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FINITE GROUPS OF SYMPLECTIC AUTOMORPHISMS OF HYPERKÄHLER MANIFOLDS OF TYPE $K3^{[2]}$

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Abstract

We determine the possible finite groups G of symplectic automorphisms of hyperkähler manifolds which are deformation equivalent to the second Hilbert scheme of a K3 surface. We prove that G has such an action if, and only if, it is isomorphic to a subgroup of either the Mathieu group M_{23} having at least four orbits in its natural permutation representation on 24 elements, or one of two groups $3^{1+4}{:}2.2^2$ and $3^4{:}A_6$ associated to $\mathcal{S}$-lattices in the Leech lattice. We describe in detail those G which are maximal with respect to these properties, and (in most cases) we determine all deformation equivalence classes of such group actions. We also compare our results with the predictions of Mathieu Moonshine.

1. Introduction

A *hyperkähler manifold* is a $4n$-dimensional compact Riemannian manifold with holonomy group contained in $\mathrm{Sp}(n)$. Such a manifold is of *type* $K3^{[2]}$ if it is deformation equivalent to the second Hilbert scheme of a K3 surface. An example of a K3 surface is the Fermat quartic $Y \subset \mathbf{CP}^3$ given by the equation $x_0^4+x_1^4+x_2^4+x_3^4=0$. An isometry of a hyperkähler manifold fixing the complex structures is called a *symplectic automorphism.* See [39] for a review of basic properties of hyperkähler manifolds.

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In the present paper, we determine and study those finite groups G which can occur as groups of symplectic automorphisms of hyperkähler manifolds of type $K3^{[2]}$. Recent work of Mongardi [50] shows that G is isomorphic to a subgroup of the Conway group Co_0, the group of isometries of the Leech lattice Λ. Moreover, the fixed-point sublattice Λ^G must have rank at least 4. Mongardi also gave (loc. cit.) restrictions on the possible automorphisms of prime order. In a well-known paper [56], Mukai showed that a finite group of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group M_{23} having at least five orbits on its defining action on 24 elements. Our main result is the following theorem, which may be regarded as a higher-dimensional analog of Mukai's result.

Theorem A. Let G be a finite group of symplectic automorphisms of a hyperkähler manifold of type $K3^{[2]}$. Then G is isomorphic to one of the following:

(a) a subgroup of M_{23} with at least four orbits in its natural action on 24 elements,

(b) a subgroup of one of two subgroups $3^{1+4}{:}2.2^2$ and $3^4{:}A_6$ of Co_0 associated to $\mathcal{S}$-lattices in the Leech lattice.

There are 13 isomorphism classes of subgroups of type (a) that are *maximal* in the poset of all such groups. We will describe them and the two maximal groups of type (b) in detail.

By explicit construction, Mukai also showed (loc. cit.) that every subgroup of M_{23} that satisfies the conditions of his theorem indeed occurs as a group of symplectic automorphisms of a $K3$ surface. These groups also act on the corresponding Hilbert schemes, thereby providing examples of groups G as in part (a) of Theorem A, and examples explicitly realizing several more of the maximal groups are known. We will establish the full analog of Mukai's result, namely:

Theorem B. Each group G in Theorem A can be realized as group of symplectic automorphisms of some hyperkähler manifold of type $K3^{[2]}$.

Hashimoto has classified [33] all the deformation equivalence classes of finite symplectic group actions on K3 surfaces. He found that for each group permitted by Mukai's theorem there is *a unique* such class, with the exception of five cases where there are two such classes. We obtain a similar result for $K3^{[2]}$:

Theorem C. There are at least 243 deformation classes of finite symplectic group actions on hyperkähler manifold of type $K3^{[2]}$.

We can deduce Theorem C from the following purely lattice-theoretic result:

Theorem D. Let L be the unique even, integral lattice of signature $(3, 20)$ and discriminant group of order 2. There are at least 243 conjugacy classes of subgroups G of the isometry group $O(L)$ of L such that the orthogonal complement L_G of the fixed-point lattice L^G in L satisfies the following three properties:

(i) L_G is negative-definite;
(ii) L_G contains no vectors of norm -2;
(iii) L_G contains no vectors v of norm -10 such that $v/2$ is contained in the dual lattice L^*.

The different classes can be read off from Tables 12, 13 and 9.

The methods used in our paper are based on ideas developed by Nikulin, Mukai, Kondō and Hashimoto for K3 surfaces [57, 56, 41, 33]. We also use fundamental results on the geometry of hyperkähler manifolds obtained by many authors in recent decades, including work on the global Torelli theorem due to Huybrechts, Markman and Verbitsky. Recent results of Mongardi are crucial in allowing us to achieve a complete classification.

We also provide a somewhat new and more conceptual proof of Mukai's original result on symplectic automorphisms of K3 surfaces [56]. To explain this, let N denote the $K3$-lattice, i.e., the unique even, unimodular, integral lattice of signature $(3, 19)$, and let G be a group of symplectic automorphisms of a $K3$ surface. Kondō showed in [41] that the lattice $N_G(-1) \oplus A_1$ can be embedded into one of the 23 Niemeier lattices with roots, and a case-by-case analysis reveals that G must be a subgroup of M_{23} with at least five orbits. Conversely, in the appendix of [41], and again by a case-by-case analysis, Mukai is able to realize each group arising from such a lattice construction as symplectic automorphisms. In [33], Hashimoto computed all possible embeddings of $N_G(-1) \oplus A_1$ into the Niemeier lattices, and it turns out that the 82 isomorphism types of group lattices (N_G, G) are in one-to-one correspondence with the combinatorial structure of symplectic group actions as determined previously in [62]. In our approach, we embed $N_G(-1)$ into the Leech lattice Λ. This essentially reduces the computation of the group lattices (N_G, G) to the *group-theoretic problem* of enumerating all conjugacy classes of subgroups $G \subseteq \mathrm{Co}_0$ such that $rk(\Lambda^G) \geq 5$, and indeed we find that there are just 82 isomorphism types of such (N_G, G). In addition, we clarify the result [33] that (N_G, G) together with N^G uniquely determine the conjugacy class of G in $O(N)$. This is achieved by an improved *group theoretical analysis* of embeddings $N_G \oplus N^G \subseteq N$. To a large extent, the above analysis of symplectic automorphisms of K3 surfaces is contained in the corresponding analysis for $K3^{[2]}$. We will mention the results for K3 surfaces, and possible necessary modifications, at relevant points in the paper.

Much of our interest in the subject matter of the present paper originates from issues surrounding moonshine. Hirzebruch suggested [36] that the Witten genus of a hypothetical 24-dimensional monster manifold could be related to monstrous moonshine. Furthermore, the equivariant denominator identity of the monster Lie algebra can be interpreted as the equivariant second quantized Witten genus of a monster manifold as noted by the first author [38]. Mathieu Moonshine [23] connects the Mathieu group M_{24} with the complex elliptic genus of a K3 surface. It seems natural to investigate geometric questions dealing with the equivariant second quantized complex elliptic genus of a K3 surface. See also [12] for a physical interpretation. Mathieu Moonshine is also closely related to a multiplicative version of Moonshine for M_{24} found by the second author [47]. Important input also came from recent work of Gaberdiel, Hohenegger and Volpato [28], where the lattice approach of Kondō for K3 surfaces was partially generalized to sigma models on K3 surfaces.

The paper is organized as follows. In Section 2 we cover required background about integral lattices and hyperkähler manifolds of type $K3^{[2]}$. In Section 3 we discuss the conjugacy classes of subgroups $G \subseteq \mathrm{Co}_0$ with $\mathrm{rk}\,(\Lambda^G) \geq 4$. Building on Mongardi's work, in Section 4 we show that there are exactly 15 conjugacy classes of elements in Co_0 (the *admissible* classes) that can occur as symplectic automorphisms of $K3^{[2]}$. This is achieved by applying the equivariant Atiyah-Singer index formula to Hirzebruch's χ_y-genus. In Section 5, a group-theoretic analysis based on this conjugacy restriction then shows that the only groups satisfying $\mathrm{rk}\,(\Lambda^G) \geq 4$ and consisting only of admissible elements are those described in parts (a) and (b) of Theorem A, or certain groups of order 12, 16, 32, 48 or 64. Apart from a certain (inevitable) amount of computer calculation, the methods here are an extension of those used in [48] to study the corresponding problem for K3 surfaces. In Section 6, we show that there are exactly 198 conjugacy classes of such groups in Co_0. In Section 7 we determine — apart from a few cases — the conjugacy classes of groups in $O(L)$ that arise from these 198 conjugacy classes, while in Section 8 we determine which of these conjugacy classes arise from symplectic group actions on some $K3^{[2]}$. In the final section, we compare the equivariant complex elliptic genus of a $K3^{[2]}$ with the predictions of Mathieu Moonshine applied to the second quantized elliptic genus. In the appendix, we describe the conjugacy classes of subgroups $G \subseteq \mathrm{Co}_0$ found in Section 6, together with additional information about G and the corresponding lattices L_G.

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2. Background on $K3^{[2]}$

2.1. Integral lattices

We introduce some notation related to integral lattices and record some results that we will need.

Let $A = (A, q)$ be a *finite quadratic space*, i.e., a finite abelian group A together with a quadratic form $q : A \longrightarrow \mathbf{Q}/2\mathbf{Z}$. We denote the corresponding orthogonal group by $O(A)$. This is the subgroup of $\mathrm{Aut}(A)$ that leaves q invariant.

Let L be an even integral lattice, with *dual lattice* L^*. The *discriminant group* L^*/L is equipped with the *discriminant form* $q_L : L^*/L \to \mathbf{Q}/2\mathbf{Z}$, $x + L \mapsto \langle x, x\rangle \bmod 2\mathbf{Z}$. This turns L^*/L into a finite quadratic space, called the *discriminant space* of L and denoted by $A_L := (L^*/L, q_L)$. We let $O(L) := \mathrm{Aut}(L)$ be the automorphism group (i.e. group of isometries) of L.

Automorphisms in $O(L)$ induce orthogonal transformations of the discriminant space A_L. This leads to the short exact sequence

$$1 \longrightarrow O_0(L) \longrightarrow O(L) \longrightarrow \overline{O}(L) \longrightarrow 1,$$

where $\overline{O}(L)$ is the subgroup of $O(A_L)$ induced by $O(L)$ and $O_0(L)$ consists of the automorphisms of L which act trivially on A_L.

Let $\mathcal{L}$ be the category whose objects are even, integral lattices and whose morphisms are *injective isometries*. The category of *group lattices* consists of objects (L, G) where L is an object of $\mathcal{L}$ and $G \subseteq \mathrm{Aut}_{\mathcal{L}}(L) = O(L)$ is a subgroup. A morphism $(L, G) \to (L', G')$ of group lattices is a pair (ι, j) where $\iota : L \to L'$ is a morphism in $\mathcal{L}$ and $j : G \to G'$ is an *injective* morphism of groups such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} L & \xrightarrow{g} & L \\ \downarrow \iota & & \downarrow \iota \\ L' & \xrightarrow{j(g)} & L' \end{array}.$$

In particular, if (L, G) is a group lattice and $\iota : L \to L'$ an *isomorphism* in $\mathcal{L}$ we set

$$\iota[G] := \{\iota \circ g \circ \iota^{-1} \mid g \in G\}.$$

Then $(L', \iota[G])$ is a group lattice isomorphic to (L, G). Upon identifying L and L', this just means that G and G' are conjugate subgroups of $O(L)$.

The *invariant* and *coinvariant* lattices of a group lattice (L, G) are respectively defined as follows:

$$\begin{aligned} L^G &= \{x \in L \mid gx = x \text{ for all } g \in G\}, \\ L_G &= \{x \in L \mid (x, y) = 0 \text{ for all } y \in L^G\}. \end{aligned}$$

These are both *primitive sublattices* of L, i.e., L/L^G and L/L_G are free abelian groups. The restriction of the G-action to L_G turns it into a group lattice (L_G, G). Moreover, G acts trivially on the discriminant group A_{L_G}.

As a matter of notation, by $L(n)$ we will mean a lattice L with norms scaled by an integer n. We also note that the *genus* of an even integral lattice L is determined by the quadratic space A_L together with the signature of L [57].

2.2. Automorphisms of $K3^{[2]}$

In this subsection we fix the following notation: X is a hyperkähler manifold of type $K3^{[2]}$, $\mathrm{Aut}(X)$ is the group of *symplectic* automorphisms of X, and $G \subseteq \mathrm{Aut}(X)$ is a *finite* subgroup.

The second integral cohomology $L := H^2(X, \mathbf{Z})$ admits a non-degenerate symmetric integral bilinear form $(. \, , \, .)$, called the *Beauville-Bogomolov* form, with respect to which L is isomorphic to the lattice $E_8(-1)^2 \oplus U^3 \oplus \langle -2 \rangle$ of signature $(3, 20)$. Here, $E_8(-1)$ denotes the unique even, unimodular, negative-definite lattice of rank 8, U the hyperbolic plane and $\langle -2 \rangle \cong A_1(-1)$ the 1-dimensional lattice generated by a vector of norm -2. The discriminant space $(A_L, q_L) \cong (A_{A_1}, -q_{A_1})$ has order 2. The group $\overline{O}(L)$ is trivial, i.e., $O(L)$ acts trivially on A_L.

There is an *injective* map ([50, 35, 5])

$$\nu : \mathrm{Aut}(X) \longrightarrow O(L)$$

by which we may, and shall, identify G with its image in $O(L)$.

Theorem 2.1 (Mongardi [51], Lemma 3.5)**.** *The coinvariant lattice L_G has the following properties:*

(i) *L_G is negative definite.*

(ii) *L_G contains no vectors of norm -2.*

It is also shown that L_G is contained in the Picard lattice of X.

Recall that the *Leech lattice* Λ is the unique positive-definite, even, unimodular lattice of rank 24 without roots. Its automorphism group is the Conway group Co_0.

Theorem 2.2. *There is an embedding of group lattices* $(L_G(-1), G) \to (\Lambda, \mathrm{Co}_0)$ *such that* $(L_G(-1), G) \cong (\Lambda_G, G)$.

Proof. The discriminant form q_L of L is the negative of the discriminant form q_{A_1} of the root lattice $A_1 = \langle 2 \rangle$. This permits us to extend the lattice $L \oplus A_1$ by the coset $(x, y) \in A_L \oplus A_{A_1}, (x \neq 0, y \neq 0)$ to the unique even unimodular lattice M of signature $(4, 20)$, thus providing a primitive embedding of L into M. Since $\overline{O}(L)$ is trivial, the G-action extends to M, thereby fixing the sublattice $L^G \oplus A_1 \subseteq M$. Because L_G is negative-definite, $L^G \oplus A_1 \subseteq M$ has signature $(4, 20 - \mathrm{rk}\, L_G)$. In particular, we can find a 4-dimensional positive-definite subspace $\Pi \subset M \otimes \mathbf{R}$ such that $L_G = \Pi^\perp \cap L$. It was shown in [28] (see also [40], Prop. 2.2) that because L_G is negative-definite, G can be embedded into the Conway group (a result first shown in [50]). Indeed, the proof actually shows that this corresponds to an embedding $(L_G(-1), G) \to (\Lambda, \mathrm{Co}_0)$ of group lattices with $(L_G(-1), G) \cong (\Lambda_G, G)$. ☐

Note that because $\mathrm{rk}\, L_G \leq 20$ then $\mathrm{rk}\, \Lambda^G \geq 4$. Theorem 2.2 allows us to identify G with a subgroup of Co_0. We will see in Section 6 that for a given group lattice (L_G, G), the resulting embedding of $G \to \mathrm{Co}_0$ is *unique* up to conjugation in Co_0.

3. $\mathcal{S}$-lattices and Subgroups of Co_0

We have seen in the previous section that a finite group G of symplectic automorphisms of a hyperkähler manifold X of type $K3^{[2]}$ defines a subgroup $G \subseteq \mathrm{Co}_0$ with the property that $\mathrm{rk}\,(\Lambda^G) \geq 4$. In this section, we will establish some general results about such G.

