,0),$$

there are

$$\mathrm{Im}\psi_- = 0,\ \frac{\partial\psi_-}{\partial t} = 0,\ \frac{\partial\psi_-}{\partial\theta} = 0,$$

and

$$\det\psi_-'' = \det\begin{bmatrix}\frac{\partial^2\psi_-}{\partial t^2} & \frac{\partial^2\psi_-}{\partial\theta\partial t}\\ \frac{\partial^2\psi_-}{\partial t\partial\theta} & \frac{\partial^2\psi_-}{\partial\theta^2}\end{bmatrix} = \det\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} = -1 \ne 0.$$

Thus the point $(x,y,t,\theta)=(p,p,1,0)$ satisfies the assumption in Proposition 2.3. Moreover, when (x,y) varies near (p,p) we can have the Melin–Sjöstrand critical value in Proposition 2.3, because the system of equations

$$\frac{\partial\tilde{\psi}_-}{\partial\tilde{t}}(x,y,\tilde{t},\tilde{\theta}) \;=\; 0$$

and

$$\frac{\partial\tilde{\psi}_-}{\partial\tilde{\theta}}(x,y,\tilde{t},\tilde{\theta}) \;=\; 0$$

also has the solution

$$(\tilde{t},\tilde{\theta}) \;=\; \Big(1, y_{2n+1}-\Phi(\mathring{x},\mathring{y})\Big)\in\mathbb{C}^2$$

where

$$\tilde{\psi}_-(x,y,\tilde{t},\tilde{\theta}) \;=\; \Big(\tilde{\theta}-y_{2n+1}+\Phi(\mathring{x},\mathring{y})\Big)\tilde{t}-\tilde{\theta}$$

is an almost analytic extension of ψ_- with respect to (t,θ). Therefore, we consider the decomposition also denoted by

$$I_1=I_7+I_8,$$

where I_7 is the one cut off by a bump function $\chi_3(t,\theta)\in\mathscr{C}_0^\infty(\mathbb{R}^2)$ and I_8 is the one cut off by $1-\chi_3$. Here, χ_3 satisfies $\chi_3\equiv 1$ near $(t,\theta)=(1,0)$, $\operatorname{supp}(\chi_3(t,\theta))\subset[\frac{1}{2},\frac{3}{2}]\times[-\frac{\pi}{2},\frac{\pi}{2}]$.

On one hand, similar to the proof of Proposition 3.10, consider a cut-off function $\tau\in\mathscr{C}_0^\infty(\mathbb{R})$, $\tau(t)\equiv 1$ when $|t|\le 1$ and $\tau(t)\equiv 0$ when $|t|\ge 2$. We also set τ_j as in (3.35), $j\in\mathbb{N}$. Again, by taking D_p small enough, we may assume that $\partial_t\psi_- := \frac{\partial\psi_-}{\partial t}=\theta-y_{2n+1}+\Phi(\mathring{x},\mathring{y})\neq 0$ on I_8. Take the decomposition $I_8:=I_8'+I_8''$, where

$$I_8' := \frac{me^{imx_{2n+1}}}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{im\psi_-(x,y,t,\theta)}\tau_0(t)(1-\chi_3)(t,\theta)\chi_1(\mathring{x},\theta) \times a_-\big((\mathring{x},\theta),y,mt\big)\chi_0(y)\,dtd\theta \qquad (3.39)$$

and

$$I_8'' := \frac{me^{imx_{2n+1}}}{2\pi}\sum_{j=1}^{\infty}\int_{-\pi}^{\pi}\int_0^\infty e^{im\psi_-(x,y,t,\theta)}\tau_j(t)(1-\chi_3)(t,\theta)\chi_1(\mathring{x},\theta) \times a_-\big((\mathring{x},\theta),y,mt\big)\chi_0(y)\,dtd\theta, \qquad (3.40)$$

By $e^{im\psi_-} = \frac{\partial}{\partial t}\left(\frac{e^{im\psi_-}}{im\partial_t\psi_-}\right)$, for (3.40), we can take integration by parts with respect to t, and combine with (3.35), (3.36) and $a_-(x,y,t) \in S^n_{\rm cl}(D\times D\times\mathbb{R}_+)$ to show that $I''_8 = O(m^{-\infty})$. Also, since χ_1 has support in $(-\pi,\pi)$, for (3.39), we can apply integration by parts with respect to t and θ as in Proposition 3.10 to show that $I'_8 = O(m^{-\infty})$.

On the other hand, by the Melin–Sjöstrand complex stationary phase formula, see Theorem 2.4, up to an element in $O(m^{-\infty})$, the I_7 can be written into

$$\begin{aligned}&\frac{m}{2\pi}e^{im(\tilde{\psi}_-(x,y,1,y_{2n+1}-\Phi(\mathring{x},\mathring{y}))+x_{2n+1})}A(x,y,m)\\ &= e^{im(x_{2n+1}-y_{2n+1}+\Phi(\mathring{x},\mathring{y}))}A(x,y,m),\end{aligned}$$

where

$$A(x,y,m) := \frac{\tilde{\chi}_3(1,y_{2n+1}-\Phi(\mathring{x},\mathring{y}))\tilde{\chi}_1(x,y_{2n+1}-\Phi(\mathring{x},\mathring{y}))\tilde{a}_-((\mathring{x},y_{2n+1}-\Phi(\mathring{x},\mathring{y})),y,m)\chi_0(y)}{\det\left(\frac{m\tilde{\psi}''_-(x,y,1,y_{2n+1}-\Phi(\mathring{x},\mathring{y}))}{2\pi i}\right)^{\frac{1}{2}}}, \tag{3.41}$$

is in the symbol space $S^n_{\rm loc}(1;D_p\times D_p)$, $\tilde{\chi}_3$, $\tilde{\chi}_1$ and $\tilde{a}_-$ is an almost analytic extensions of χ_3, χ_1 and a_- in the varaible (t,θ), respectively, and $\tilde{\psi}''_- := \begin{bmatrix}\frac{\partial^2\tilde{\psi}_-}{\partial\tilde{t}^2} & \frac{\partial^2\tilde{\psi}_-}{\partial\tilde{\theta}\partial\tilde{t}}\\ \frac{\partial^2\tilde{\psi}_-}{\partial\tilde{t}\partial\tilde{\theta}} & \frac{\partial^2\tilde{\psi}_-}{\partial\tilde{\theta}^2}\end{bmatrix}$. Note that by basic properties of symbol space, we have the expansion $A(x,y,m) \sim \sum_{j=0}^{\infty} m^{n-j}A_j(x,y)$ in $S^n_{\rm loc}(1;D_p\times D_p)$, where $A_j(x,y) \in \mathscr{C}^\infty(D_p\times D_p)$, and by the construction of almost analytic extension, we can find that

$$A_0(p,p) = (a_-)_0(p,p) = \frac{\det\mathcal{L}_p}{2\pi^{n+1}}. \tag{3.42}$$

We sum up the discussion so far as the following local result:

Proposition 3.12. *Let $p\in Y$, D be a BRT coordinates patch as in Theorem 2.2 near p and take an small enough $D_p\subset D$ such that $\overline{D}_p\subset D$. For any $(x,y)\in D_p\times D_p$, the m-th Fourier component of the Szegő kernel on lower energy functions is*

$$\Pi^{(0)}_{\leq\lambda,m}(x,y) \equiv e^{im(x_{2n+1}-y_{2n+1}+\Phi(\mathring{x},\mathring{y}))}A(x,y,m) \bmod O(m^{-\infty}).$$

Here, $\mathring{x}:=(x_1,\ldots,x_{2n})$, $\mathring{y}:=(y_1,\ldots,y_{2n})$*; the function* $\Phi(\mathring{x},\mathring{y})$ *is a complex-valued function satisfying:*

$$\mathrm{Im}\Phi(\mathring{x},\mathring{y})\geq C|\mathring{x}-\mathring{y}|^2,\ \textit{for some constant } C>0,$$

and

$$\Phi(\mathring{x},\mathring{y})=-\overline{\Phi(\mathring{y},\mathring{x})},\ \Phi(\mathring{x},\mathring{y})=0 \textit{ if and only if } \mathring{x}=\mathring{y}$$

for all $(x,y)\in D_p\times D_p$*; and* $A(x,y,m)\in S^n_{\mathrm{loc}}(1;D_p\times D_p)$ *with asymptotic expansion*

$$A(x,y,m)\sim\sum_{j=0}^{\infty}m^{n-j}A_j(x,y)\ \textit{in } S^n_{\mathrm{loc}}(1;D_p\times D_p),$$
$$A_j(x,y)\in\mathscr{C}^\infty(D_p\times D_p), A_0(p,p)=\frac{\det\mathcal{L}_p}{2\pi^{n+1}}.$$

Before preceding, note that in view of Proposition 3.9, we see that $\Pi^{(0)}_{\leq\lambda}$ is smoothing away from the diagonal. From this observation, we can eother use integration by parts with respect to θ in (3.27) for the case near Y or apply Thoerem 3.1 for the case away from Y to show that :

Proposition 3.13. *For* $(q_1,q_2)\in X\times X$ *such that* q_1 *and* q_2 *are not in the same* S^1*-orbit, then on* $U\times V$, $\Pi^{(0)}_{\leq\lambda,m}(x,y)=O(m^{-\infty})$*, where* U *and* V *are some open neighborhoods of* q_1 *and* q_2*, respectively.*

Now, for $x\in D$, if $(e^{i\theta_1},\ldots,e^{i\theta_d})\circ x\in D$, let

$$x':=x'(\theta_1,\ldots,\theta_d):=(\mathring{x}',x'_{2n+1}):=(e^{i\theta_1},\ldots,e^{i\theta_d})\circ(\mathring{x},x_{2n+1}).$$

Recall that

$$\begin{aligned}&\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y)\\&=\frac{1}{(2\pi)^d}\int_{T^d}\Pi^{(0)}_{\leq\lambda,m}\left((e^{i\theta_1},\ldots,e^{i\theta_d})\circ x,y\right)e^{-im\sum_{j=1}^d p_j\theta_j}d\theta_1,\ldots d\theta_d.\end{aligned}\tag{3.43}$$

Let $p\in Y$ with BRT coordinates patch D near p as in Theorem 2.2 and $(x,y)\in D\times D$. We first observe that $(e^{i\theta_1},\ldots,e^{i\theta_d})\circ x\notin D$, then

$$\Pi^{(0)}_{\leq\lambda,m}\left((e^{i\theta_1},\ldots,e^{i\theta_d})\circ x,y\right)=O(m^{-\infty}).$$

One way to see this is by writing

$$\Pi^{(0)}_{\leq\lambda,m}\Big((e^{i\theta_1},\dots,e^{i\theta_d})\circ x, y\Big) = e^{imy_{2n+1}}\Pi^{(0)}_{\leq\lambda,m}\Big((e^{i\theta_1},\dots,e^{i\theta_d})\circ x, \hat{y}\Big).$$

Clearly, $(e^{i\theta_1},\dots,e^{i\theta_d})\circ x$ and $\hat{y}$ must not in the same S^1-orbit, otherwise there exists a $\mathring{\theta}\in(-\pi,\pi)$ such that $(e^{i\theta_1},\dots,e^{i\theta_d})\circ x = e^{i\mathring{\theta}}\circ\hat{y} = (\mathring{y},\mathring{\theta})\in D$ leading to a contradiction. From Proposition 3.13, this implies that

$$\Pi^{(0)}_{\leq\lambda,m}\Big((e^{i\theta_1},\dots,e^{i\theta_d})\circ x, y\Big) = O(m^{-\infty}).$$