Recall [14] that the 2^{24} cosets comprising $\Lambda/2\Lambda$ have representatives v which may be chosen to be *short vectors*, i.e., $(v, v) \leq 8$. More precisely, if $(v,v) \leq 6$ then $\{v, -v\}$ are the *only* short representatives of $v + 2\Lambda$; if $(v,v) = 8$ then the short vectors in $v + 2\Lambda$ comprise a *coordinate frame* $\{\pm w_1, \ldots, \pm w_{24}\}$, where the w_j are pairwise orthogonal vectors of norm 8. In particular, if $u \in \Lambda$ then $u = v + 2w$ for some v, $w \in \Lambda$ and v a short vector, and if $(v,v) \leq 6$ then v is *unique* up to sign.

It is well-known that Co_0 acts *transitively* on coordinate frames, the (setwise) stabilizer of one such being the *monomial group* $2^{12}{:}M_{24}$.

A sublattice $S \subseteq \Lambda$ is an *$\mathcal{S}$-lattice* if, for every $u \in S$, the corresponding short vector v satisfies $(v,v) \leq 6$ and furthermore $w \in S$. This concept, introduced by Curtis [18], will be very useful.

The following theorem depends in an essential way on a result of Allcock [1]. See also [28].

Theorem 3.1. *Suppose that $G \subseteq \mathrm{Co}_0$ is a subgroup with $\mathrm{rk}\,\Lambda^G \geq 4$. One of the following holds.*

(a) *G leaves a coordinate frame invariant,*

(b) *Λ^G is an $\mathcal{S}$-lattice of rank 4,*

(c) *Λ^G is contained in a G-invariant $\mathcal{S}$-lattice of rank larger than 4.*

Proof. Let $L := \Lambda^G$. We may assume that (a) does not hold. Suppose that $u \in L$ with $u = v + 2w$ and $(v,v) \leq 8$. Then $v + 2\Lambda = u + 2\Lambda$ is G-invariant, and since G leaves no coordinate frame invariant then we have $(v,v) \leq 6$. Thus G acts on $\{\pm v\}$. We claim that $v \in L \cup L^{\perp}$. For if $v \notin L$ there is a $g \in G$ such that $g(v) = -v$. Then for $x \in L$ we obtain $(x,v) = (g(x), g(v)) = (x, -v)$, showing that $v \in L^{\perp}$.

We use results of Allcock [1], especially (a special case of) Lemma 4.8 (loc. cit.) which we state as follows: suppose that $L \subseteq \Lambda$ is a primitive sublattice of rank at least 4 with the property that if $u \in L$ with $u = v + 2w$ and $(v,v) \leq 8$, then $(v,v) \leq 6$ and $v \in L \cup L^{\perp}$. Then L is *contained* in an $\mathcal{S}$-lattice. The previous paragraph establishes that $L = \Lambda^G$ satisfies these properties, so L is contained in an $\mathcal{S}$-lattice. Because the family of $\mathcal{S}$-lattices containing L is closed under intersection and G-conjugation, there is a G-*invariant* $\mathcal{S}$-lattice that contains L. Let S be such a lattice.

If $\mathrm{rk}\,S = 4$ then $L \subseteq S$ has finite index because of our assumption that $\mathrm{rk}\,L \geq 4$. Then $L = S$ because L is primitive, and we are in case (b) of the

Theorem. Otherwise $\operatorname{rk} S \geq 5$ and (c) holds. This completes the proof of the Theorem. $\square$

We now draw some more detailed conclusions concerning the subgroups $G \subseteq \mathrm{Co}_0$ using Theorem 3.1. This depends on Curtis's classification of $\mathcal{S}$-lattices [18]. See also [29] for a related discussion.

Theorem 3.2. *There are exactly six conjugacy classes of subgroups $G \subseteq \mathrm{Co}_0$ such that $\operatorname{rk} \Lambda^G \geq 4$, G is the full (pointwise) stabilizer of Λ^G in Co_0, and G fixes* no *coordinate frame. The group G and Λ^G are described as follows.*

(i) *Λ^G is an $\mathcal{S}$-lattice of rank 6 and $G \cong 3^{1+4}.2$,*
(ii) *$\operatorname{rk} \Lambda^G = 5$ and $G \cong 3^{1+4}.2.2$,*
(iii) *Λ^G is an $\mathcal{S}$-lattice of rank 4 and $G \cong 3^4.A_6$,*
(iv) *Λ^G is an $\mathcal{S}$-lattice of rank 4 and $G \cong 5^{1+2}.4$,*
(v) *$\operatorname{rk} \Lambda^G = 4$ and $G \cong 3^{1+4}.2.2^2$,*
(vi) *$\operatorname{rk} \Lambda^G = 4$ and $G \cong 3^{1+4}.2.2$.*

Proof. Set $L := \Lambda^G$. By Theorem 3.1, there is a G-invariant $\mathcal{S}$-lattice S with $L \subseteq S$. Let N be the (pointwise) stabilizer of S. Because S is G-invariant then G normalizes N, so since N fixes L pointwise then $N \trianglelefteq G$. Set $\widetilde{G} := G/N \subseteq \mathrm{Aut}_{\mathrm{Co}_0}(S)$.

The possibilities for S are as follows ([18], [15]):

rk S	N	$\mathrm{Aut}_{\mathrm{Co}_0}(S)$
4	$3^4.A_6$	$2 \times (S_3 \times S_3).2$
4	$5^{1+2}.4$	$2 \times S_5$
6	$3^{1+4}.2$	$2 \times U_4(2).2$

If S/L is *finite* then $L = S$ because L is primitive, so we have $G = N$ and cases (i), (iii) or (iv) apply. Thus from now on we will assume that S/L is *not* finite. In particular we have $\operatorname{rk} S \geq 5$, whence S is the $\mathcal{S}$-lattice of rank 6 as we see from the table, moreover $\operatorname{rk} L = 4$ or 5. We also have $3^{1+4} \cong O^2(N) \trianglelefteq G$ and $|\widetilde{G}| > 1$.

From the table, the group of isometries of S induced within Co_0 is the group $\{\pm 1\} \times W(E_6)$ (which is actually the *full group of isometries*). Indeed, a generator of the direct factor ± 1 acts on S as -1, and $W(E_6) \cong U_4(2).2$ is the Weyl group of type E_6. The lattice $\frac{1}{\sqrt{3}}S$ is isometric to the weight lattice of type E_6, the roots corresponding to short vectors of norm 6 in S.

If rk $L = 5$ then $\widetilde{G}$ fixes a hyperplane pointwise, so that $\tilde{G} \cong \mathbf{Z}_2$ is generated by a reflection in a hyperplane of S orthogonal to a norm 6 vector, and any two such hyperplanes are conjugate in the Weyl group. This is case (ii).

The case rk $L = 4$ requires more care. If L is an $\mathcal{S}$-lattice then from the table, it must be that L is of the first kind, i.e., with stabilizer $3^4.A_6$. Although there is a containment of $\mathcal{S}$-lattices of this kind ([18] or Tables 12 and 13 below), in our set-up we have $3^{1+4} \trianglelefteq G$, whereas $3^4.A_6$ has no such normal subgroup. Thus L is *not* an $\mathcal{S}$-lattice, and in the proof of Theorem 3.1 we showed that in this situation we can find a nonzero short vector $v \in L^{\perp}$. Then in the orthogonal $GF(2)$-space $\bar{\Lambda} := \Lambda/2\Lambda$, v maps onto a nonzero element $\bar{v} \in rad(\overline{L})$, so that $\overline{L}$ is a 4-dimensional *degenerate* subspace of the nondegenerate 6-dimensional orthogonal space $\overline{S}$. ($\overline{L}$, $\overline{S}$ are the images of L, S respectively in $\overline{\Lambda}$.) The pointwise stabilizer of such a degenerate subspace in the full isometry group $O_6^-(2)$ of $\overline{S}$ is a 2-group, and as a result it follows that $\widetilde{G}$ is also a 2-group. Since no element of order 4 in $\mathrm{Aut}_{\mathrm{Co}_0}(S)$ fixes a rank 4 sublattice pointwise, then $\widetilde{G} \cong \mathbf{Z}_2^k$ for some $k \geq 1$.

Suppose that $\widetilde{G} \not\subseteq W(E_6)$. There is a unique conjugacy class of involutions t (the product of the -1-involution and an involution in $U_4(2)$ of type $2A$) in $\{\pm 1\} \times W(E_6) \setminus W(E_6)$ fixing a sublattice in S of rank ≥ 4 ([15]) and the rank is exactly 4. Moreover $2A$ is a central involution in $U_4(2).2$ and its centralizer (mod $\langle 2A \rangle$) acts faithfully on its -1-eigenspace, which corresponds to the fixed-space of $-2A$. It follows that $\tilde{G}$ has order 2 in this case, which is case (vi) of the Theorem.

The remaining possibility is $\widetilde{G} \subseteq W(E_6)$. Involutions in the Weyl group fixing a sublattice of S of rank ≥ 4 pointwise are those of type $2B$ and $2C$ ([15]), the fixed-point ranks being 4 and 5 respectively. Moreover, the product of a pair of distinct commuting Weyl reflections (type $2C$) is of type $2B$. It follows that L is the sublattice of S fixed pointwise by an involution of type $2B$ (so that all such sublattices are conjugate in $\mathrm{Aut}_{\mathrm{Co}_0}(S)$), or equivalently by a pair of commuting Weyl reflections (so that $|\widetilde{G}| \geq 4$). Moreover, any subgroup of $\mathrm{Aut}_{\mathrm{Co}_0}(S)$ strictly containing $\widetilde{G}$ has fixed sublattice of rank *no greater* than 3, whence $\widetilde{G} \cong \mathbf{Z}_2^2$. This is case (v).

From what we have proved so far, it follows that the pointwise stabilizers of rank 4 sublattices of S as in cases (v) and (vi) are *not* conjugate in the (setwise) stabilizer of S. On the other hand, if they are conjugate by an element $g \in \mathrm{Co}_0$, then g must normalize their common normal subgroup $3^{1+4} = O^2(N)$. Since the normalizer of this group *is* the setwise stabilizer of S, then g must belong to this group, a contradiction. It follows that the groups corresponding to cases (v) and (vi), and then even to all six cases, are *not* conjugate in Co_0, and the proof of the theorem is complete. □

4. Geometric conditions

In this section, we use geometric arguments to obtain strong restrictions on the group-theoretic properties of finite groups of symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$.

We determine which conjugacy classes of Co_0 can arise as symplectic automorphisms. We refer to these conjugacy classes, and the elements in them, as the *admissible conjugacy classes* and *admissible elements* respectively. The remaining conjugacy classes and elements are called *inadmissible.* The main result (Theorem 4.9) asserts that there are just 15 admissible conjugacy classes. We also determine the structure of the fixed-point set of the admissible elements and the action on the normal bundle. Finally, we show that a 2-group of symplectic automorphisms has order at most 2^7.

As explained in Section 2, a finite group G of symplectic automorphisms of a hyperkähler manifold of type $K3^{[2]}$ can be identified with a subgroup of Co_0. After making this identification, the primitive embedding of $L_G(-1)$ into the Leech lattice Λ is such that $\mathrm{rk}\,(\Lambda^G) \geq 4$. Obviously then, we have $\mathrm{rk}\,(\Lambda^g) \geq 4$ for every $g \in G$.

We start with Table 1, which lists the 42 conjugacy classes $[g]$ of Co_0 that satisfy the condition $\mathrm{rk}\,(\Lambda^g) \geq 4$, together with some supplementary data. It transpires that such a $[g]$ is uniquely specified by the triple (order of g, Trace (g), Trace (g^2)), and this is the entry in the first column of the table. In what follows, we often identify a conjugacy class using this triple. The second column is the *Frame shape*[1] of g, the third column gives $\mathrm{rk}\,(\Lambda^g)$, the fourth column the *torsion-invariants* of $A_{\Lambda^g} = (\Lambda^g)^*/\Lambda^g$, and the fifth column ('powers') the nontrivial prime powers of g. Column six records whether g belongs (up to conjugacy) to the monomial subgroup $2^{12}{:}M_{24} \subseteq \mathrm{Co}_0$ ($*$ indicates that it *does*). Finally, in the seventh column ('excluded') the symbol 4.x refers to the Lemma or Theorem 4.x below by which the inadmissible elements are excluded.

Columns 1, 2, 3, 5 and 6 in Table 1 can be read-off from the Atlas [15]. The structure of the fixed-point lattice has been investigated in [43, 44, 45, 30]. The table was confirmed using Magma [46] together with a realization of Co_0 as a matrix group.

Since the triple (order of g, Trace(g), Trace(g^2)) uniquely determines the Co_0 conjugacy class, the Co_0 conjugacy class $[g]$ associated to a finite symplectic automorphism g is uniquely determined.

The following observation is clear:

Remark 4.1. If a conjugacy class $[g]$ is inadmissible, then so is $[h]$ whenever g is a power of h.

The lattice-theoretic set-up leads to a condition on the discriminant group.

Lemma 4.2. *Let $A_{\Lambda^g} = (\Lambda^g)^*/\Lambda^g$ be the discriminant group of Λ^g with quadratic form $q_{\Lambda^g} : A_{\Lambda^g} \to \mathbf{Q}/2\mathbf{Z}$. Suppose that $\mathrm{rk}\,(A_{\Lambda^g}) = \mathrm{rk}\,(\Lambda^g)$. Then the following hold:*

(a) *The discriminant group of L^g has index 2 in A_{Λ^g}.*

(b) *q_{Λ^g} has one of the values $\frac{1}{2}$, $\frac{3}{2}$ in its image.*

We defer the proof until Section 7.

Lemma 4.3. *Conjugacy classes of type $(2,-8,24)$, $(3,0,0)$, $(4,0,0)$, $(4,8,-8)$, $(6,-1,-3)$, $(6,0,0)$, $(6,-4,6)$, $(6,-2,6)$, $(6,4,6)$, $(8,0,8)$, $(8,0,0)$, $(10,-2,4)$, $(10,2,4)$ and $(12,2,-2)$ are inadmissible.*

[1] g has *Frame shape* $1^{m_1}2^{m_2}\dots$ if its characteristic polynomial (considered as a linear transformation of $\Lambda \otimes \mathbf{R}$) is $(t-1)^{m_1}(t^2-1)^{m_2}\dots$. Each $m_i \in \mathbb{Z}$.

Table 1: Conjugacy classes of Co_0 with at least four-dimensional fixed-point lattice.

class $[g]$	Frame shape	rk Λ^g	A_{Λ^g}	powers	$2^{12}{:}M_{24}$	excluded
$(1,24,24)$	1^{24}	24	1		$*$	–
$(2,8,24)$	$1^8 2^8$	16	2^8		$*$	–
$(2,0,24)$	2^{12}	12	2^{12}		$*$	4.11
$(2,-8,24)$	$2^{16}/1^8$	8	2^8		$*$	4.3, 4.11
$(3,6,6)$	$1^6 3^6$	12	3^6		$*$	–
$(3,0,0)$	3^8	8	3^8		$*$	4.3
$(3,-3,-3)$	$3^9/1^3$	6	3^5		No	–
$(4,8,-8)$	$1^8 4^8/2^8$	8	2^8	$(2,-8,24)$	$*$	4.1, 4.3
$(4,4,8)$	$1^4 2^2 4^4$	10	$2^2 4^4$	$(2,8,24)$	$*$	–
$(4,0,8)$	$2^4 4^4$	8	$2^4 4^4$	$(2,8,24)$	$*$	4.13
$(4,0,-8)$	$4^8/2^4$	4	$2^2 4^2$	$(2,-8,24)$	$*$	4.1
$(4,0,0)$	4^6	6	4^6	$(2,0,24)$	$*$	4.1, 4.3
$(4,-4,8)$	$2^6 4^4/1^4$	6	$2^2 4^4$	$(2,8,24)$	$*$	4.13
$(5,4,4)$	$1^4 5^4$	8	5^4		$*$	–
$(5,-1,-1)$	$5^5/1$	4	5^3		No	4.16
$(6,5,-3)$	$1^5.3.6^4/2^4$	6	3^5	$(2,8,24)$, $(3,-3,-3)$	No	–
$(6,4,6)$	$1^4.2.6^5/3^4$	6	$2^5 6^1$	$(2,-8,24)$, $(3,6,6)$	$*$	4.1, 4.3
$(6,2,6)$	$1^2 2^2 3^2 6^2$	8	6^4	$(2,8,24)$, $(3,6,6)$	$*$	–
$(6,0,6)$	$2^3 6^3$	6	$2^3 6^3$	$(2,0,24)$, $(3,6,6)$	$*$	4.1
$(6,0,0)$	6^4	4	6^4	$(2,0,24)$, $(3,0,0)$	$*$	4.1, 4.3
$(6,-2,6)$	$2^4 6^4/1^2 3^2$	4	$2^2 6^2$	$(2,-8,24)$, $(3,6,6)$	$*$	4.1, 4.3
$(6,-1,-3)$	$3^3 6^3/1.2$	4	$3^2 6^2$	$(2,8,24)$, $(3,-3,-3)$	No	4.3
$(6,-4,6)$	$2^5 3^4.6/1^4$	6	$2^1 6^5$	$(2,8,24)$, $(3,6,6)$	$*$	4.3
$(7,3,3)$	$1^3 7^3$	6	7^3		$*$	–
$(8,4,0)$	$1^4 8^4/2^2 4^2$	4	$2^2 4^2$	$(4,0,-8)$	$*$	4.1
$(8,0,8)$	$2^4 8^4/4^4$	4	4^4	$(4,8,-8)$	$*$	4.1, 4.3
$(8,2,4)$	$1^2.2.4.8^2$	6	$2^1 4^1 8^2$	$(4,4,8)$	$*$	–
$(8,0,0)$	$4^2 8^2$	4	$4^2 8^2$	$(4,0,8)$	$*$	4.1, 4.3
$(8,-2,4)$	$2^3.4.8^2/1^2$	4	$2^1 4^1 8^2$	$(4,4,8)$	$*$	4.14
$(9,3,3)$	$1^3 9^3/3^2$	4	$3^2 9^1$	$(3,-3,-3)$	No	–
$(10,3,-1)$	$1^3.5.10^2/2^2$	4	5^3	$(2,8,24)$, $(5,-1,-1)$	No	4.1
$(10,2,4)$	$1^2.2.10^3/5^2$	4	$2^3 10^1$	$(2,-8,24)$, $(5,4,4)$,	$*$	4.1, 4.3
$(10,0,4)$	$2^2 10^2$	4	$2^2 10^2$	$(2,0,24)$, $(5,4,4)$	$*$	4.1
$(10,-2,4)$	$2^3 5^2 10/1^2$	4	$2^1 10^3$	$(2,8,24)$, $(5,4,4)$	$*$	4.3
$(11,2,2)$	$1^2 11^2$	4	11^2		$*$	–
$(12,2,2)$	$1^2.4.6^2 12/3^2$	4	$2^2 4^1 12^1$	$(4,-4,8)$, $(6,2,6)$	$*$	4.1
$(12,2,-2)$	$1^2 3^2 4^2 12^2/2^2 6^2$	4	$2^2 6^2$	$(4,8,-8)$, $(6,-2,6)$	$*$	4.1, 4.3
$(12,1,5)$	$1.2^2.3.12^2/4^2$	4	$3^1 6^2$	$(4,4,8)$, $(6,5,-3)$	No	–
$(12,0,2)$	$2.4.6.12$	4	$2^2 12^2$	$(4,0,8)$, $(6,2,6)$	$*$	4.1
$(12,-2,2)$	$2^2 3^2.4.12/1^2$	4	$2^1 6^1 12^2$	$(4,4,8)$, $(6,2,6)$	$*$	4.17
$(14,1,3)$	$1.2.7.14$	4	14^2	$(2,8,24)$, $(7,3,3)$	$*$	–
$(15,1,1)$	$1.3.5.15$	4	15^2	$(3,6,6)$, $(5,4,4)$	$*$	–

Proof. Inspection of the structure of A_{Λ^g} in Table 1, together with Lemma 4.2 (a) eliminates types $(3,0,0)$, $(4,0,0)$, $(6,-1,-3)$, $(6,0,0)$, $(8,0,8)$ and $(8,0,0)$. The additional types $(2,-8,24)$, $(4,8,-8)$, $(6,-4,6)$, $(6,-2,6)$, $(6,4,6)$, $(10,-2,4)$, $(10,2,4)$, and $(12,2,-2)$ are excluded by Lemma 4.2 (b) since a computer calculation shows that q_{Λ^g} has neither the value $\frac{1}{2}$ nor $\frac{3}{2}$ in its image.

Suppose that g is a finite symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$. We say that g is of *$K3$-type* if there is a symplectic automorphism h of a $K3$ surface such that g is conjugate in Co_0 to the element defined by h. We also say that the elements and conjugacy classes in Co_0 corresponding to g are themselves of $K3$-type. For symplectic automorphisms of K3 surfaces we have $\mathrm{rk}\,(\Lambda^g) \geq 5$ [56], while the analog of part a) Lemma 4.2 is $\mathrm{rk}\,(A_{\Lambda^g}) \leq \mathrm{rk}\,(\Lambda^g) - 2$. Then examination of Table 1 establishes the next Remark:

Remark 4.4. There are 8 conjugacy classes in Co_0 of $K3$-type, namely $(1,24,24)$, $(2,8,24)$, $(3,6,6)$, $(4,4,8)$, $(5,4,4)$, $(6,2,6)$, $(7,3,3)$ and $(8,2,4)$.

Nikulin [57] first proved that the order of a (finite order) symplectic automorphism of $K3$ is at most 8. See also [48], [56].

To exclude further cases beyond Lemma 4.3, we apply the equivariant Atiyah-Singer theorem to the Hirzebruch χ_y-genus of the hyperkähler manifold X of type $K3^{[2]}$. Let $g \in \mathrm{Aut}(X)$ have finite order n and let

$$\chi_y(g;X) := \sum_{p,\,q=0}^{4} (-1)^q \,\mathrm{Tr}(g|H^{p,q}(X))\, y^p$$

be the equivariant χ_y-genus.

Lemma 4.5. *Let* $t = \mathrm{Tr}(g|H^{1,1}(X))$ *and* $s = \mathrm{Tr}(g^2|H^{1,1}(X))$*. Then*

$$\chi_y(g;X) = 3 - 2t\,y + \frac{6+t^2+s}{2}\,y^2 - 2t\,y^3 + 3\,y^4.$$

Proof. Inspection of the Hodge diamond of X shows that the only nontrivial contributions one has to know are those coming from $H^{1,1}(X)$ and $H^{2,2}(X)$. The remainder then follow from the symmetries of the Hodge diamond, which holds equivariantly. But $H^{2,2}(X) \cong \mathbf{C} \oplus S^2H^{1,1}(X)$. Together with the formula for the character of a symmetric square, this gives the result. For further details, see Camere [11]. □

We note that $\mathrm{Tr}(g|\Lambda) = \mathrm{Tr}(g|H^{1,1}(X)) + 3$. Moreover from Table 1, we see that t and s are rational integers.

There is a basic result regarding the structure of the fixed-point set X^g.

Theorem 4.6 (cf. Camere [11], Proposition 3)**.** *The fixed-point set of a finite symplectic automorphism of a compact hyperkähler manifold is the disjoint union of finitely many components, which are themselves hyperkähler manifolds. The centralizer of such an automorphism acts by symplectic automorphisms on the fixed-point set.* □

Since the only connected 4-dimensional hyperkähler manifolds are K3-surfaces and complex 2-tori, it follows that the fixed-point set of a non-trivial finite symplectic automorphism on an 8-dimensional hyperkähler manifold consists of isolated points, complex 2-tori and K3 surfaces.

The occurrence of 2-tori can sometimes be excluded by the following geometric result of Mongardi.

Theorem 4.7 (Mongardi [50], Proposition 5.1.4)**.** *Let g be a symplectic automorphism of finite order of a hyperkähler manifold X of type $K3^{[2]}$, and suppose that X^g contains a torus. Then* $\mathrm{rk}\,(L^g) \leq 6$.

Corollary 4.8. *Suppose that g lies in one of the Conway classes* $(2,8,24)$, $(2,0,24)$, $(3,6,6)$, $(3,0,0)$, $(4,8,-8)$, $(4,4,8)$, $(4,0,8)$, $(5,4,4)$ *or* $(6,2,6)$. *Then the components of X^g are isolated fixed-points or $K3$ surfaces.*

Proof. For all of these choices of g one has $\mathrm{rk}\,L^g = \mathrm{rk}\,\Lambda^g - 1 \geq 7$ (cf. column 3 of Table 1) and the Theorem applies. □

To compute $\chi_y(g;X)$ using the equivariant Atiyah-Singer index theorem, we have to know the possible eigenvalues for the action of g on the normal bundle in X of a component F of X^g. Let $\zeta = e^{2\pi i/n}$ (where g has order n).

Since the structure group of the tangent bundle of X can be reduced to $\mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{U}(4)$ there are the following possibilities:

F is an isolated fixed-point p. The possible eigenvalues for g acting on T_pX are $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$, $0 < i \leq j \leq n/2$.

F is a K3 surface or 2-torus. The possible eigenvalues for g acting on the normal bundle N of F in X are (ζ^i, ζ^{-i}), $0 < i \leq n/2$.

By the equivariant fixed-point theorem (cf. [36]) one has the following formula:

$$\chi_y(g;X) = \sum_{F\subset X^g} \prod_{k=1}^{\dim_{\mathbf{C}} F} \frac{x_k(1+ye^{-x_k})}{1-e^{-x_k}} \prod_{k=1}^{4-\dim_{\mathbf{C}} F} \frac{1+y\lambda_k e^{-x'_k}}{1-\lambda_k e^{-x'_k}}[F], \quad (1)$$

the sum running over the components F of X^g. The x_k and x'_k are the formal roots of the total Chern classes of the tangent and normal bundle of F respectively, and the λ_k are the eigenvalues of g acting on the normal bundle. We will evaluate the right-hand-side for each type of fixed-point component and g-action on the normal bundle.

For an isolated fixed-point p of type $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$, the contribution is

$$f_{i,j} := \frac{1+y\zeta^i}{1-\zeta^i}\cdot\frac{1+y\zeta^{-i}}{1-\zeta^{-i}}\cdot\frac{1+y\zeta^j}{1-\zeta^j}\cdot\frac{1+y\zeta^{-j}}{1-\zeta^{-j}}.$$

For a complex 2-dimensional surface F, the total Chern class of F is $c(TF) = (1+x_1)(1+x_2) = 1+c_2(TF)$ since $c_1(TF) = 0$. A short calculation expanding e^{-x_k} up to order 2 shows that the contribution from the tangent bundle in (1) is the factor

$$h_0 := \frac{x_1(1+ye^{-x_1})}{1-e^{-x_1}}\cdot\frac{x_2(1+ye^{-x_2})}{1-e^{-x_2}} = (1+y)^2 + (2-20y+2y^2)\cdot\frac{c_2(TF)}{12}.$$

For the normal bundle N with g acting with eigenvalues (ζ^i, ζ^{-i}), we have to express

$$h_i := \frac{1+y\zeta^i e^{-x'_1}}{1-\zeta^i e^{-x'_1}}\cdot\frac{1+y\zeta^{-i}e^{-x'_2}}{1-\zeta^{-i}e^{-x'_2}}$$

in the total Chern class $c(N) = (1+x'_1)(1+x'_2)$ of the normal bundle. One obtains

$$\begin{aligned} h_i \;=\; & -\frac{\zeta^i + y + \zeta^{2i}y + \zeta^i y^2}{(\zeta^i-1)^2} - \frac{\zeta^i(\zeta^i+1)(y+1)^2}{(\zeta^i-1)^3}\cdot x'_1 \\ & -\frac{\zeta^i(\zeta^{2i}+4\zeta^i+1)(y+1)^2}{2\,(\zeta^i-1)^4}\cdot (x'_1)^2, \end{aligned}$$

where we have also used that $x_1' + x_2' = c_1(N) = c_1(TX)|_F - c_1(TF) = 0$. Note that if g is an involution, the linear term for x_1' vanishes. Otherwise, N splits canonically into two eigenspace bundles and x_1' is well-defined in this case.

Assume that there are $a_{i,j}$ isolated fixed-points of type $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$ and b_i fixed-point components which are K3 surfaces of type (ζ^i, ζ^{-i}). The right hand side of (1) equals

$$\sum_{i,j} a_{i,j} \cdot f_{i,j} \; + \sum_i \sum_{F \subset \Phi_i} h_0 h_i[F] \; + \sum_i \sum_{F \subset \Psi_i} h_0 h_i[F],$$

where Φ_i and Ψ_i denote the union of fixed-point components $F \subset X^g$ which are K3-surfaces resp. 2-tori of type (ζ^i, ζ^{-i}). Using $-(x_1')^2 = c_2(N) = c_2(TX)|_F - c_2(TF)$ and $c_2(TF)[F] = 24$ for F a K3 surface resp. $c_2(TF)[F] = 0$ for F a 2-tori, we see that for fixed i, the sum $\sum_{F \subset \Phi_i \cup \Psi_i} h_0 h_i[F]$ depends only on b_i and the sum $C_i := \sum_{F \subset \Phi_i \cup \Psi_i} c_2(TX|_F)[F]$.

Thus (1) becomes

$$\chi_y(g; X) = \sum_{0 < i \leq j \leq n/2} a_{i,j} \cdot f_{i,j} + \sum_{0 < i \leq n/2} b_i \cdot \beta_i + \sum_{0 < i \leq n/2} C_i \cdot \gamma_i, \qquad (2)$$

with explicit polynomials β_i and γ_i in y and ζ. This gives $3\,\varphi(n)$ linear equations (possibly trivial and linearly dependent) for the integers $a_{i,j}$, b_i and C_i since there are 3 independent rational coefficients in the palindromic polynomial $\chi_y(g; X)$, and the right-hand-side is a polynomial in y with coefficients in the cyclotomic field $\mathbf{Q}(\zeta)$ of degree $\varphi(n)$ over $\mathbf{Q}$. In addition, the $a_{i,j}$ and b_i are non-negative.

If h is a power of g then the fixed-point configurations of g and h and the actions on the normal bundles are related. As before, let n be the order of g and let $h = g^k$ for $k|n$, $k < n$. Consider an isolated fixed-point p for which g acts with eigenvalues $(\zeta^i, \zeta^{-i}, \zeta^j, \zeta^{-j})$ in the tangent space. Then p is also a fixed-point of h and h acts with eigenvalues $(\zeta^{ik}, \zeta^{-ik}, \zeta^{jk}, \zeta^{-jk})$ in the tangent space. If both, ζ^{ik} and ζ^{jk}, are different from 1 then p is also an isolated fixed-point of h. If one of them is equal to 1, then p belongs to a 4-dimensional fixed-point set, i.e. a K3 surface or a 2-torus. The case that $\zeta^{ik} = \zeta^{jk} = 1$ is impossible, since otherwise $h = 1$. If p is a fixed-point of g belonging to a higher-dimensional fixed-point component F of g, then h

acts with the $(\zeta^{ik}, \zeta^{-ik})$ in the normal bundle and ζ^{ik} must be different from 1, i.e., ζ^i is necessarily a primitive n-th root of unity. Note that h can have additional fixed-point components besides the one described above.

It will turn out that for all classes of Table 1, there is at most one possible fixed-point configuration. Thus by analyzing the fixed-point structure and the action of g on the normal bundle, we can apply the information previously obtained to all non-trivial powers of g. This gives several restrictions on the possible fixed-point components and the eigenvalues.

For a given conjugacy class, our approach now is to solve the resulting system of linear equations and inequalities. This can be done in a straightforward way with the help of a computer, although the number of equations and variables will become quite large. From these calculations, together with some additional geometric results, we obtain the following main theorem:

Theorem 4.9. *A symplectic automorphism g of finite order of a hyperkähler manifold of type $K3^{[2]}$ belongs to one of the* 15 Co_0 *conjugacy classes* $(1,24,24)$, $(2,8,24)$, $(3,6,6)$, $(3,-3,-3)$, $(4,4,8)$, $(5,4,4)$, $(6,2,6)$, $(6,5,-3)$, $(7,3,3)$, $(8,2,4)$, $(9,3,3)$, $(11,2,2)$, $(12,1,5)$, $(14,1,3)$, $(15,1,1)$. *If g is of type* $(2,8,24)$, *the fixed-point set contains a unique $K3$-surface; if g is of type* $(3,-3,-3)$, *the fixed-point set contains a unique* 2*-torus; for all other $g \neq 1$, the fixed-point set consists of isolated fixed-points. The complete description of the fixed-point sets and the action on the normal bundles is given in Table* 2.