In view of (3.43), for simplicity, we assume

$$(e^{i\theta_1},\dots,e^{i\theta_d})\circ x\in D \text{ for all } (\theta_1,\dots,\theta_d)\in T^d \text{ and } x\in D. \tag{3.44}$$

from now on. Moreover,

Proposition 3.14. *Let $p\in Y$ and $D_p := \tilde{D}_p\times(-\epsilon,\epsilon)$, where ϵ is a small number, as in Proposition 3.12, for $(x,y)\in D_p\times D_p$, if $(e^{i\theta_1},\dots,e^{i\theta_d})\circ x\notin D_p$, then $\Pi^{(0)}_{\leq\lambda,m}((e^{i\theta_1},\dots,e^{i\theta_d})\circ x,y) = O(m^{-\infty})$.*

Proof. If $(e^{i\theta_1},\dots,e^{i\theta_d})\circ x$ and y are in the same S^1-orbit, i.e. there is a $\mathring{\theta}\in(-\pi,\pi)$ such that $(e^{i\theta_1},\dots,e^{i\theta_d})\circ x = e^{i\mathring{\theta}}\circ\hat{y} = (\mathring{y},\mathring{\theta})\notin D_p$, then there must be $|\theta|>\epsilon>|y_{2n+1}|$. Take cut-off functions $\chi_0,\chi_1\in\mathscr{C}^\infty_0(D)$, where $\chi_0\equiv 1$ on D_p and $\chi_1\equiv 1$ on $\mathrm{supp}(\chi)$. Pick $\tau\in\mathscr{C}^\infty_0(\mathbb{R})$, where $\tau(t)\equiv 1$ when $|t|\leq 1$ and $\tau(t)\equiv 0$ when $|t|\geq 2$, and set $\tau_j(t) := \tau(2^{-j}t)-\tau(2^{1-j}t)$, $j\in\mathbb{N}$, $\tau_0 := \tau$. From (3.29), (3.30), (3.31) and (3.32) we can find

$$\begin{aligned}
&\Pi^{(0)}_{\leq\lambda,m}((e^{i\theta_1},\dots,e^{i\theta_d})\circ x,y) = \Pi^{(0)}_{\leq\lambda,m}((\mathring{y},\theta),y)\\
&= \frac{e^{im\mathring{\theta}}}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{im\big((\theta-y_{2n+1}+\Phi(\mathring{y},\mathring{y}))t-\theta\big)}\chi_1(\mathring{y},\theta)a_-((\mathring{y},\theta),y,t)\chi_0(y)dtd\theta\\
&= \frac{e^{im\mathring{\theta}}}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{im\big((\theta-y_{2n+1})t-\theta\big)}\chi_1(\mathring{y},\theta)a_-((\mathring{y},\theta),y,t)\chi_0(y)dtd\theta\\
&= \frac{e^{im\mathring{\theta}}}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{im\big((\theta-y_{2n+1})t-\theta\big)}\tau_0(t)\chi_1(\mathring{y},\theta)a_-((\mathring{y},\theta),y,t)\chi_0(y)dtd\theta\\
&\quad+\sum_{j=1}^\infty\frac{e^{im\mathring{\theta}}}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{im\big((\theta-y_{2n+1})t-\theta\big)}\tau_j(t)\chi_1(\mathring{y},\theta)a_-((\mathring{y},\theta),y,t)\chi_0(y)dtd\theta\\
&=: I_9+I_{10}.
\end{aligned}$$

Since $\theta - y_{2n+1} \neq 0$, we can apply integration by parts with respect to t to show that $I_{10} = O(m^{-\infty})$; moreover, since χ_1 has compact support in θ, we can also apply integration by parts with respect to θ to show that the boundary term appeared in partial integration with respect to t in I_9 is also $O(m^{-\infty})$. So when $(e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x$ and y are in the same S^1-orbit, $\Pi^{(0)}_{\leq\lambda,m}((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x, y) = O(m^{-\infty})$. And if $(e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x$ and y are not in the same S^1-orbit, this proposition follows from Proposition 3.13. □

Again, from (3.43), for simplicity, we assume

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x \in D_p \text{ for all } (\theta_1, \ldots, \theta_d) \in T^d \text{ and } x \in D_p. \tag{3.45}$$

from now on. From (3.43), we can write

$$\Pi^{(0)}_{\leq\lambda,m}((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x, y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{im\Psi(x,y,\theta_1,\ldots\theta_d)} A(x', y, m) d\theta_1 \cdots d\theta_d \tag{3.46}$$

where the phase function is

$$\Psi(x, y, \theta_1, \ldots \theta_d) := x'_{2n+1} - y_{2n+1} + \Phi(\mathring{x}', w) - \sum_{j=1}^{d} p_j \theta_j,$$

and the symbol

$$B(x, y, \theta_1, \ldots, \theta_d, m) := A(x', y, m).$$

We shall also notice that when $(\theta_1, \ldots, \theta_d) \neq 0$, there must be $\mathring{x}' \neq \mathring{x}$, otherwise we will have

$$1 \circ (e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x = e^{i\theta_0} \circ (1, \ldots, 1) \circ x$$

where $\theta_0 := x'_{2n+1} - x_{2n+1} \bmod (-\pi, \pi)$, contradicting Assumption 1.2. Thus,

$$\mathrm{Im}\Psi(x, y, \theta_1, \ldots, \theta_d) = \mathrm{Im}\Phi(\mathring{x}', y) > 0 \text{ for } (\theta_1, \ldots, \theta_d) \neq 0.$$

We hence consider a cut-off function $\chi(\theta_1, \ldots, \theta_d) \in \mathscr{C}^\infty_0(\mathbb{R}^d)$ such that $\chi = 1$ near $(\theta_1, \ldots, \theta_d) = (0, \ldots, 0)$. Write

$$\frac{1}{(2\pi)^d} \int_{T^d} e^{im\Psi(x,y,\theta_1,\ldots\theta_d)} A(x', y, m) d\theta_1 \cdots d\theta_d = I_9 + I_{10}$$

where we cut the integrand of I_{11} by $\chi(\theta_1,\ldots,\theta_d)$ and the one for I_{12} by $1-\chi(\theta_1,\ldots,\theta_d)$. Note that in the integrand of I_{12} we have $\mathrm{Im}\Psi > 0$, so $I_{12} = O(m^{-\infty})$ by the elementary inequality that for any $m, N \in \mathbb{N}_{\not 0}$, $m^N e^{-m} \le C_N$ for some constant $C_N > 0$. As for the I_{11}, we shall apply the Melin–Sjöstrand stationary phase formula Proposition 2.3 to establish the asymptotic expansion for torus equivariant Szegő kernel.

We now fix a point $p \in Y$ and a small enough BRT patch D_p containing p. We claim that the point $(x, y, \theta_1, \ldots, \theta_d) = (p, p, 0, \ldots, 0)$ satisfies the requirement in the Proposition 2.3. To see this, first note that in real coordinates $(x, y) = (\mathring{x}, x_{2n+1}, \mathring{y}, y_{2n+1})$ we have

$$\frac{\partial\Psi}{\partial\theta_j} = -\frac{\partial x'_{2n+1}}{\partial\theta_j} + \frac{\partial\Phi(\mathring{x}',\mathring{y})}{\partial\mathring{x}'}\frac{\partial\mathring{x}'}{\partial\theta_j} - p_j.$$

Under the local expression of the phase function [21, Theorem 3.4] in terms of canonical coordinates, we have relations

$$\phi_-(x,y) = x_{2n+1} - y_{2n+1} + \Phi(\mathring{x},\mathring{y}) \text{ and } d_x\phi_-(x,y)|_{x=y} = -\omega_0(x). \quad (3.47)$$

We can hence get

$$-\omega_0(p) = \left(\frac{\partial\phi_-(x',y)}{\partial x'}dx'\right)\Bigg|_{\substack{x=y=p\\ \theta=0}} = \left(-dx_{2n+1} + \frac{\partial\Phi(\mathring{x}',\mathring{y})}{\partial\mathring{x}'}d\mathring{x}\right)(p,p,0) \quad (3.48)$$

and for $j = 1, \ldots, d$,

$$\begin{aligned} T_j(p) &:= \frac{\partial}{\partial\theta_j}\left((1,\ldots,e^{i\theta_j},\ldots,1)\circ x\right)\Big|_{\substack{x=p\\ \theta_j=0}} \\ &= \frac{\partial}{\partial\theta_j}\left((e^{i\theta_1},\ldots,e^{i\theta_j},\ldots,e^{i\theta_d})\circ x\right)\Big|_{\substack{x=p\\ \theta_1=\cdots=\theta_d=0}} \\ &= \frac{\partial x'}{\partial\theta_j}\frac{\partial}{\partial x}\Big|_{\substack{x=p\\ \theta_1=\cdots=\theta_d=0}} = \left(\frac{\partial x'_{2n+1}}{\partial\theta_j}\frac{\partial}{\partial x_{2n+1}} + \frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial}{\partial\mathring{x}}\right)(p,0). \quad (3.49)\end{aligned}$$

By the definition of Y and (3.48) and (3.49), we know that for each $j = 1, \ldots, d$,

$$p_j = \langle -\omega_0(p), T_j(p)\rangle = \left(-\frac{\partial x'_{2n+1}}{\partial\theta_j} + \frac{\partial\Phi(\mathring{x}',\mathring{y})}{\partial\mathring{x}'}\frac{\partial\mathring{x}'}{\partial\theta_j}\right)(p,p,0).$$

Thus

$$\frac{\partial\Psi}{\partial\theta_j}(p,p,0)=0 \text{ for all } j=1,\ldots,d.$$

Now, for $j,k=1,\ldots,d$, we need to show that

$$\left(\frac{\partial^2\Psi}{\partial\theta_j\partial\theta_k}\right)_{j,k=1}^{d}=\left(-\frac{\partial^2 x'_{2n+1}}{\partial\theta_j\partial\theta_k}+\frac{\partial^2\Phi}{\partial\mathring{x}'^2}\frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial\mathring{x}'}{\partial\theta_k}+\frac{\partial\Phi}{\partial\mathring{x}'}\frac{\partial^2\mathring{x}'}{\partial\theta_j\partial\theta_k}\right)_{j,k=1}^{d},$$

where $\frac{\partial^2\Phi}{\partial\mathring{x}'^2}\frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial\mathring{x}'}{\partial\theta_k}:=\sum_{a,b=1}^{2n}\frac{\partial^2\Phi}{\partial x'_a\partial x'_b}\frac{\partial x'_a}{\partial\theta_j}\frac{\partial x'_b}{\partial\theta_k}$ and $\frac{\partial\Phi}{\partial\mathring{x}'}\frac{\partial^2\mathring{x}'}{\partial\theta_j\partial\theta_k}:=\sum_{c=1}^{2n}\frac{\partial\Phi}{\partial x'_c}\frac{\partial^2 x'_c}{\partial\theta_j\partial\theta_k}$, $j,k=1,\ldots,d$, is a non-singular matrix at $(x,y,\theta_1,\ldots,\theta_d)=(p,p,0,\ldots,0)$. Under BRT coordinates, see Theorem 2.2, write $(z,w)=(z_1,\ldots,z_n,w_1,\ldots,w_n)$, where $z_j:=x_{2j_1}+ix_{2j}$ and $w_j:=y_{2j-1}+y_{2j}$. In [21, Theorem 3.6], it suggests that the term $\Phi(\mathring{x},\mathring{y})$ in the phase function is in the form of

$$\Phi(\mathring{x},\mathring{y})=i(\phi(z)+\phi(w))-2i\sum_{|\alpha|+|\beta|\le N}\frac{\partial^{\alpha+\beta}\phi}{\partial z^\alpha\partial\overline{z}^\beta}(0)\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!}+O(|(z,w)|^{N+1}), \tag{3.50}$$