In the following, we discuss the proof in more detail.

For involutions, the fixed-point formula was first used by Camere. She obtained the following result, which we verified with our program.

Theorem 4.10 (Camere [11])**.** *Let g be a symplectic involution of a hyperkähler manifold X of type $K3^{[2]}$. Then g is of type* $(2,0,24)$, $(2,6,24)$ *or* $(2,8,24)$ *and the corresponding fixed-point sets are as follows:*

$(2,0,24)$: 12 *isolated fixed-points and at least one complex torus,*

$(2,6,24)$: 36 *isolated fixed-points and at least one complex torus,*

$(2,8,24)$: 28 *isolated fixed-points, one K3 surface and an undetermined number of complex tori.*

Combining this with the other information, we obtain

Table 2: Admissible classes and corresponding fixed point configurations.

class of g	# of components of a certain type	prime powers
(1,24,24)*	X	
(2,8,24)*	28×(-1,-1), $K3$	
(3,6,6)*	$27\times(\zeta_3,\zeta_3)$	
(3,-3,-3)†	T^2	
(4,4,8)*	$8\times[(i,i),\ (i,-1)]$	(2,16,8)*
(5,4,4)*	$(\zeta_5,\zeta_5),\ (\zeta_5^2,\zeta_5^2),\ 12\times(\zeta_5,\zeta_5^2)$	
(6,2,6)*	$(\zeta_6,\zeta_6),\ 6\times(\zeta_6,\zeta_6^2)$	(2,8,24)*, (3,6,6)*
(6,5,-3)†	$10\times(\zeta_6,\zeta_6^3),\ 6\times(\zeta_6^2,\zeta_6^3)$	(2,8,24)*, (3,-3,-3)†
(7,3,3)*	$3\times[(\zeta_7,\zeta_7^2)\,(\zeta_7,\zeta_7^3)\,(\zeta_7^2,\zeta_7^3)]$	
(8,2,4)*	$2\times[(i,\zeta_8),\ (i,\zeta_8^3),\ (\zeta_8,\zeta_8^3)]$	(4,4,8)*
(9,3,3)†	$3\times[(\zeta_9,\zeta_9^3),\ (\zeta_9^2,\zeta_9^3),\ (\zeta_9^3,\zeta_9^4)]$	(3,-3,-3)†
(11,2,2)	$(\zeta_{11},\zeta_{11}^3),(\zeta_{11},\zeta_{11}^4),(\zeta_{11}^2,\zeta_{11}^3),(\zeta_{11}^2,\zeta_{11}^5),(\zeta_{11}^4,\zeta_{11}^5)$	
(12,1,5)†	$(\zeta_{12},\zeta_{12}^3),(\zeta_{12}^3,\zeta_{12}^5),2\times(\zeta_{12}^2,\zeta_{12}^3)$	(6,5,-3)†,(4,4,8)*
(14,1,3)	$(\zeta_{14},\zeta_{14}^4),\ (\zeta_{14}^2,\zeta_{14}^3),\ (\zeta_{14}^5,\zeta_{14}^6)$	(2,8,24)*,(7,3,3)*
(15,1,1)	$(\zeta_{15},\zeta_{15}^4),(\zeta_{15}^2,\zeta_{15}^7)$	(3,6,6)*,(5,4,4)*

Notation: * means element is of K3-type and contained in M_{24}; † means element is not in $2^{12}{:}M_{24}$.

Lemma 4.11 (Mongardi [50], Theorem 6.2.3)**.** *The symplectic automorphisms of order* 2 *have type* $(2,8,24)$*.*

Proof. The case $(2,-8,24)$ of Table 1 cannot occur by Camere's theorem, $(2,6,24)$ cannot occur since it is absent from Table 1, and $(2,0,24)$ is excluded by Corollary 4.8. The only remaining possibility from Table 1 is $(2,8,24)$. □

Another application of Corollary 4.8 shows if g is an involution, then the components of X^g are either isolated fixed-points or $K3$ surfaces. Therefore we have:

Lemma 4.12. *If g is a symplectic automorphism of X of even order, then the components of X^g are either isolated fixed-points or $K3$ surfaces.*

Next we consider symplectic automorphisms of order 4.

Proposition 4.13. *Let g be an order 4 symplectic automorphism of a hyperkähler manifold X of type $K3^{[2]}$. Then g is of type $(4,4,8)$, and X^g consists of 16 isolated fixed-points. There are 8 fixed-points with eigenvalues $(i,-i,i,-i)$ and 8 fixed-points with eigenvalues $(i,-i,-1,-1)$.*

Proof. Since g^2 has order 2, we know from Lemma 4.11 that g^2 has type $(2,8,24)$, whence g is of type $(4,4,8)$, $(4,0,8)$ or $(4,-4,8)$ (cf. Table 1). By Lemma 4.12, the only fixed-point components are isolated fixed-points or K3 surfaces. Since g^2 is not the identity we have $a_{2,2}=b_2=0$. Thus we must solve eqn. (2) for the four variables $a_{1,1}$, $a_{1,2}$, b_1 and C_1. For the right-hand-side of (2) we obtain

$$\frac{a_{1,1}}{4}(1+y^2)^2+\frac{a_{1,2}}{8}(-1+y)^2(1+y^2)-b_1(11(1+y^4)+58(y+y^3)+70y^2)+\frac{C_1}{8}(1+y)^4.$$

The left-hand-side is given by Lemma 4.5, where the value for t and s can be read off from the type $(n,t+3,s+3)$ of g.

For g of type $(4,4,8)$ one obtains three linear equations

$$\begin{aligned} \tfrac{1}{4}\,a_{1,1}+\tfrac{1}{8}\,a_{1,2}-11\,b_1+\tfrac{1}{2}\,C_1 &= 3,\\ -\tfrac{1}{4}\,a_{1,2}-58\,b_1+2\,C_1 &= -2,\\ \tfrac{1}{2}\,a_{1,1}+\tfrac{1}{4}\,a_{1,2}-70\,b_1+3\,C_1 &= 6 \end{aligned}$$

with the solutions $a_{1,1}=\frac{2}{3}(C_1+12)$, $a_{1,2}=\frac{1}{3}(24-5C_1)$, and $b_1=\frac{1}{24}C_1$. Using the inequalities $a_{1,1}\geq 0$, $a_{1,2}\geq 0$, $b_1\geq 0$ for integral C_1 shows that $C_1\in\{0,1,2,3,4\}$. Only for $C_1=0$ we obtain integer solutions $a_{1,1}=a_{1,2}=8$ and $b_1=0$. In particular, g must have isolated fixed-points.

The same approach for g of type $(4,0,8)$ or $(4,-4,8)$ gives no solutions. (Moreover, this also holds for the cases of type $(4,8,-8)$, $(4,0,-8)$, and $(4,0,0)$, though they are already excluded.) This completes the proof of the proposition. □

We remark that the 8 fixed-points of an order four element g with eigenvalues $(i,-i,-1,-1)$ necessarily lie on the $K3$ fixed-point component of g^2, and g acts on this K3 surface with 8 isolated fixed-points. This is in agreement with results of Nikulin and Mukai for symplectic automorphisms of K3 surfaces.

A similar approach will handle the elements of order 8:

Proposition 4.14. *Let g be a symplectic automorphism of a hyperkähler manifold X of type $K3^{[2]}$ of order 8. Then g is of K3-type $(8,2,4)$, acting with 6 isolated fixed-points and eigenvalues as in Table 2.*

Proof. By Proposition 4.13, g^2 has type $(4,4,8)$, whence g is of type $(8,2,4)$ or $(8,-2,4)$ by Table 1. Moreover, since g^2 has to act with isolated fixed-points, the same is true for g, and $a_{1,4}=a_{2,4}=a_{3,4}=a_{4,4}=0$. Since $g^4 \neq 1$ then $a_{2,2}=0$. Solving eqn. (2) for the remaining variables $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,3}$, and $a_{3,3}$ gives for g of type $(8,2,4)$ a unique solution $a_{1,1}=0$, $a_{1,2}=2$, $a_{1,3}=2$, $a_{2,3}=2$, $a_{3,3}=0$, and for g of type $(8,2,-4)$ a unique solution $a_{1,1}=1$, $a_{1,2}=2$, $a_{1,3}=-4$, $a_{2,3}=2$, $a_{3,3}=1$. The latter case is impossible since $a_{1,3}$ is negative, and the Proposition is proved. □

Note that for an element g of order 8, g^4 is an involution which has a K3 surface as a fixed-point component on which g acts with 4 fixed-points with eigenvalues $(i,-i)$, as required.

The cases when g has order 3, 5, 7 or 11 are covered by Mongardi [50] (the fixed-point sets are also described) and we merely state the result in these cases. By our method based on equation (2), we can determine the eigenvalues for the action of g on the normal bundle in all cases. Mongardi's result for g of order 3 is as follows:

Proposition 4.15 (Mongardi [50], Theorem 6.2.4, Proposition 6.2.8)**.** *The admissible elements g of order 3 are the K3-type $(3,6,6)$ acting with 27 isolated fixed-points, and type $(3,-3,-3)$ acting with a complex 2-torus as fixed-point set and eigenvalues as in Table 2.*

Note that type $(3,0,0)$ is inadmissible by Lemma 4.3. Mongardi's proof for the type $(3,-3,-3)$ (and also if g has order 7 or 11) uses Theorem 1.2 of [7]. The eigenvalues are all uniquely determined.

At this point, application of Remark 4.1, Lemmas 4.3 and 4.11, and Propositions 4.13 and 4.14 leaves only types $(5,-1,-1)$, $(10,3,-1)$ and $(12,-2,2)$ from Table 1 to be eliminated.

To deal with the prime 5 we use:

Proposition 4.16 (Mongardi [50], Thm. 6.2.9)**.** *Let g be a symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$ of order 5. Then g is* *of type* $(5,4,4)$ *acting with* 14 *isolated fixed-points and eigenvalues as in Table* 2*.*

The eigenvalues for g have been implicitly determined in the proof of [50], Thm. 6.2.9. We confirmed the calculation with our computer program.

Mongardi's result means that elements of type $(5,-1,-1)$ are inadmissible, and those of type $(10,3,-1)$ are then also inadmissible because they have squares of type $(5,-1,-1)$ (cf. Table 1).

Proposition 4.17. *Let g be a symplectic automorphism of a hyperkähler manifold of type $K3^{[2]}$ of order* 12*. Then g is of type* $(12,1,5)$ *acting with* 4 *isolated fixed-points and eigenvalues as in Table* 2*.*

Proof. The only conjugacy classes for g having g^3 of type $(4,4,8)$ and g^2 of type $(6,2,6)$ or $(6,5,-3)$ are those of type $(12,-2,2)$ and $(12,1,5)$, respectively. Since g^3 acts by isolated fixed-points the same holds for g. Since g^2 acts by isolated fixed-points (see Proposition 4.18 below) we have $a_{i,6}=0$ $(1\leq i\leq 6)$. Since $g^6\neq 1$, $a_{2,2}$, $a_{2,4}$, $a_{4,4}$ also vanish, and since $g^4\neq 1$ then $a_{3,3}=0$. In addition, if g is of type $(12,-2,2)$ then g^4 has only isolated fixed-points and $a_{1,3}=a_{2,3}=a_{3,4}=a_{3,5}=0$.

For g of type $(12,-2,2)$, equation (2) has no solution such that the remaining seven variables are non-negative. For g of type $(12,1,5)$, the only solution such that the remaining eleven variables are non-negative and integral is $a_{1,3}=1$, $a_{3,5}=1$, $a_{2,3}=2$ and 0 for the other eight variables. The proposition follows. □

For an order 12 element g, its power g^6 is an involution with a K3 surface as one of the fixed-point components. On this, g acts with 2 fixed-points, as required. The power g^4 is an order 3 element and has a 2-torus as one fixed-point component on which g acts with 4 fixed-points.

This completes the proof that the admissible classes of symplectic automorphisms are those listed in Theorem 4.9.

Next we discuss the fixed-point configuration for the classes in Table 2 not yet covered.

Proposition 4.18. *The admissible g of order* 6 *are the $K3$-type* $(6,2,6)$ *and type* $(6,5,-3)$*, each acting with isolated fixed-points and eigenvalues as in Table* 2*.*

Proof. We have already shown that the only possibilities are types $(6,2,6)$ and $(6,5,-3)$.

In the first case, g^2 is of type $(3,6,6)$ acting with isolated fixed points. Hence g must act by isolated fixed points, too. In the second case, g^2 is of type $(3,-3,-3)$ and has a 2-torus as fixed-point set. Furthermore, g^3 is of type $(2,8,24)$ and contains only isolated fixed points or a K3 surface in its fixed-point set. Thus g can only have isolated fixed points.

We also see that the numbers $a_{2,2} = a_{3,3} = 0$ for g (otherwise g^2 or g^3 would be the identity).

For g of type $(6,2,6)$, equation (2) has the solution $a_{1,1} = 1 + a_{1,3}/8$, $a_{1,2}=6-9\,a_{1,3}/8$, $a_{2,3}=0$, implying that $a_{1,3}=0$, $a_{1,1}=1$, $a_{1,2}=6$. For type g of type $(6,2,6)$, we get the solution $a_{1,1}=-5/4+a_{1,3}/8$, $a_{1,2}=45/4-9\,a_{1,3}/8$ and $a_{2,3}=6$, which implies $a_{1,3}=10$, $a_{1,1}=0$ and $a_{1,2}=0$. □

In both cases, g^3 has a K3 surface as a fixed-point component on which g acts with 6 isolated fixed-points, as required. For g of type $(6,5,-3)$, g^2 has a 2-torus as one fixed-point component on which g acts with 16 isolated fixed-points.

Proposition 4.19. *An admissible elements of order* 9 *has type* $(9,3,3)$, *acting with* 9 *isolated fixed-points and eigenvalues as in Table* 2.

Proof. From Table 1 there is only one possible type of element g of order 9.

Since g^3 is of type $(3,-3,-3)$ with a 2-torus as fixed-point set, we have to consider the nine variables $a_{1,3}$, $a_{2,3}$, $a_{3,4}$, b_1, b_2, b_4, C_1, C_2 and C_4 which might be nonzero. Equation (2) provides us with the unique solution $a_{1,3} = a_{2,3} = a_{3,4} = 3$, $b_1 = b_2 = b_4 = 0$ and $C_1 = C_2 = C_4 = 0$ for non-negative $a_{i,j}$ and b_i. □

Proposition 4.20. *An admissible element of order* 15 *has type* $(15,1,1)$, *acting with* 2 *isolated fixed-points and eigenvalues as in Table* 2.

Proof. The statement about the type is clear. By considering g^5 of type $(3,6,6)$ and g^3 of type $(5,4,4)$ it is also clear that g acts with isolated fixed-points and $a_{1,3}$, $a_{2,3}$, $a_{3,3}$, $a_{3,4}$, $a_{3,5}$, $a_{3,6}$, $a_{3,7}$, $a_{1,6}$, $a_{2,6}$, $a_{4,6}$, $a_{5,6}$, $a_{6,6}$, $a_{6,7}$ and $a_{1,5}$, $a_{2,5}$, $a_{3,5}$, $a_{4,5}$, $a_{5,5}$, $a_{5,6}$, $a_{5,7}$ all vanish. Equation (2) has the general solution $a_{1,1} = (-a_{4,7} - a_{7,7})/5$, $a_{1,2} = a_{4,7}$, $a_{1,4} = (5 - 4\,a_{4,7} + 26\,a_{7,7})/5$, $a_{1,7} = (a_{4,7} - 24\,a_{7,7})/5$, $a_{2,2} = a_{7,7}$, $a_{2,4} = (a_{4,7} - 24\,a_{7,7})/5$, $a_{2,7} = (5 - 6\,a_{4,7} + 14\,a_{7,7})/5$, $a_{4,4} = (-a_{4,7} - a_{7,7})/5$. The only non-negative integral solution occurs when $a_{4,7} = a_{7,7} = 0$, leading to the eigenvalues as in Table 2. □

The two fixed-points of g are the two distinguished fixed-points of g^3 with eigenvalues $(\zeta_5, \zeta_5^4, \zeta_5, \zeta_5^4)$ and $(\zeta_5^2, \zeta_5^3, \zeta_5^2, \zeta_5^3)$.