for every $N\in\mathbb{N}$, where the function ϕ is given by $\phi(z)=\sum_{j=1}^n\lambda_j|z_j|^2+O(|z|^3)$ and $\lambda_1,\ldots,\lambda_n>0$ are the eigenvalues of the Levi form $\mathcal{L}_p$ at the point p. Hence, at $(x,y,\theta_1,\ldots,\theta_d)=(p,p,0,\ldots,0)$, we have

$$\frac{\partial\Phi}{\partial\mathring{x}'}(p,p,0)=0.$$

Observe an easy fact from linear algebra: if A and B are real symmetric matrix, and B is positive definite, then $C:=A+iB$ is non singular. (Consider the orthogonal decomposition $B=P^tP$, and $Q:=P^{-1}$, then $Q^tCQ=Q^tAQ+i\mathrm{Id}$. Suppose $\det C=0$, then $-i$ is an eigenvalues of A, contradicting the fact that all the eigenvalues of A are real). So it remains to show that

$$\mathrm{Im}\left(\frac{\partial^2\Phi}{\partial\mathring{x}'^2}\frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial\mathring{x}'}{\partial\theta_k}\right)(p,p,0) \text{ is positive definite.}$$

Since $\Phi(\mathring{x},\mathring{y})$ has leading term $\sum_{j=1}^n\lambda_j|z_j-w_j|^2$ where $\lambda_j>0$, $j=1,\ldots,d$, we know that

$$\mathrm{Im}\frac{\partial^2\Phi}{\partial\mathring{x}'^2}(p,p,0) \text{ is positive definite.}$$

To examine whether the submatrix

$$\mathrm{Im}\left(\frac{\partial^2\Phi}{\partial\mathring{x}'^2}\frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial\mathring{x}'}{\partial\theta_k}\right)(p,p,0)$$

is positive definite, it sufficies to take any $0 \neq u \in M_{d\times 1}(\mathbb{R})$ and $D := \begin{bmatrix} \frac{\partial x_1'}{\partial\theta_1} & \cdots & \frac{\partial x_1'}{\partial\theta_d} \\ \vdots & & \vdots \\ \frac{\partial x_{2n}'}{\partial\theta_1} & \cdots & \frac{\partial x_{2n}'}{\partial\theta_d} \end{bmatrix}(p,0)$ and check that

$$u^t\mathrm{Im}\left(\frac{\partial^2\Phi}{\partial\mathring{x}'^2}\frac{\partial\mathring{x}'}{\partial\theta_j}\frac{\partial\mathring{x}'}{\partial\theta_k}\right)(p,p,0)u = (Du)^t\mathrm{Im}\frac{\partial^2\Phi}{\partial\mathring{x}'^2}(p,p,0)(Du) > 0$$

which is equivalent to examine whether D is of full rank. And this is in fact guaranteed by assumption that the torus action is free near Y and the assumption that $p \in Y$ is a regular level set. One way to see this is to take the map $\sigma_x : T_eT^d \to T_xX$ by $\frac{\partial}{\partial\theta_j} \mapsto T_j = \frac{\partial}{\partial\theta_j}\Big|_{\theta_j=0}(1,\ldots,e^{i\theta_j},\ldots,1)\circ x$. For the torus action is free, i.e. $\{g \in T^d : g\circ x = x, x \text{ near } Y\} = \{e\}$, the map σ_x is injective. So the column vectors

$$[T_1,\ldots,T_j](p) = \begin{bmatrix} \frac{\partial x_1'}{\partial\theta_1} & \cdots & \frac{\partial x_1'}{\partial\theta_d} \\ \vdots & & \vdots \\ \frac{\partial x_{2n}'}{\partial\theta_1} & \cdots & \frac{\partial x_{2n}'}{\partial\theta_d} \\ \frac{\partial x_{2n+1}'}{\partial\theta_1} & \cdots & \frac{\partial x_{2n+1}'}{\partial\theta_d} \end{bmatrix}(p,0)$$

has rank d. Also, since p is in a regular level set, we know that

$$[d(T_1\lrcorner\omega_0),\ldots,d(T_d\lrcorner\omega_0)](p) = \begin{bmatrix} \frac{\partial\langle\omega_0,T_1\rangle_x}{\partial x_1} & \cdots & \frac{\partial\langle\omega_0,T_d\rangle_x}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial\langle\omega_0,T_1\rangle_x}{\partial x_{2n}} & \cdots & \frac{\partial\langle\omega_0,T_d\rangle_x}{\partial x_{2n}} \\ \frac{\partial\langle\omega_0,T_1\rangle_x}{\partial x_{2n+1}} & \cdots & \frac{\partial\langle\omega_0,T_d\rangle_x}{\partial x_{2n+1}} \end{bmatrix}(p)$$

is of rank d. Since the one form ω_0 is torus invariant, we know that $\mathcal{L}_{T_j}\omega_0 = 0$ for all $j = 1,\ldots,d$. So the Cartan formula $\mathcal{L}_{T_j}\omega_0 = T_j\lrcorner d\omega_0 + d(T_j\lrcorner\omega_0)$ suggests that the one forms $T_j\lrcorner d\omega_0(p) \neq 0$ for all $j = 1,\ldots,d$, otherwise for all $j = 1,\ldots,d$, $d(T_j\lrcorner\omega_0)(p) = 0$, leading to a contradiction. We also note that $T_0\lrcorner d\omega_0 \equiv 0$. One way to see this is that under BRT coordinates