Proposition 4.21 (Mongardi [50], Proposition 6.2.15)**.** *An admissible element of order* 7 *has type* $(7,3,3)$, *acting with* 9 *isolated fixed-points with eigenvalues as in Table* 2.

Proposition 4.22. *An admissible element g of order* 14 *has type* $(14,1,3)$, *acting with* 3 *isolated fixed-points and eigenvalues as in Table* 2.

Proof. The statement about the type is again clear. By considering g^7 of type $(2,8,24)$ and g^2 of type $(7,3,3)$ it follows that equation (2) contains the variables $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{1,4}$, $a_{1,5}$, $a_{1,6}$, $a_{2,3}$, $a_{2,5}$, $a_{3,3}$, $a_{3,4}$, $a_{3,5}$, $a_{3,6}$, $a_{4,5}$, $a_{5,5}$, $a_{5,6}$. The solution of the corresponding system of linear equations depends on the variables $a_{1,5}$, $a_{1,6}$, $a_{3,4}$, $a_{3,5}$, $a_{3,6}$, $a_{4,5}$, $a_{5,5}$, $a_{5,6}$. The only non-negative integral solution is obtained for $a_{1,4} = a_{2,3} = a_{5,6} = 1$, with the remaining variables vanishing. This verifies the entry for g in Table 2. □

If g has order 14 then g^7 is an involution with a K3 surface as a fixed-point component on which g acts with 3 fixed-points, as required.

Proposition 4.23 (Mongardi [50], Proposition 6.2.16)**.** *An admissible element of order* 11 *has type* $(11,2,2)$, *acting with* 5 *isolated fixed-points and eigenvalues as in Table* 2.

This completes the proof of Theorem 4.9.

In Section 8, we will show that there are hyperkähler manifolds X of type $K3^{[2]}$ realizing symplectic automorphisms of each of the fifteen admissible types described in Theorem 4.9 and Table 2.

For K3 surfaces, it follows from [56] that a finite 2-group of symplectic automorphisms is isomorphic to a subgroup of the 2-Sylow subgroup of M_{23} and thus has order at most 2^7. A short proof is given in [48]: starting from the fact that a symplectic involution on a $K3$ surface has exactly 8 isolated fixed-points, it is shown that the centralizer of an involution in a group of symplectic automorphisms group must be a subgroup of $\hat{A}_8$, the unique non-split extension of the alternating group A_8 by $\mathbf{Z}_2$. The result then follows courtesy of the fact that the 2-Sylow subgroups of $\hat{A}_8$ and M_{23} are *isomorphic.*

Let G be a 2-group of symplectic automorphisms acting on a hyperkähler manifold X of type $K3^{[2]}$. Let $t \in G$ be an involution, and consider the fixed-point set X^t. By Theorem 4.9 we know that X^t has a unique component that is a $K3$ surface, call it Y. We claim that $C_G(t)/\langle t\rangle$ acts *faithfully* on Y.

By way of contradiction, assume that $C_G(t)/\langle t\rangle$ does *not* act faithfully on Y. Then there is $t \in A \subseteq G$ such that $|A| = 4$ and A acts trivially on Y. The group A cannot be cyclic, because elements of order 4 have discrete fixed-points by Theorem 4.9. Therefore, $A = \langle s, t\rangle \cong \mathbf{Z}_2^2$. Since both s and t leave Y fixed pointwise, they each act on the tangent space at a point $p \in Y$, and this consists of the tangent space of Y at p and the restriction of the normal bundle. Both s and t act trivially on the first piece, and as -1 on the second. Therefore st acts trivially on both, and hence is the identity. As s and t are involutions then $s = t$, which is the required contradiction.

Theorem 4.24. *Let G be a 2-group such that for every involution t, the quotient $C_G(t)/\langle t\rangle$ is isomorphic to a subgroup of M_{23}. Then $|G| \le 2^7$, and G belongs to one of* 70 *possible isomorphism types.*

Proof. We verified this by checking the condition for all 2-groups of order at most 256 [10] using Magma. □

5. Isomorphism Classes of Admissible Subgroups in Co_0

In this section we describe the isomorphism classes of subgroups $G \subseteq \mathrm{Co}_0$ with the following two properties:

1. G consists of admissible elements,
2. $\operatorname{rk}\Lambda^G \ge 4$.

We call these subgroups *admissible.* We note that an admissible element generates an admissible cyclic subgroup.

The admissible subgroups G are of three distinct types according to whether G is a 2-group; all conjugacy classes of G meet the monomial group $2^{12}{:}M_{24}$ and G is *not* a 2-group; or G contains elements in one of the four admissible classes $(3,-3,-3)$, $(6,5,-3)$, $(9,3,3)$ or $(12,1,5)$ which do *not* meet $2^{12}{:}M_{24}$. One of the main results of this section is a sufficient condition for G to be isomorphic to a subgroup of M_{23}: namely, that G is of the second type. If G *is* a 2-group then it lies in $2^{12}{:}M_{24}$ just because the monomial group contains a Sylow 2-subgroup of Co_0.

The maximal subgroups in the first two cases classes will be explicitly described. In the third case, G has to be one of the four groups described in Theorem 3.2 (i), (ii), (iii) or (v).

5.1. Admissible subgroups related to M_{23}

In this subsection we consider subgroups $G \subseteq \mathrm{Co}_0$ with the following properties:

1. G is admissible,
2. G contains *no* elements of type $(3,-3,-3)$, (3)
3. G is *not* a 2-group.

As a main result we have:

Theorem 5.1. *Assume that G satisfies the hypotheses of* (3). *Then G is isomorphic to a subgroup of one of the following* 13 *groups:*

(a) $L_2(11)$,

(b) $\mathbf{Z}_2 \times L_2(7)$,

(c) $\mathbf{Z}_2^3{:}L_2(7)$,

(d) A_7,

(e) $L_3(4)$,

(f) $(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$,

(g) $\mathbf{Z}_2^4{:}A_6$,

(h) $\mathbf{Z}_2^4{:}S_5$,

(i) M_{10},

(j) S_6,

(k) $\mathbf{Z}_3^2{:}QD_{16}$,
(l) $\mathbf{Z}_2^4{:}(S_3 \times S_3)$,
(m) $Q(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$, $|Q| = 2^6$.

Of the 15 admissible classes of elements enumerated in Theorem 4.9 (cf. Table 1), those of types $(3,-3,-3)$, $(6,5,-3)$, $(9,3,3)$ and $(12,1,5)$ all have some power of type $(3,-3,-3)$. Since this latter type is excluded by assumption, all four of these classes are excluded. There remain 11 admissible conjugacy classes, these being precisely the admissible classes that meet the monomial group $2^{12}{:}M_{24}$. Indeed, each of these classes meets M_{23} and the class is characterized by the order n of the elements that it contains ($n = 1$, …, 8, 11, 12, 15).

The upshot is that for $g \in G$, the character $\chi(g)$ of the representation of G on $V := \Lambda \otimes \mathbf{Q}$ is equal to the trace of the corresponding element in M_{23} in the usual permutation representation of degree 24. Explicitly, if $g \in G$ has order n then

$$\chi(g) = \varepsilon(n) := 24\Big(n\prod_{p|n}(1+\frac{1}{p})\Big)^{-1}. \tag{4}$$

Following Mukai [56], we call a 24-dimensional representation ρ of a finite group H a *Mathieu representation* if $\mathrm{Tr}\rho(g) = \varepsilon(n)$ whenever $g \in H$ has order n. Consequently, if G is as in the statement of Theorem 5.1 then V furnishes a Mathieu representation of G over $\mathbf{Q}$ with $\dim V^G \geq 4$.

The following general result was obtained in [56], Thm. (3.22):

Theorem 5.2 (Mukai)**.** *Suppose that H has a Mathieu representation over $\mathbf{Q}$. Then $|H|$ divides $|M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.*

In our case there are no admissible elements of order 23, whence $|G|\,|\,2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. We also recover the upper bound of 2^7 of Theorem 4.24.

The following formula is well-known:

$$\dim V^G = |G|^{-1}\sum_{g\in G}\chi(g). \tag{5}$$

We refer to this result as the *invariant subspace formula* (i.s.f.).

We turn to the proof of Theorem 5.1, which will be divided into several cases. Let G be as in the statement of the Theorem. In particular, since G is admissible then $\dim V^G \geq 4$. We use this inequality frequently in tandem with the i.s.f. to eliminate a number of possible groups.

Case 1. $11||G|$. We will show in this easiest case that

$$G \text{ is isomorphic to a subgroup of } L_2(11).$$

Let $P \subseteq G$ be a Sylow 11-subgroup. Thus $P \cong \mathbf{Z}_{11}$, and since there are no elements of order $11k$ $(k \geq 2)$ then $C_G(P) = P$. The i.s.f. shows that there are no dihedral subgroups of order 22, whence $|N_G(P)| = 11$ or 55.

In the first case, G has a normal 11-complement by Burnside's normal p-complement theorem, call it Q. Thus $G = QP$ with $\gcd(|Q|, 11) = 1$. Since $C_G(P) = P$ then there must be a prime $p = 2$, 3, 5 or 7 and a subgroup $E \subseteq Q$ with $E \cong \mathbf{Z}_p^k$ for some $k \geq 0$ such that P normalizes and acts faithfully on E. Moreover, if $Q \neq 1$ then we can choose $E \neq 1$. Setting $H = EP$, the i.s.f. applied to H shows that the only possibility is $E = 1$, so that $H = 1$ and $G = P$ is a Sylow 11-subgroup of $L_2(11)$.

The second possibility is $|N_G(P)| = 55$. The previous argument shows that P cannot normalize *any* nontrivial subgroups of G of order coprime to 11. So if G is *solvable* then $G = N(P)$ has order 55 and is the normalizer of a Sylow 11-subgroup of $L_2(11)$. Finally, assume that G is *nonsolvable*. A minimal normal subgroup $N \trianglelefteq G$ cannot have order coprime to 11, so it is simple and contains P. By the Frattini argument, $G = NN_G(P)$. If $N_G(P)$ is not contained in N then P is self-normalizing in N, so N cannot be simple by the same Burnside theorem, contradiction. Thus $N_G(P) \subseteq N$ and $G = N$ is simple. Because $|G|$ divides $2^6.3^2.5.7.11$, any one of several classification theorems will tell us that $G \cong L_2(11)$. This completes Case 1.

Case 2. $7||G|$. We will establish

$$G \text{ is isomorphic to a subgroup of } \mathbf{Z}_2 \times L_2(7),\ \mathbf{Z}_2^3{:}L_2(7),\ A_7 \text{ or } L_3(4).$$

Let $P \cong \mathbf{Z}_7$ be a Sylow 7-subgroup.

Lemma 5.3. *G has no subgroup H of any of the following types: dihedral of order* 14*, order* 28*, order* $2^4.7$*.*

Proof. The i.s.f. eliminates the dihedral group. Suppose that $|H| = 28$. Since there are no elements of order 28 and no dihedral group of order 14, then H is either $\mathbf{Z}_2 \times \mathbf{Z}_{14}$ or $\mathbf{Z}_7{:}\mathbf{Z}_4$. The first possibility is eliminated by the i.s.f. The second possibility does not hold either, because calculation (e.g. by Magma) shows that χ *cannot* be the character of a rational representation of such a group (the Schur index is greater than 1). Suppose that $|H| = 2^4.7$. Since there are no dihedral groups of order 14 or abelian groups of order 28 then a 2-Sylow-subgroup $Q \subseteq H$ is normal (Burnside's theorem again), moreover $|C_Q(P)| = 2$. The only possibility is $Q \cong \mathbf{Z}_2^4$, and one verifies by applying the i.s.f. to QP that this is impossible. □

Now suppose that $Q \neq 1$ is a subgroup of order coprime to 7 and normalized by P. Then P leaves invariant a Sylow r-subgroup of Q for each prime divisor r of $|Q|$. Let R be such a Sylow r-subgroup. If $r \geq 3$ then $|Q| \leq r^2$, so P acts trivially on R, thereby producing elements of order $7r$, contradiction. So $Q = R$ is a 2-group. We have $|C_Q(P)| \leq 2$ by Lemma 5.3 (there are no subgroups of order 28). Suppose that Q is extra-special. Then P acts faithfully on Q, and we have $|Q| = 2^{1+6k}$. The (unique) faithful irreducible representation of Q has dimension 2^{3k}, so we must have $k = 1$, and the i.s.f. applied to QP yields a contradiction. Therefore Q is not extra-special. So if $|Q| \geq 4$, there is a noncyclic characteristic elementary abelian 2-group of Q, call it E. Because $|C_E(P)| \leq 2$ then P acts faithfully on E, so $|E| \geq 8$. If P centralizes $Z(Q)$ then $EZ(Q)P$ contains a group of order $2^4.7$, contradiction. So P does not centralize $Z(Q)$, whence we may, and shall, take $E \subseteq Z(Q)$. The same argument then shows that $C_Q(P) = 1$. If $|Q| \geq 16$ we must then have $|Q| = 2^{3k}, k \geq 2$, and the i.s.f. again yields a contradiction. So in fact $Q = E \cong \mathbf{Z}_2^3$. Thus we have so far shown:

> Suppose there is $1 \neq Q \subseteq G$ normalized by P with $\gcd(|Q|, 7) = 1$.
> Then $Q \cong \mathbf{Z}_2$ or $\mathbf{Z}_2^3$, and in the second case $C_Q(P) = 1$.

Now assume that $\mathbf{Z}_2^3 \cong Q \trianglelefteq G$. If G is solvable we easily find (there being no dihedral subgroup of order 14 by Lemma 5.3) that $G \cong \mathbf{Z}_2^3{:}\mathbf{Z}_7$ or $\mathbf{Z}_2^3{:}(\mathbf{Z}_7{:}\mathbf{Z}_3)$. Suppose that G is nonsolvable. Since $C_G(Q)$ has order coprime to 7 we must have $C_G(Q) = Q$. Since $\mathrm{Aut}(Q) \cong L_3(2) \cong L_2(7)$ is a minimal simple group, the only possibility is $G/Q \cong L_2(7)$, so that G occurs in a short exact sequence $1 \to \mathbf{Z}_2^3 \to G \to L_2(7) \to 1$. We assert that this extension splits. Indeed, let $H \subseteq G$ be the subgroup that commutes with a given nonzero element of Q so that $H/Q \cong S_4$. Use the i.s.f. to see that $O_2(H) \backslash Q$ contains involutions, from which it follows that $O_2(H) \cong Q_8 * Q_8$. Then we can check directly that H is in fact a *split extension* of S_4 by Q, and in particular Q splits in a Sylow 2-subgroup of G. Therefore, the extension G of $L_2(7)$ by Q also splits. Note that the possible solvable groups with $\mathbf{Z}_2^3 \trianglelefteq G$ are all contained in this nonsolvable example.

Assume that $|Q| = 2$. If G is solvable, it follows from what has already been established that G is isomorphic to a subgroup of $\mathbf{Z}_2 \times (\mathbf{Z}_7{:}\mathbf{Z}_3)$. Finally, suppose G is nonsolvable. Then G/Q is simple of order divisible by 7, so $G/Q \cong L_2(7), A_7, A_8, L_3(4)$. Since there are no elements of order 10, only the first possibility survives. Therefore, $G \cong \mathbf{Z}_2 \times L_2(7)$ or $SL_2(7)$. In the second case the i.s.f. fails, leaving only the first possibility. Note once again that all solvable possibilities are contained in the nonsolvable case.