Theorem 2.2 $T_0 = \frac{\partial}{\partial x_{2n+1}}$ and $d\omega_0$ is independent of x_{2n+1}. Write

$$T_j(p) = \sum_{k=1}^{2n+1} \frac{\partial x'_k}{\partial \theta_j}(p,0)\frac{\partial}{\partial x_k},$$

and if we suppose that

$$B := \begin{bmatrix} \frac{\partial x'_1}{\partial \theta_1} & \cdots & \frac{\partial x'_1}{\partial \theta_d} \\ \vdots & & \vdots \\ \frac{\partial x'_{2n}}{\partial \theta_1} & \cdots & \frac{\partial x'_{2n}}{\partial \theta_d} \end{bmatrix}(p,0)$$

has rank less than d, we can find numbers $(\alpha_1, \ldots, \alpha_d) \in \dot{\mathbb{R}}^d$ such that $\sum_{j=1}^d \alpha_k \frac{\partial x'_k}{\partial \theta_j}(p,0) = 0$ for all $k = 1 \cdots, 2n$ and $\sum_{j=1}^d \alpha_j \frac{\partial x'_{2n+1}}{\partial \theta_j}(p,0) \neq 0$ (recall $[T_1, \ldots, T_d](p)$ has rank d). However, this means that $T_0(p) = \frac{\sum_{j=1}^d \alpha_j T_j(p)}{\sum_{j=1}^d \alpha_j \frac{\partial x'_{2n+1}}{\partial \theta_j}(p,0)}$, which leads to a contradiction $(T_0 \lrcorner d\omega_0)(p) = 0$. So the matrix $D := \begin{bmatrix} \frac{\partial x'_1}{\partial \theta_1} & \cdots & \frac{\partial x'_1}{\partial \theta_d} \\ \vdots & & \vdots \\ \frac{\partial x'_{2n}}{\partial \theta_1} & \cdots & \frac{\partial x'_{2n}}{\partial \theta_d} \end{bmatrix}(p,0)$ has full rank d, and we conclude that

$$\det\left(\frac{\partial^2 \Psi}{\partial \theta_j \partial \theta_k}\right)_{j,k=1}^d (p,p,0) \neq 0.$$

The above discussion implies that the point $(x, y, \theta_1, \ldots, \theta_d) = (p, p, 0, \ldots, 0)$ satisfies the assumption in Proposition 2.3, where $p \in Y$ is fixed.

Let $\tilde{\theta}_1(x,y), \ldots, \tilde{\theta}_d(x,y)$ be the solution of the equations

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{\theta}_j}\left(x, y, \tilde{\theta}_1(x,y), \ldots, \tilde{\theta}_d(x,y)\right) = 0,\ j = 1, \ldots, d,$$

in a neighborhood of $(x,y) = (p,p) \in \mathbb{C}^{2n}$ with $\tilde{\theta}_j(x,y) = 0$ at $(x,y) = (p,p)$ for all $j = 1, \ldots, d$, where $\tilde{\Psi}$ ia an almost analytic extension of Ψ in the variable θ_j's and $\tilde{\theta}_j$'s are allowed to be complex near $(0, \ldots, 0) \in \mathbb{C}^d$. Accordingly, by complex stationary phase formula Proposition 2.4 and the basic properties of almost analytic extension, up to an element in $O(m^{-\infty})$, the torus equivariant Szegő kernel $\Pi^{(0)}_{\leq\lambda,m,mp_1,\dots,mp_d}(x,y)$ is

$$(2\pi)^{-d}e^{im\tilde{\Psi}(x,y,\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y))}\left(\det\frac{m\tilde{\Psi}''_{\tilde{\theta}\tilde{\theta}}(x,y,\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y))}{2\pi i}\right)^{-\frac{1}{2}} \tag{3.51}$$
$$\tilde{\chi}\left(\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y)\right)\tilde{A}(\widetilde{x'}(\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y)),y,m).$$

We let

$$b(x,y,m):=\frac{\tilde{\chi}\left(\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y)\right)\tilde{A}(\widetilde{x'}(\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y)),y,m)}{\left(\det\frac{m\tilde{\Psi}''_{\tilde{\theta}\tilde{\theta}}(x,y,\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y))}{2\pi i}\right)^{\frac{1}{2}}}\neq 0,$$

and we can check that $b(x,y,m)\in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1;D_p\times D_p)$. Moreover, for the asymptotic sum

$$b(x,y,m)\sim\sum_{j=0}^{\infty}m^{n-\frac{d}{2}-j}b_j(x,y)\text{ in }S^{n-\frac{d}{2}}_{\mathrm{loc}}(1;D_p\times D_p),$$

we can find that for $p\in Y$, by construction of almost analytic functions, for some constant $C>0$,

$$b_0(p,p)=\frac{1}{(2\pi)^{\frac{d}{2}}}\frac{A_0(p,p)}{\left(\det\frac{\Psi''_{\theta\theta}(p,p,0,\dots,0)}{i}\right)^{\frac{1}{2}}}\geq C\frac{|\det\mathcal{L}_p|}{2^{\frac{d}{2}+1}\pi^{n+1+\frac{d}{2}}}>0. \tag{3.52}$$

Here, we choose the branch as in Proposition 2.4 such that $\left(\det\frac{\Psi''_{\theta\theta}(p,p,0,\dots,0)}{i}\right)^{\frac{1}{2}}$ >0. To finish the proof of Theorem 3.3, it establish the followings:

Proposition 3.15. *Let* $f(x,y):=\tilde{\Psi}(x,y,\tilde{\theta}_1(x,y),\dots,\tilde{\theta}_d(x,y))$*, then* $\mathrm{Im}f\geq 0$*. Moreover,*

$$f(x,x)=0,\ d_xf(x,x)=-\omega_0(x),\ d_yf(x,x)=\omega_0(x),\ b_0(x,x)>0$$
$$\text{for all }x\in Y\cap D_p.$$