The last possibility is that there is *no* nontrivial normal subgroup of G of order coprime to 7. Then a minimal normal subgroup $N \trianglelefteq G$ is simple and contains P. The only possibilities for N (taking into account that $|G|$ divides $2^6.3^2.5.7$ with no elements of order 9) are

$$N \cong L_2(7),\ A_7,\ A_8,\ L_3(4).$$

We claim that A_8 cannot occur. Indeed, $A_8 \subseteq M_{23}$ acts with only 3 orbits on the 24 points. Therefore, applying the i.s.f. to our abstract group $G \cong A_8$ will certainly yield $\dim V^G = 3$, contradiction. We assert that $G = N$ in the other three cases, so assume this is false. If $N \cong L_3(4)$, we have $V = V^G \oplus U$ where U is G-invariant and irreducible of dimension 20. ($L_3(4)$ has no irreducible representation of dimension less than 20 [15].) Since $|G|$ is not divisible by 27, we must have $|G/N| | 4$, and every nontrivial coset of N in G contains an involution t such that $\mathrm{Tr}_U(t) \neq 4$ (loc. cit). Because $8 = \chi(t) = 4 + \mathrm{Tr}_U(t)$, this is a contradiction. On the other hand, if $N \cong L_2(7)$ or A_7 then $G \cong \mathrm{PGL}_2(7)$ or S_7 respectively, and since both of these groups contain a dihedral group of order 14 they cannot occur either. This completes the proof that $G = N \cong L_2(7)$, A_7 or $L_3(4)$. Because $L_2(7) \subseteq A_7$ there is no need to include $L_2(7)$ on the list of possible groups, and the analysis of Case 2 is done.

Before taking up Case 3 we interpolate a Lemma.

Lemma 5.4. *Suppose that* $|G| = 2^k.5$. *Then either* $G \cong \mathbf{Z}_5{:}\mathbf{Z}_4$ *or* $O_2(G) \cong \mathbf{Z}_2^4$.

Proof. Let P be a Sylow 5-subgroup of G. Since G is necessarily solvable, and assuming that the first stated possibility does not hold we have $Q := O_2(G) \neq 1$. Since there are no elements of order 10 then $Z(Q)$ contains a P-invariant subgroup $E \cong \mathbf{Z}_2^4$. We will obtain a contradiction if $Q \neq E$. Indeed, in the contrary case we have $|Q| \geq 2^8$ because $C_Q(P) = 1$, and this is impossible because a Sylow 2-subgroup has order $\leq 2^7$. □

Case 3. $5||G|$ and there exists $\mathbf{Z}_2^4 \cong N \trianglelefteq G$. We will establish

$$G \text{ is isomorphic to a subgroup of } \mathbf{Z}_2^4{:}S_5 \text{ or } \mathbf{Z}_2^4{:}A_6.$$

First, we can check using the i.s.f. that $\dim V^N = 9$ and $\dim V^E = 10$ for every hyperplane $E \subseteq N$. This implies that if $V = W \oplus V^N$ is a G-invariant decomposition, then $\dim W = 15$ and (considered as N-module) W is the sum of the 15 distinct nontrivial irreducible characters of N. Therefore, we can choose a 1-dimensional subspace of V^G, call it V_0 so that $U := W \oplus V_0$ is both G-invariant and a *free* N-module. There is a unique set of 1-dimensional spaces $V_i := \mathbf{C}v_i$ $(0 \leq i \leq 15)$ spanning $W \oplus V_0$ and afford the irreducible characters of N, and G permutes the V_i among themselves. Furthermore, we readily find that $G = N{:}H$ is a *split* extension and that G acts as a transitive permutation group on U with point-stabilizer H. Applying Lemma 5.4 with G replaced by $O_2(G)P$ (P a Sylow 5-subgroup of G), we conclude that $N = O_2(G)$, i.e., $O_2(H) = 1$. Furthermore, the i.s.f. shows that G cannot contain any abelian groups of order 2^5 and rank at least 4. In particular, N is self-centralizing in G whence H is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong L_4(2) \cong A_8$. From the analysis of Case 2 it also follows that $\gcd(|H|, 7) = 1$, and by the i.s.f. there is no subgroup isomorphic to $\mathbf{Z}_2^4{:}\mathbf{Z}_{15}$.

From these reductions, it follows that if G is solvable then G is isomorphic to a subgroup of $\mathbf{Z}_2^4{:}(\mathbf{Z}_5{:}\mathbf{Z}_4)$, which is itself a subgroup of $\mathbf{Z}_2^4{:}S_5$. If G is nonsolvable then H has a normal subgroup $K \cong A_5$ or A_6, and in the latter case we have $H \cong A_6$ or S_6. We can eliminate the latter possibility much as in the $L_3(4)$ case handled earlier. Indeed, if $H \cong S_6$ then V decomposes into a transitive permutation representation of K of dimension 15 (corresponding to its action on the involutions of N), plus a 4-dimensional fixed-point subspace, plus a 5-dimensional irreducible in V^N. Let $t \in H$ be an involution acting on this 5-dimensional space with trace -1 (such a t always exists). t has trace 7 or 3 on the 15-dimensional space, so that $\chi(t) = 10$ or 6, contradiction. Suppose that $K \cong A_5$. Because there are no subgroups of the form $\mathbf{Z}_2^4 : \mathbf{Z}_{15}$ then G is isomorphic to a subgroup $\mathbf{Z}_2^4{:}S_5$. This completes Case 3.

Case 4. $5||G|$, $\gcd(|G|, 77) = 1$. We show in this case that either Case 3 holds, or

$$G \text{ is isomorphic to a subgroup of } (\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2,\ S_6 \text{ or } M_{10}.$$

If $O_2(G) \neq 1$ then by Lemma 5.4 we are in Case 3 and there is nothing to prove. Hence, we may assume that $O_2(G) = 1$. If there is a nontrivial normal subgroup of order prime to 5 it must then be a 3-group, call it $R \trianglelefteq G$. Then $P \cong \mathbf{Z}_5$ acts trivially on R since $|R| \leq 9$. There is no abelian subgroup of order 45 by the i.s.f. so $R = O_3(G) \cong \mathbf{Z}_3$. Let M/R be a minimal normal subgroup of $C_G(R)/R$. If (gcd$(|M|, 5) = 1$ then M/R is a 2-group and it centralizes R and therefore descends to yield a nontrivial normal 2-subgroup of G, contradiction. Thus $P \subseteq M$. If M is solvable then $M = P \times R \cong \mathbf{Z}_{15}$ and $G \subseteq \mathbf{Z}_{15}{:}\mathbf{Z}_4$. If M is nonsolvable then $M/R \cong A_5, M \cong \mathbf{Z}_3 \times A_5$, and G is isomorphic to a subgroup of $(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$. This contains the solvable group described earlier in the paragraph.

Finally, assume that a minimal nontrivial normal subgroup N contains P. If it is equal to P then G is isomorphic to a subgroup of $\mathbf{Z}_5{:}\mathbf{Z}_4$. Otherwise, N is simple, so that $N \cong A_5$ or A_6 and $G = A_5$, S_5 or $A_6 \subseteq G \subseteq \mathrm{Aut}(A_6)$. In the latter case, since there are no elements of order 10 then we cannot have $G = PGL_2(9)$ or $\mathrm{Aut}(A_6)$. Therefore, $G \cong A_6$, M_{10} or S_6. (The latter two groups correspond to the two cosets of $\mathrm{Aut}(A_6)/A_6$ distinct from that corresponding to $PGL_2(9)$). This completes the analysis of Case 4.

We have now completed the proof of Theorem 5.1 in case $|G|$ is divisible by one of 5, 7 or 11.

Case 5. $|G| = 2^f 3^2$. We will show that either G is contained in one of the groups occurring in Cases 1–4, or else it is a subgroup of one of

$$\mathbf{Z}_3^2{:}QD_{16},\ \mathbf{Z}_2^4{:}(S_3 \times S_3) \text{ or } Q{:}(\mathbf{Z}_3^2{:}\mathbf{Z}_2) \text{ with } |Q| = 2^6.$$

First note that G is solvable of 3-length 1. Because there are no elements of order 9 then a Sylow 3-subgroup $R \subseteq G$ is isomorphic to $\mathbf{Z}_3^2$ and $G = RT$ where T is a Sylow 2-subgroup of G. Suppose first that $O_2(G) = 1$. Then T acts faithfully on R and is therefore isomorphic to a subgroup of $\mathrm{Aut}(R) = GL_2(3)$. This latter group has Sylow 2-subgroup QD_{16} (quasidihedral), so G is isomorphic to a subgroup of $\mathbf{Z}_3^2{:}QD_{16}$.

Next assume that $O_2(G) \neq 1$, $S := O_3(G) \neq 1$. Since there is no abelian subgroup of order 18 by the i.s.f. we have $|Q| = 3$. Similarly, because there is no subgroup $\mathbf{Z}_3 \times A$ with $|A| \geq 16$ by the i.s.f., we have $|Q| = 4$. Then G is isomorphic to a subgroup of $(\mathbf{Z}_3 \times A_4){:}\mathbf{Z}_2$, and this group is contained in $(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$.

Now suppose that $O_3(G) = 1$. Because R is self-centralizing and there is no $\mathbf{Z}_3 \times A$, $|A| = 16$ as before, then $Q := O_2(G)$ has order 4^k where $k = 2$ or 3 is the number of subgroups $U \subseteq R$ of order 3 satisfying $C_Q(U) \neq 1$ (in which case $C_Q(U) \cong \mathbf{Z}_2^2$). Since there is no subgroup isomorphic to $\mathbf{Z}_2^5$, it follows easily that either $k = 2$ and $Q \cong \mathbf{Z}_2^4$, or else $k = 3$ and $Z(Q) \cong \mathbf{Z}_2^2$. Moreover, $G = QH$ where $H := N_G(R)$.

Lemma 5.5. *If* $\mathbf{Z}_3 \cong U \subseteq R$ *then either* $C_H(U) = R$ *or* $C_Q(U) = 1$.

Proof. Assume this is false. Set $Q_0 := C_Q(U) \cong \mathbf{Z}_2^2$ and $R = U \times U_0$. Then U_0 acts on Q_0, and since $C_G(R) = R$ then $C_{Q_0}(U_0) = 1$. Now $8 \| C_G(U)|$. Since $C_Q(R) = 1$ we can choose an involution $t \in C_H(U)$. Since $R \trianglelefteq H$ then t normalizes R and we may, and shall, choose t so that it normalizes U_0. Since t commutes with U but not R then t must act as the inverting automorphism of U_0. Now $U_0\langle t\rangle \cong S_3$ acts on Q_0. Since t commutes with Q_0 then so does $U_0 = [U_0, t]$, that is $Q_0 = C_{Q_0}(U_0)$. This contradicts the earlier statement that $C_{Q_0}(U_0) = 1$, and the proof of the Lemma is complete. □

We now consider the two possibilities for the integer k defined, as before, by the equality $|Q| = 4^k$.

Case 5(a). $k = 2$. Here, $Q \cong \mathbf{Z}_2^4$ and H is isomorphic to a subgroup of $\mathbf{Z}_3^2{:}QD_{16}$. However, because $k = 2$ then a Sylow 2-subgroup T_0 of H *cannot* act transitively (by conjugation) on the subgroups of R of order 3. As a consequence, T_0 is neither QD_{16} itself, nor is it $\mathbf{Z}_8$ or Q_8. It is thus a subgroup of D_8. If it is D_8 then we can find a $\mathbf{Z}_3 \cong U \subseteq R$ which satisfies $C_Q(U) \neq 1, C_H(U) \neq R$, against Lemma 5.5. Therefore, $T_0 \cong \mathbf{Z}_2, \mathbf{Z}_4$ or $\mathbf{Z}_2^2$. Furthermore, the conditions of Lemma 5.5 have to be satisfied. If $T_0 \cong \mathbf{Z}_4$ then $G = \mathbf{Z}_2^4{:}(\mathbf{Z}_3^2 : \mathbf{Z}_4)$ is a subgroup of $\mathbf{Z}_2^4{:}A_6$, which occurs in Case 3. If $T_0 \cong \mathbf{Z}_2^2$ then $G = \mathbf{Z}_2^4{:}(S_3 \times S_3)$, while if $T_0 \cong \mathbf{Z}_2$, then G is isomorphic to a subgroup of $G = \mathbf{Z}_2^4{:}(S_3 \times S_3)$. This completes Case 5(a).

Case 5(b). $k = 3$. Here, Q is the product of three subgroups $Q_i := C_Q(U_i) \cong \mathbf{Z}_2^2$ ($i = 0, 1, 2$) for three distinct subgroups $U_i \subseteq R$ or order 3. We may, and shall, take $Q_0 = Z(Q)$. Suppose that QR is a *proper* subgroup of G. Then the 2-Sylow subgroup T_0 of H is nontrivial. T_0 normalizes $U_0 = C_R(Z(Q))$, and no nonidentity element can act trivially by Lemma 5.5. Thus $T_0 = \langle t \rangle$ has order 2. Now t must also normalize the unique subgroup $U_3 \subseteq R$ of order 3 satisfying $C_Q(U_3) = 1$. We claim that t also inverts U_3. Otherwise, t centralizes U_3, so U_3 acts on $C_Q(t)$. Then $C_Q(t)$ has order 16 and contains $Z(Q)$, from which it follows that $C_Q(t)$ is abelian. Then $\langle t \rangle \times C_Q(t)$ is abelian of order 32, and the fixed-point formula shows that this is not possible. This completes the proof that t inverts U_3. (It follows from these facts that the Sylow 2-subgroup $T = Q\langle t \rangle$ of G is isomorphic to a Sylow 2-subgroup of M_{23}.) In any case, $G \cong Q{:}(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$, and Case 5(b) is finished.

This completes the discussion of the case when $3^2 || G|$.

Case 6. $|G| = 2^f.3$. We choose a computational approach for the case when $3||G|$. By Theorem 5.2 we have $|G||384$. We use the library of small groups [6] in Magma to find those G satisfying $3||G||384$ and possessing a 20-dimensional character $\chi - 4 \cdot \mathbf{1}$, $\chi(g)$ being determined by the order of g. With the exception of a single group X_{24} of order 24 (library entry #7), all others are contained in one of the 13 groups of Theorem 5.1. See also Table 3.

X_{24} has eight 1-dimensional and four 2-dimensional irreducible characters. In the representation with character $\chi - 4 \cdot \mathbf{1}$, all irreducible characters appear with multiplicity 1 apart from one 2-dimensional character which has multiplicity 3. However, two of the 2-dimensional irreducible representations

Table 3: Isomorphism classes of $2^f.3$-groups

order	3	6	12	24	48	96	192	384	total
no. of groups	1	2	5	15	52	231	1543	20169	22018
and correct representation	1	2	4	5	8	6	8	4	38
contained in M_{23}	1	2	4	4	8	6	8	4	37
others contained in Co_0	0	0	0	0	0	0	0	0	0

have Schur index 2, implying that the representation affording $\chi-4\cdot\mathbf{1}$ *cannot* be rational. Consequently, G cannot be contained in Co_0.

This completes the proof of the Theorem. □

There is another characterization of the 13 isomorphism types of groups occurring in Theorem 5.1 which we state by reformulating the result in terms of the usual permutation action of M_{23} on a set Ω of 24 elements.

Theorem 5.6. *The following sets coincide:*

(a) *The* 13 *isomorphism classes of groups (a)–(m) in Theorem 5.1.*

(b) *The isomorphism classes of subgroups of* M_{23} maximal *subject to having* at least 4 *orbits on* Ω.

(c) *The isomorphism classes of subgroups of* M_{23} maximal *subject to having* exactly 4 *orbits.*

Proof. In the following it is convenient to identify a group with its isomorphism class.

That each of the 13 groups G listed in Theorem 5.1 is isomorphic to a subgroup of M_{23} with at least four orbits on Ω is implicit in the proof of Theorem 5.1 (and can be verified by a Magma calculation). So certainly G is *contained* in a group in (b).

On the other hand, a Sylow 2-subgroup of M_{23} is strictly contained in the group $Q(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$, so *no* 2-subgroup $G \subseteq M_{23}$ can be maximal subject to having at least 4 orbits. It is then clear that every group G in (b) satisfies the assumptions, and therefore also the conclusions of Theorem 5.1, so G is contained in one of the groups of part (a). Together with the previous paragraph, and because there are no containments between the groups (a)–(m) in Theorem 5.1, this shows that the sets in parts (a) and (b) coincide.

Finally, each G in (a) has *exactly* 4 orbits, for otherwise it would have at least 5 orbits and would appear in Mukai's list [56] — but it does not. Therefore, by the same argument as the last paragraph, the groups in (a) and (c) also coincide. This completes the proof of the Theorem. $\square$

In the following, we let $\mathcal{T}$ denote the isomorphism classes of groups satisfying any (and hence all) of the properties (a)–(c) in the last theorem.

Remark 5.7. The M_{23}-conjugacy classes of the groups G in $\mathcal{T}$ are *not* unique. For the groups A_7, $\mathbf{Z}_2^3{:}L_2(7)$ and $\mathbf{Z}_2^4{:}S_5$ there are two conjugacy classes. For $\mathbf{Z}_2^4{:}S_5$, the orbit structure on Ω is different for the two conjugacy classes, whence they are also not conjugate in M_{24}.

Remark 5.8. The proof of Theorem 5.1 together with Theorem 5.6 actually shows that if a finite group G has a Mathieu representation V over $\mathbf{Q}$ with $\dim V^G \geq 4$ and is not a 2-group then G is contained in a group from $\mathcal{T}$.

Remark 5.9. The subgroups of M_{23} with at most four orbits also appear in the work of Dolgachev and Keum [20] on finite groups G of symplectic automorphisms of algebraic K3 surfaces Y defined over an algebraic closed field k of positive characteristic p. Dolgachev and Keum show that if $p \not| \, |G|$ then G has elements of order no bigger than 8, and affords a Mathieu representation on $V = H^*_{\text{et}}(Y, \mathbf{Q}_\ell)$ for $\ell \neq p$ prime, with $\dim V^G \geq 4$.

Using further geometric arguments, they found ten isomorphism classes of such groups which could potentially act on a K3 surface and be maximal, i.e. not contained in a larger such group ([20], Thm. 5.2). These are precisely the ten groups of our Theorem 5.1 containing no elements of order greater than 8 (a condition which excludes $L_2(11)$, $\mathbf{Z}_2 \times L_2(7)$ and $(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2)$. Our proof of Theorem 5.1 assumes only that G has a rational Mathieu representation with $\dim V^G \geq 4$ and is not a 2-group. On the other hand, there are nine types of 2-groups *not* contained in a 2-Sylow subgroup of M_{23}, as we will show in the next Subsection.

Dolgachev and Keum also list all isomorphism types of groups which can occur in their situation and which do *not* appear in Mukai's classification. These are necessarily subgroups of the ten maximal types. They list 28 such groups ([20], Thm. 5.2). These are in essential agreement with our list of such groups, which can be read off from Table 12. (The only ambiguity is that our nonisomorphic groups No. 144 and No. 146 of order 192 are both of type $\Gamma_{13}a_1{:}3$ and hence both correspond to the group (xxxiii) from [20].)

If $p||G|$, further groups G can be realized (cf. Section 6 in [20] and [42]). Some of these, such as $L_2(11)$, M_{11} and M_{22}, are subgroups of M_{23}. Others, such as $U_4(3)$, are not. It follows from the calculations described in Section 6 that M_{11}, M_{22} and $U_4(3)$ cannot be groups of symplectic automorphisms of any hyperkähler manifold of type $K3^{[n]}$, and likewise cannot be groups of symplectic autoequivalences for a complex K3 surface as considered in [40].

5.2. Admissible 2-groups

Suppose G only has elements of orders 1, 2, 4 or 8. Then G is necessarily a 2-group, and we already know from Theorem 5.2 that $|G| \leq 2^7$. To describe the possible 2-groups we will describe a computational approach, which seems unavoidable if one wants a complete proof of the result that is not too long.

Proposition 5.10. *Let G be a 2-group of exponent at most 8 having a complex Mathieu representation V with $\dim V^G \geq 4$. Then G is isomorphic either to a subgroup of M_{23}, or to one of 9 additional groups of order 16, 32 or 64 described in Table 4.*

Table 4: The nine non-excluded 2-groups

No.	order	Group library	Symbol	A	G/A
1	16	#4	$\Gamma_2 c_2$	$\mathbf{Z}_4 \times \mathbf{Z}_2$	$\mathbf{Z}_2$
2	16	#5	$\mathbf{Z}_8 \times \mathbf{Z}_2$	$\mathbf{Z}_8 \times \mathbf{Z}_2$	1
3	16	#10	$\mathbf{Z}_4 \times \mathbf{Z}_2^2$	$\mathbf{Z}_4 \times \mathbf{Z}_2^2$	1
4	32	#8	$\Gamma_7 a_3$	$\mathbf{Z}_4 \times \mathbf{Z}_2$	$\mathbf{Z}_4$
5	32	#30	$\Gamma_4 c_1$	$\mathbf{Z}_4 \times \mathbf{Z}_2^2$	$\mathbf{Z}_2$
6	32	#32	$\Gamma_4 c_3$	$\mathbf{Z}_4^2$	$\mathbf{Z}_2$
7	32	#35	$\Gamma_4 a_3$	$\mathbf{Z}_4^2$	$\mathbf{Z}_2$
8	32	#50	$\Gamma_5 a_2$	$\mathbf{Z}_4 \times \mathbf{Z}_2$	$\mathbf{Z}_2^2$
9	64	#36	$\Gamma_{23} a_3$	$\mathbf{Z}_4^2$	$\mathbf{Z}_4$

Proof. We first use the library of small groups in Magma to find the 2-groups G of order at most 256 and exponent at most 8 [10]. Then we check to see if G has a 20-dimensional representation of the correct type.[2]

[2]Magma Commands: `D:=SmallGroupDatabase(); value:=[20,4,99,0,99,99,99,-2]; [ #[ G : G in [SmallGroup(o, n): n in [1..NumberOfSmallGroups(o)]] | ({x[1]: x in`

Table 5: Isomorphism classes of 2-groups

order	1	2	4	8	16	32	64	128	256	total
no. of 2-groups	1	1	2	5	14	51	267	2328	56092	58761
of exponent ≤ 8	1	1	2	5	13	45	234	2093	53529	55923
and correct representation	1	1	2	5	12	12	6	1	0	40
contained in M_{23}	1	1	2	5	9	7	5	1	0	31
others contained in Co_0	0	0	0	0	1	2	1	0	0	4

The result is given in Table 5. As expected, there are *no* such groups of order 256, and a *unique* group of order 128, namely a Sylow 2-subgroup of M_{23}. We also list in Table 5 the number of 2-groups contained in M_{23}.

To explain Table 4, recall that a normal subgroup $A \trianglelefteq G$ which is maximal with respect to being abelian, is necessarily self-centralizing. Thus G/A is isomorphic to a subgroup of $\mathrm{Aut}(A)$. In the present situation, the existence of the complex representation means that A is necessarily isomorphic to a subgroup of one of $\mathbf{Z}_2^4$, $\mathbf{Z}_4 \times \mathbf{Z}_2^2$, $\mathbf{Z}_4^2$ or $\mathbf{Z}_8 \times \mathbf{Z}_2$. Table 4 lists a choice of A for each of the nine isomorphism classes of G *not* contained in M_{23}. Some choices of A (elementary abelian and cyclic) are not represented in Table 4, meaning that the corresponding G is contained in M_{23}. The group library number refers to [6], the symbol for the non-abelian groups is as in [32]. □

Four of the nine groups in Table 4 satisfy $\dim V^G \geq 5$ and have been described in [56], Prop. (6.3.).

Lemma 5.11. *The number of conjugacy classes of abelian subgroups* $A \cong \mathbf{Z}_4 \times \mathbf{Z}_2$, $\mathbf{Z}_8 \times \mathbf{Z}_2$, $\mathbf{Z}_4^2$ *and* $\mathbf{Z}_4 \times \mathbf{Z}_2^2$ *in* $2^{12}{:}M_{24}$ *containing only admissible elements is* 8, 0, 6 *and* 7, *respectively.*

Proof. We construct all abelian subgroups of rank at most 3 by the following procedure. We first select representatives a for each of the 7 conjugacy classes in $2^{12}{:}M_{24}$ of the correct Conway types. For each of those elements, we select representatives b for the conjugacy classes of its centralizer in $2^{12}{:}M_{24}$ and check if the subgroups $H = \langle a, b \rangle$ generated by a and b contain only elements of admissible Conway types. In total, there are 26 such conjugacy classes of subgroups H. Then we select representatives c for the conjugacy classes of

```
Classes(G)} subset {1,2,4,8}) and IsCharacter(CharacterRing(G) ! [value[x[1]] : x
in Classes(G)]) ] : o in [1,2,4,8,16,32,64,128,256] ];
```

the centralizer of the subgroup H and determine the conjugacy classes of subgroups $K = \langle H, c\rangle$ in $2^{12}{:}M_{24}$. There are 41 such conjugacy classes of subgroups K containing only admissible elements. The resulting number of conjugacy classes for each choice of A is as stated in the Lemma. □

Theorem 5.12. *Suppose G is an admissible 2-group. Then either G is isomorphic to a subgroup of M_{23}, or is isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_2^2$, or is contained in a group of order 32 or 64 described below (Table 4, nos. 8 and 9).*

Proof. We have to check which of the nine groups G of Proposition 5.10 (cf. Table 4) are isomorphic to subgroups of Co_0 and contain only admissible elements. Note that because the monomial subgroup $2^{12}{:}M_{24}$ contains a Sylow 2-subgroup of Co_0, the relevant calculations can be carried out in the monomial group.

For each conjugacy class of abelian subgroups $A \subseteq 2^{12}{:}M_{24}$ as in Lemma 5.11, we compute the normalizer $N(A)$ in $2^{12}{:}M_{24}$ and check if it contains a group G as in Table 4 containing only admissible elements. This is done by selecting a representative d of each conjugacy class of elements in $N(A)$ such that d^4 is in A and determining the isomorphism type of $\langle A, d\rangle$. In the case of group no. 8 in Table 4 we also select in addition all elements e of $N(A)$ such that e^2 and the commutator $[e, d]$ is in A and determine the isomorphism type of $\langle A, d, e\rangle$. It transpires that only groups no. 3, 4, 8, and 9 occur. Moreover, group no. 4 is a subgroup of group no. 9. □

6. Conjugacy Classes of Admissible Subgroups in Co_0

In the previous section, we determined by largely theoretical arguments the abstract isomorphism types of admissible subgroups of Co_0.

For the classification of group lattices (Λ_G, G), we will enumerate the conjugacy classes of admissible subgroups $G \subseteq \mathrm{Co}_0$. The following main result of this section depends heavily on computer calculations:

Theorem 6.1. *There are 198 conjugacy classes of admissible subgroups $G \subseteq \mathrm{Co}_0$. In particular, there are exactly 22 classes which are* maximal *(with respect to containment), which are described as follows:*

(a) *thirteen subgroups of M_{23} (Table 6);*

(b) *two groups $3^4.A_6$ and $3^{1+4}{:}2.2^2$ related to $\mathcal{S}$-lattices;*

(c) *two groups of order* 48 *and five* 2*-groups* (*Table* 7).

Detailed information about all of these classes of groups can be found in Table 12 in the appendix. The corresponding group lattices (Λ_G, G), which we will discuss in the next section, are pairwise nonisomorphic (cf. Table 13 in the appendix).

Remark. There are just 82 admissible conjugacy classes of G containing only elements of $K3$-type. The corresponding group lattices (L_G, G) were first determined by Hashimoto [33], who used the 23 Niemeier lattices with roots rather than the Leech lattice that we use here. These 82 classes are in 1-1 correspondence with the combinatorial structure of symplectic group actions on a K3 surface, as determined by Xiao [62].

In the following, we describe our method of computing admissible conjugacy classes of subgroups $G \subseteq \mathrm{Co}_0$ using Magma. We first realized Co_0 as a group of integral 24×24 matrices starting from an explicit description of the Leech lattice. This realization was used to determine the conjugacy classes of Co_0 and to compute the dimension of Λ^G for a subgroup $G \subseteq \mathrm{Co}_0$. In addition, a realization as permutation group on the 196560 minimal vectors of Λ together with an explicit isomorphism with the matrix group realization was constructed. This realization was used to check if two subgroups of Co_0 are conjugate. Computations with both realizations are relatively time consuming and had to be minimized.

By Theorem 3.2, an admissible subgroup G is either conjugate to a subgroup of the monomial group $2^{12}{:}M_{24}$ or a subgroup of one of three groups related to $\mathcal{S}$-lattices. We identified $2^{12}{:}M_{24}$ inside the permutation representation of Co_0 by computing the stabilizer of 1104 norm 4 vectors $(\pm u \pm v)/2$, where $\pm u$ and $\pm v$ run through a coordinate frame of Λ. For the $\mathcal{S}$-lattices, we determined the pointwise stabilizers by using the explicit realizations given by Curtis [18]. For the $\mathcal{S}$-lattice of rank 6, we computed in addition the normalizer of the stabilizer in Co_0. For calculations inside $2^{12}{:}M_{24}$, we used a realization as a permutation group on the 48 elements. We also determined an explicit isomorphism between this permutation realization of $2^{12}{:}M_{24}$ and the above described frame stabilizer inside Co_0. For calculations inside the three $\mathcal{S}$-lattice groups, we constructed a permutation representation of much smaller degree together with an explicit isomorphism with the corresponding subgroups in Co_0. This allowed us to determine their complete subgroup lattice. For each conjugacy class of subgroups we constructed the corresponding subgroup of Co_0 and selected the admissible one. Finally, we determined the Co_0-conjugacy classes.

For the subgroups $G \subseteq 2^{12}{:}M_{24}$ we distinguished two cases: 2-groups and non 2-groups. For the non 2-groups, we were able to construct the corresponding part of the subgroup lattice of $2^{12}{:}M_{24}$ completely. Starting from $2^{12}{:}M_{24}$, we constructed inductively via the order all maximal subgroups of fixed order of the already found conjugacy classes. Then we checked for a given order all the found maximal subgroups for conjugacy in $2^{12}{:}M_{24}$. Since the number of required conjugacy checks is growing almost quadratically with the number of such subgroups, we first determined for each subgroup the size of each of its conjugacy classes, using this as a numerical indicator for non-conjugacy and thereby reducing the number of conjugacy checks in $2^{12}{:}M_{24}$. Overall, there are $279,343$ classes of such subgroups in $2^{12}{:}M_{24}$. Just 280 of these classes contain only admissible elements and of these, 241 classes are admissible. They belong to 94 Co_0-conjugacy classes.

It follows from Theorem 5.6 that the admissible subgroups of $2^{12}{:}M_{24}$ of order divisible by 3, 5, 7, or 11 are isomorphic to subgroups of the set $\mathcal{T}$ of isomorphism classes of subgroups of M_{23} maximal among having at least four orbits on Ω. In most cases, there exist several $2^{12}{:}M_{24}$ conjugacy classes for a group in $\mathcal{T}$. With respect to Co_0-conjugacy however, our calculations show:

Theorem 6.2. *For each of the groups in $\mathcal{T}$, there exists a unique* Co_0-*conjugacy class inside* $2^{12}{:}M_{24}$.

However, for groups of order $2^f.3$ there are further possibilities:

Theorem 6.3. *Suppose that $G \subseteq 2^{12}{:}M_{24}$ is admissible and* not *a 2-group. Then G is conjugate in* Co_0 *to a subgroup of either a group in the set $\mathcal{T}$ of subgroups of M_{23} (cf. Theorem* 5.6)*, or one of two conjugacy classes of groups of order* 48.

Although the two groups of order 48 project isomorphically to subgroups in $M_{23} \subseteq M_{24}$, they are not conjugate.

We collect basic information about the groups in $\mathcal{T}$ in Table 6 and the two groups of order 48 in Table 7. The entries of the first four columns is self-explanatory, the column A_{Λ^G} gives the structure of the discriminant group $(\Lambda^G)^*/\Lambda^G$. The last three columns give information on all $2^{12}{:}M_{24}$-conjugacy classes of G. To explain this, let $E = \mathbf{Z}_2^{12}$ and $M = M_{24}$, so that the monomial group is $E{:}M$. Then in Table 6 and 7, $A = G \cap E$ and $P = G/A \subseteq M$. The seventh column describes the orbits of P on Ω, and a star in the last column indicates that $G \cong A.P$ is *not* a subgroup $A{:}P$ of $E{:}M$.

For the 2-subgroups of $2^{12}{:}M_{24}$, we were unable to compute the corresponding part of the subgroup lattice because the number of conjugacy classes became too large. Instead we determined only the admissible one. First we selected a 2-Sylow subgroup $P \subseteq 2^{12}{:}M_{24}$ and hence Co_0. All P-conjugacy classes of admissible 2-groups $G \subseteq P$ were determined, starting with the trivial group and successively adding further elements. Finally, we tested for $2^{12}{:}M_{24}$-conjugacy and Co_0-conjugacy. The numbers of such conjugacy classes of groups for a given order are listed in Table 8.