Proof. First of all, by Proposition 2.3 and 2.4, $\mathrm{Im}f\geq 0$ holds. Second, for $p\in Y$ and a small BRT patch D_p near p, by the construction of almost analytic extension and (3.50), we can find

$$f(p,p)=\tilde{\Psi}(p,p,0)=\Phi(0,0)=0. \tag{3.53}$$

By continuity, we may assume that $|f(x,y)|<\frac{1}{2}$ on $D_p\times D_p$ by taking D_p small enough. Also, for $k=1,\ldots,2n+1$,

$$\begin{aligned}\frac{\partial f}{\partial x_k}(p,p) &= \frac{\partial\tilde{\Psi}}{\partial x_k}(p,p,0)+\sum_{l=1}^{d}\frac{\partial\tilde{\Psi}}{\partial\tilde{\theta}_l}(p,p,0)\frac{\partial\tilde{\theta}}{\partial x_k}(p,p,0)\\ &= \frac{\partial\Phi}{\partial x_k}(0,0)=\begin{cases}0: k=1,\ldots,2n.\\ 1: k=2n+1.\end{cases}\end{aligned}$$

and similarly

$$\frac{\partial f}{\partial y_k}(p,p)=\begin{cases}0: k=1,\ldots,2n.\\ -1: k=2n+1.\end{cases}$$

We can check that on D_p, $\omega_0(x)=-dx_{2n+1}+i\sum_{j=1}^{n}\left(\frac{\partial\phi}{\partial z_j}dz_j-\frac{\partial\phi}{\partial d\overline{z}_j}d\overline{z}_j\right)$, and hence

$$d_xf(p,p)=-\omega_0(p),\ d_yf(p,p)=\omega_0(p). \tag{3.54}$$

Second, for all $x\in Y\cap D_p$, we now prove

$$f(x,x)=0 \text{ and } d_xf(x,x)=-\omega_0(x),\ d_yf(x,x)=\omega_0(x).$$

For $p\in Y$ and a small BRT patch D_p near p, if we take any other $q\in Y\cap D_p$ and another small BRT patch D_q near q, by the discussion in this section, we can write

$$\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y)\equiv e^{imf(x,y)}b(x,y,m) \bmod O(m^{-\infty}) \text{ on } D_p\times D_p,$$

where $f(p,p)=0$, $df(p,p)=0$, $|f(x,y)|<\frac{1}{2}$ on $D_p\times D_p$,

$$b(x,y,m)\sim\sum_{j=0}^{\infty}m^{n-\frac{d}{2}-j}b_j(x,y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1;D_p\times D_p),$$

$b_j(x,y)\in\mathscr{C}^\infty(D_p\times D_p)$, $b_0(p,p)>0$

and

$$\Pi^{(0)}_{\leq\lambda,m,mp_1,\ldots,mp_d}(x,y) \equiv e^{imf_1(x,y)}b_1(x,y,m) \bmod O(m^{-\infty}) \text{ on } D_q \times D_q,$$

where $f_1(q,q)=0$, $df_1(q,q)=0$, $|f_1(x,y)|<\frac{1}{2}$ on $D_q\times D_q$,

$$b_1(x,y,m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j}(b_1)_j(x,y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; D_q\times D_q),$$

$(b_1)_j(x,y)\in\mathscr{C}^\infty(D_q\times D_q)$, $(b_1)_0(q,q)>0$.

By consinuity, we may assume that $|b_0(x,y)|>0$ on $D_p\times D_p$. We arrange

$$e^{imf(x,y)}b(x,y,m) = e^{imf_1(x,y)}b_1(x,y,m)+R(x,y,m),\ R(x,y,m)=O(m^{-\infty}). \tag{3.55}$$

into

$$e^{im(f-f_1)(x,y)}b(x,y,m) = b_1(x,y,m)+e^{-imf_1(x,y)}R(x,y,m), \tag{3.56}$$

and after evaluating at the point $(x,y)=(q,q)$, we get

$$e^{im(f-f_1)(q,q)}b(q,q,m)=b_1(q,q,m).$$

Since

$$\lim_{m\to\infty} e^{-m\mathrm{Im}f(q,q)} = \lim_{m\to\infty}\frac{|b_1(q,q,m)|}{|b(q,q,m)|} = \frac{|(b_1)_0(q,q)|}{|b_0(q,q)|},$$

which is a non-zero finite number, we can conclude that $\mathrm{Im}f(q,q)=0$. Moreover, notice that

$$\lim_{m\to\infty} e^{imf(q,q)} = \lim_{m\to\infty}\frac{b_1(q,q,m)}{b(q,q,m)} = \frac{(b_1)_0(q,q)}{b_0(q,q)},$$

i.e. the limit exists. However, the limit

$$\lim_{m\to\infty} e^{imf(q,q)} = \lim_{m\to\infty} e^{im\mathrm{Re}f(q,q)} = \lim_{m\to\infty}(\cos(m\mathrm{Re}f(q,q))+i\sin(m\mathrm{Re}f(q,q)))$$

does not exists if $|\mathrm{Re}f(q,q)|<\frac{1}{2}$, $\mathrm{Re}f(q,q)\neq 0$. Hence, we conclude that

$$f(x,x)=0, \text{ for all } x\in Y\cap D_p. \tag{3.57}$$

Next, take derivative in both sides of (3.55) with respect to x_j, $j = 1, \ldots$, $2n+1$, and evaulate at $(x, y) = (q, q)$, then we have

$$im\frac{\partial}{\partial x_j}(f - f_1)(q,q)b(q,q,m) + \frac{\partial}{\partial x_j}b(q,q,m) = \frac{\partial}{\partial x_j}b_1(q,q,m) + \mathring{R}(x,y,m),$$

where $\mathring{R}(x,y,m) := \frac{\partial}{\partial x_j}R(q,q,m) - im\frac{\partial}{\partial x_j}f_1(q,q)R(q,q,m) = O(m^{-\infty})$. Therefore, for some constant $C > 0$,

$$\begin{aligned}\left|\frac{\partial}{\partial x_j}(f - f_1)(q,q)\right| &= \lim_{m\to\infty}\frac{\left|\left(\frac{\partial}{\partial x_j}(b_1 - b) + \mathring{R}\right)(q,q,m)\right|}{m|b(q,q,m)|}\\ &\leq \lim_{m\to\infty}\frac{Cm^{n-\frac{d}{2}}}{|b_0(q,q)|m^{n-\frac{d}{2}+1}} = 0\end{aligned}$$

Hence, for all $q \in Y \cap D_p$,

$$\frac{\partial}{\partial x_j}\left(f - f_1\right)(q,q) = 0,\ j = 1, \ldots, 2n+1, \tag{3.58}$$

Similarly, for all $q \in Y \cap D_p$,

$$\frac{\partial}{\partial y_j}\left(f - f_1\right)(q,q) = 0,\ j = 1, \ldots, 2n+1. \tag{3.59}$$

Combine (3.54), (3.58) and (3.59), we establish

$$d_x f(x,x) = -\omega_0(x),\ d_y f(x,x) = \omega_0(x) \text{ for all } x \in Y \cap D_p. \tag{3.60}$$

In the last, from (3.55), by evaluating at the point $(x,y) = (q,q)$, we can find

$$b(q,q,m) = b_1(q,q,m) + R(q,q,m),$$

where $R(q,q,m) = O(m^{-\infty})$. Accordingly,

$$1 = \lim_{m\to\infty}\frac{b(q,q,m)}{b_1(q,q,m)} = \frac{b_0(q,q)}{(b_1)_0(q,q)}.$$

Thus, we can conclude that $b_0(q,q) = b_1(q,q) > 0$, and since this holds for all $q \in Y \cap D_p$, we complete the proof of this proposition. $\square$

Combine all the discussion in this section, the proof of Theorem 3.2 is completed.

Acknowledgment

The methods of microlocal analysis on CR manifolds with group action used in this work are marked by the influence of Professor Chin-Yu Hsiao and Professor George Marinescu. The author in particular wishes to express his hearty thanks to them for discussions on similar subjects and for giving the idea of the proof.

References

1. M. S. Baouendi and L. P. Rothschild and F. Tréves, CR structures with group action and extendability of CR functions, *Inventiones Mathematicae*, **82** (1985), no.2, 359-396.

2. R. Beals and P. C. Greiner, Calculus on Heisenberg manifolds, *Annals of Mathematics Studies*, vol.**119**, Princeton University Press, 1988.

3. Louis Boutet de Monvel and Victor Guillemin, The spectral theory of Toeplitz operators, *Annals of Mathematics Studies*, vol. **99**, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.

4. Boutet de Monvel, Louis and Sjöstrand, Johannes, Sur la singularité des noyaux de Bergman et de Szegő, *Journées Équations aux Dérivées Partielles*, (1975), 123-164. Astérisque, No.34-35.

5. J.-H. Cheng and C.-Y. Hsiao and I-H. Tsai, Heat kernel asymptotics, local index theorem and trace integrals for Cauchy-Riemann manifolds with S^1 action Mémoires de la Société Mathématique de France, **162** (2019), 140.

6. E. B. Davies, Spectral theory and differential operators, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge University Press, 1995.

7. M. Dimassi and J. Sjöstrand, Spectral asymptotics in the semi-classical limit, Vol.268, Cambridge University Press, 1999.

8. K. Fritsch, H. Herrmann and C.-Y. Hsiao, G-equivariant embedding theorems for CR manifolds of high codimension, arXiv:1810.09629, 2018.

9. A. Grigis and J. Sjöstrand, Microlocal analysis for differential operators: An introduction, Lecture Note Series, vol.196, London Mathematical Society, 1994.

10. V. Guillemin, Star products on compact pre-quantizable symplectic manifolds, *Letters in Mathematical Physics*, **35** (1995), no.1, 85-89.

11. H. Herrmann and C.-Y. Hsiao and X. Li, Szegő kernels and equivariant embedding theorems for CR manifolds, arXiv:1710.04910v2, 2017.

12. H. Herrmann and C.-Y. Hsiao and X. Li, Szegő kernel expansion and equivariant embedding of CR manifolds with circle action, *Annals of Global Analysis and Geometry*, **52** (2017), no.3, 313-340.

13. H. Herrmann and C.-Y. Hsiao and X. Li, Torus equivariant Szegő kernel asymptotics on strongly pseudoconvex CR manifolds, arXiv:1808.01927, 2018.

14. L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Reprint of the second (1990) edition, Springer, Berlin, 2003.

15. C.-Y. Hsiao, Projections in several complex variables, *Mémoires de la Société Mathématique de France*, **123** (2010), 131.

16. C.-Y. Hsiao and R.-T. Huang, G-invariant Szegő kernel asymptotics and CR reduction, arXiv:1702.05012v3, 2017.

17. C.-Y. Hsiao, X. Ma, and G. Marinescu, Geometric quantization on CR manifolds, arXiv:1906.05627, 2019.

18. C.-Y. Hsiao and G. Marinescu, Szegő kernel asymptotics and Morse inequalities on CR manifolds, *Mathematische Zeitschrift*, **271** (2012), no.1-2, 509-553.

19. C.-Y. Hsiao and G. Marinescu, Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles, *Communications in Analysis and Geometry*, **22** (2017), no.1, 1-108.

20. C.-Y. Hsiao and G. Marinescu, Berezin–Toeplitz quantization for lower energy forms, *Communications in Partial Differential Equations*, **42** (2017), no.6, 895-942.

21. C.-Y. Hsiao and G. Marinescu, On the singularities of the Szegő projections on lower energy forms, *Journal of Differential Geometry*, **107** (2017), no.1, 83-155.

22. X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, Birkhäuser Verlag, 2007.

23. A. Melin and J. Sjöstrand, Fourier integral operators with complex-valued phase functions, *Fourier Integral Operators and Partial Differential Equations*, 120-223. Lecture Notes in Mathematics, Vol.459, Springer, Berlin, 1975.

24. S. Zelditch, Szegő kernels and a theorem of Tian, *International Mathematics Research Notices*, **6** (1998), 317-331.