In particular, we obtained the following result:

Theorem 6.4. *Let $G \subseteq \mathrm{Co}_0$ be an admissible 2-group. Then G is conjugate to a subgroup of either a group in the set $\mathcal{T}$ of subgroups of M_{23} as described in Theorem 5.6, or of one of 5 conjugacy classes of 2-groups listed in Table 7.*

Remarks. Theorem 6.4 implies Theorem 5.12. Since the computations for Theorem 6.4 take much longer and both computations are independent, we have presented both of them.

Although the 2-group $\mathbf{Z}_4^2$ is one of the 5 maximal groups in Theorem 6.4, it is also isomorphic to a subgroup of M_{23}.

As a final step, we took all the admissible conjugacy classes of Co_0 obtained from the $\mathcal{S}$-lattices and from $2^{12}{:}M_{24}$, rechecked for Co_0-conjugacy, and determined the corresponding subgroup lattice structure. The result is Theorem 6.1, and the information given in Table 12 in the appendix.

7. Subgroups of $O(L)$

In this section, we investigate which conjugacy classes of subgroups G and lattices Λ_G found in the previous section can indeed be realized as symmetries of the lattice

$$L = H^2(X, \mathbf{Z}) \cong E_8(-1)^2 \oplus U^3 \oplus \langle -2 \rangle.$$

For most of the realizable groups G, we will also determine the exact number of corresponding isomorphism classes of embeddings $(L_G, G) \to (L, O(L))$.

For each of the 198 Co_0-classes of admissible groups G, we first determine the isomorphism classes of coinvariant lattices $L_G \cong \Lambda_G(-1)$. This is easily done with Magma. There are 69 such lattices, listed in Table 13 in the appendix.

The lattices K in Table 13 are naturally divided into three main types depending on the relation between $\operatorname{rk} K$ and $\operatorname{rk} A_K$. Setting

$$\alpha(K) := 24 - \operatorname{rk} K - \operatorname{rk} A_K,$$

we find the following possibilities:

(a) 54 lattices with $\alpha(K) \geq 2$: these are the 13 lattices corresponding to the maximal subgroups of M_{23} with at least four orbits, and the 41 lattices coming from symplectic group actions on K3 surfaces as classified in [33].

(b) 4 lattices with $\alpha(K) = 1$: two of them correspond to the two maximal $\mathcal{S}$-lattice groups, while the other two correspond to certain subgroups.

(c) 11 lattices with $\alpha(K) = 0$: they correspond to 2-groups or groups of order $2^f.3$ in $2^{12}{:}M_{24}$ which are *not* conjugate to a subgroup of M_{23}.

Apart from the non-canonically defined 2-adic genus symbol, the table in Section 10.2 of [33] seems to be in complete agreement with our Tables 12 and 13.

For certain lattices K, there is more than one group such that $K \cong L_G$. For such K we list all classes of groups G with $K \cong L_G$ in Table 14. For the $K3$ cases, this again is in accord with the Table in Section 10.4 of [33]. Let $\mathcal{G}(K)$ be the classes of groups G from Table 12 which have the same $K \cong L_G$. By inspection we find:

Theorem 7.1. *The set* $\mathcal{G}(K)$ *contains a* unique *maximal class* $G_{\max}(K)$*. The other classes* $H \in \mathcal{G}(K)$ *correspond to subgroups of* $G_{\max}(K)$*. More precisely, the classes in* $\mathcal{G}(K)$ *can be identified with the* $O(K)$*-conjugacy classes of subgroups* H *of* $O_0(K)$ *which have a trivial fixed-point lattice* K^H *and are contained in a conjugacy class* $G_{\max}(K)$*. Furthermore, for the lattices of type* (a) *and* (b) *one has* $G_{\max}(K) = O_0(K)$*.*

To study embeddings $(L_G, G) \to (L, O(L))$, we have to find a lattice L^G of rank $23 - \operatorname{rk} L_G$ and an extension of $L_G \oplus L^G$ to L. Such an extension is described by a *glue code* $C \subseteq A_{L_G} \oplus A_{L^G}$ which has to be an isotropic subspace with respect to the quadratic form $q_{L_G} \oplus q_{L^G}$ for $A_{L_G} \oplus A_{L^G}$. The resulting extension $K_C \supseteq L_G \oplus L^G$ is isomorphic to L precisely when the

Table 6: Maximal admissible groups contained in M_{23}

					2^{12}:M_{24}-classes		
No.	G	$\|G\|$	$\mathrm{rk}\Lambda^G$	A_{Λ^G}	$(\|P\|, \|A\|)$	orbits	
1	$\mathrm{L}_2(11)$	660	4	11^2	$[660, 1]$	$[1, 1, 11, 11]$	
2	$\mathrm{L}_3(4)$	20160	4	$2^1 42^1$	$[20160, 1]$	$[1, 1, 1, 21]$	
3	A_7	2520	4	105^1	$[2520, 1]$	$[1, 1, 7, 15]$	
4	$\mathbf{Z}_2^3$:$\mathrm{L}_2(7)$	1344	4	$4^1 28^1$	$[1344, 1]$	$[1, 7, 8, 8]$	
					$[1344, 1]$	$[1, 1, 8, 14]$	
					$[168, 8]$	$[1, 1, 1, 7, 14]$	
5	$\mathbf{Z}_2 \times \mathrm{L}_2(7)$	336	4	14^2	$[336, 1]$	$[1, 2, 7, 14]$	
					$[168, 2]$	$[1, 1, 7, 7, 8]$	
6	$\mathbf{Z}_2^4$:A_6	5760	4	$4^1 24^1$	$[5760, 1]$	$[1, 1, 6, 16]$	
					$[360, 16]$	$[1, 1, 1, 6, 15]$	
7	$\mathbf{Z}_2^4$:S_5	1920	4	$4^1 40^1$	$[1920, 1]$	$[1, 1, 2, 20]$	
					$[1920, 1]$	$[1, 2, 5, 16]$	
					$[120, 16]$	$[1, 1, 2, 5, 15]$	
8	S_6	720	4	$6^1 30^1$	$[720, 1]$	$[2, 6, 6, 10]$	
					$[720, 1]$	$[1, 2, 6, 15]$	
9	M_{10}	720	4	$2^1 60^1$	$[720, 1]$	$[1, 1, 10, 12]$	
10	$(\mathbf{Z}_3 \times A_5)$:$\mathbf{Z}_2$	360	4	15^2	$[360, 1]$	$[1, 3, 5, 15]$	
11	$Q(\mathbf{Z}_3^2$:$\mathbf{Z}_2)$	1152	4	$8^1 24^1$	$[1152, 1]$	$[1, 3, 4, 16]$	
					$[288, 4]$	$[1, 3, 4, 4, 12]$	*
					$[72, 16]$	$[1, 1, 3, 3, 4, 12]$	
12	$\mathbf{Z}_2^4$:$(S_3 \times S_3)$	576	4	$12^1 24^1$	$[576, 1]$	$[2, 3, 3, 16]$	
					$[576, 1]$	$[1, 3, 8, 12]$	
					$[36, 16]$	$[1, 2, 3, 3, 6, 9]$	
13	$\mathbf{Z}_3^2$:QD_{16}	144	4	$6^1 36^1$	$[144, 1]$	$[1, 2, 9, 12]$	

Table 7: Maximal admissible groups contained in $2^{12}{:}M_{24}$ but not in M_{23}

No.	G	$\lvert G\rvert$	$\mathrm{rk}\Lambda^G$	A_{Λ^G}	$2^{12}{:}M_{24}$-classes $(\lvert P\rvert, \lvert A\rvert)$	orbits	
1	#49	48	5	$2^3 6^1 12^1$	$[48,1]$	$[2,2,6,6,8]$	$*$
					$[24,2]$	$[1,1,3,3,8,8]$	$*$
					$[24,2]$	$[1,1,2,6,6,8]$	$*$
					$[6,8]$	$[1,1,2,2,3,3,6,6]$	$*$
2	#32	48	4	$2^2 4^1 12^1$	$[48,1]$	$[2,6,8,8]$	$*$
					$[24,2]$	$[1,1,6,8,8]$	$*$
					$[24,2]$	$[1,1,2,6,6,8]$	$*$
					$[24,2]$	$[1,1,2,6,6,8]$	$*$
3	$\mathbf{Z}_4\times\mathbf{Z}_2^2$	16	7	$2^4 4^3$	$[16,1]$	$[2,2,2,2,4,4,8]$	$*$
					$[8,2]$	$[1,1,1,1,2,2,8,8]$	$*$
					$[8,2]$	$[1,1,2,2,2,4,4,8]$	$*$
					$[8,2]$	$[1,1,2,2,2,2,2,2,2,8]$	$*$
					$[8,2]$	$[1,1,2,2,2,2,2,2,2,8]$	$*$
					$[4,4]$	$[1,1,1,1,2,2,2,2,2,2,4,4]$	$*$
					$[2,8]$	$[1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2]$	$*$
4	$\mathbf{Z}_4^2$	16	6	$2^2 4^4$	$[16,1]$	$[4,4,4,4,4,4]$	$*$
					$[8,2]$	$[1,1,2,2,2,4,4,8]$	$*$
					$[4,4]$	$[1,1,1,1,2,2,2,2,2,2,4,4]$	$*$
5	$\Gamma_5 a_2$	32	6	$2^2 4^4$	$[8,4]$	$[1,1,2,2,2,2,2,2,2,8]$	$*$
6	$\Gamma_7 a_3$	32	4	$2^2 8^2$	$[8,4]$	$[1,1,2,2,2,4,4,8]$	$*$
					$[8,4]$	$[1,1,2,2,2,4,4,8]$	$*$
7	$\Gamma_{23} a_3$	64	4	$2^3 16^1$	$[16,4]$	$[1,1,1,1,4,8,8]$	$*$

Table 8: Conjugacy classes of admissible 2-groups

order	1	2	4	8	16	32	64	128	256	total
# P-classes	1	27	317	1312	2190	803	239	30	0	4919
# $2^{12}{:}M_{24}$-classes	1	2	7	27	63	34	20	3	0	157
# Co_0-classes	1	1	2	6	15	10	6	1	0	42

discriminant form on A_{K_C} is isomorphic to the discriminant form on A_L, since there is only one lattice in the genus of L. Two lattices K_C and $K_{C'}$ determine $O(L)$-conjugate sublattices L_G inside L if, and only if, C and C' are in the same orbit for the action of $\overline{O}(L_G)\times\overline{O}(L^G)$ on $A_{L_G}\oplus A_{L^G}$ induced by the natural action of $O(L_G)\times O(L^G)$. For a fixed lattice K_C, the $O(L)$-conjugacy classes of $G\subseteq O_0(L_G)$ are given by the F-conjugacy classes where $F\subseteq O(L_G)$ is the image of the projection on the first factor of the stabilizer of C under the action of $O(L_G)\times O(L^G)$.

Since L^G and L_G are both primitive sublattices of L, the code C must have the form $C = \{(x, y) \mid x \in A_{L_G},\ y \in A_{L^G}\}$ and satisfy $(x, 0) \in C \Rightarrow x = 0$ and $(0, y) \in C \Rightarrow y = 0$. Since L has a discriminant group A_L of order 2, this implies that $2|C|^2 = |A_{L_G}||A_{L^G}|$, and either

(1) $C = \{(x, \gamma(x)) \mid x \in A_{L_G}\}$ where $\gamma : A_{L_G} \to A_{L^G}$ is a group monomorphism with $q_{L^G} \circ \gamma = -q_{L_G}$, or

(2) $C = \{(\gamma'(y), y) \mid y \in A_{L^G}\}$ where $\gamma' : A_{L^G} \to A_{L_G}$ is a group monomorphism with $q_{L_G} \circ \gamma' = -q_{L^G}$.

Assume we are in case (1). Then $|A_{L^G}/\gamma(A_{L_G})| = 2$. Since $\gamma(A_{L_G})$ is a non-degenerate subspace of A_{L^G} with respect to q_{L^G}, there is an orthogonal decomposition $A_{L^G} = \gamma(A_{L_G}) \oplus \gamma(A_{L_G})^\perp$ and $\gamma(A_{L_G})^\perp$ is generated by an element $v_\gamma \in A_{L^G}$ of order 2 satisfying $q|_{\langle v_\gamma \rangle} \cong q_L$. Thus in order to describe the $\overline{O}(L_G) \times \overline{O}(L^G)$-orbits of codes C, we first determine the $\overline{O}(L^G)$-orbits of splittings $A_{L^G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$ and then the $\overline{O}(L_G) \times S$-orbits of maps $\gamma : A_{L_G} \longrightarrow \gamma(A_{L_G})$ as in (1), where S is the stabilizer of v_γ in $\overline{O}(L^G)$ acting on $\gamma(A_{L_G})$. Fixing some γ allows us to identify S with a subgroup of $O(A_{L_G})$ and the $\overline{O}(L_G) \times S$-orbits with the double cosets of the pair $(\overline{O}(L_G), S)$ in $O(A_{L_G})$.

For case (2), $\gamma'(A_{L^G})$ is a subgroup of index 2 in A_{L_G} and there is an orthogonal decomposition $A_{L_G} = \gamma'(A_{L^G}) \oplus \gamma'(A_{L^G})^\perp$ where $\gamma'(A_{L^G})^\perp$ is generated by an element $w_{\gamma'} \in A_{L^G}$ of order 2 with $q|_{\langle w_{\gamma'} \rangle} \cong q_L$. So to describe the $\overline{O}(L_G) \times \overline{O}(L^G)$-orbits of codes C, we first determine the $\overline{O}(L_G)$-orbits of splittings $A_{L_G} = \gamma'(A_{L^G}) \oplus \langle w_{\gamma'} \rangle$ and then the $S \times \overline{O}(L^G)$-orbits of maps $\gamma' : A_{L^G} \longrightarrow \gamma'(A_{L^G})$ as in (2), where now S is the stabilizer of $w_{\gamma'}$ in $\overline{O}(L_G)$ acting on $\gamma'(A_{L^G})$. Fixing a γ' again permits us to identify $\overline{O}(L^G)$ with a subgroup of $O(\gamma'(A_{L^G}))$ and the $S \times \overline{O}(L^G)$-orbits with the double cosets of the pair $(S, \overline{O}(L^G))$ in $O(\gamma'(A_{L^G}))$.

For the above discussion see also [57], especially Proposition 1.5.1.

We are now ready for the proof of Lemma 4.2. We will apply the preceding discussion with $G = \langle g \rangle$.

Proof. We have

$$\begin{aligned} \alpha(L_g) &:= 24 - \operatorname{rk} L_g - \operatorname{rk} A_{L_g} \;=\; 24 - \operatorname{rk} \Lambda_g - \operatorname{rk} A_{\Lambda_g} \\ &= \operatorname{rk} \Lambda^g - \operatorname{rk} A_{\Lambda^g} \;=\; \operatorname{rk} A_{\Lambda^g} - \operatorname{rk} A_{\Lambda^g} \;=\; 0. \end{aligned}$$

This means that L_g is a lattice of type (c), and the glueing of L_g with L^g is described by case (2), i.e., by an injective map $\gamma' : A_{L^g} \to A_{L_g} \cong A_{\Lambda^g}$. This proves part (a) of Lemma 4.2.

For part (b), note that $A_{L^g} = \gamma'(A_{L^g}) \oplus \langle w_{\gamma'} \rangle$. Since $q_L(w_{\gamma'}) = \frac{3}{2}$, it follows that q_{L_g}, and therefore also q_{Λ^g}, has $\frac{3}{2}$ in its image. □

Returning to a general group G, after fixing a pair (L_G, G) we proceed along the lines of the preceding discussion according to the following five steps:

1. Fix one of the constructions (1) or (2).
2. Determine all lattices in the genus for L^G (which is uniquely determined by the genus of L_G).
3. Determine the $\overline{O}(L^G)$-orbits of splittings $A_{L^G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$ resp. the $\overline{O}(L_G)$-orbits of splittings $A_{L_G} = \gamma'(A_{L^G}) \oplus \langle w_{\gamma'} \rangle$.
4. Determine the $\overline{O}(L_G)$–S double cosets in $O(A_{L_G})$ for construction (1), and the S–$\overline{O}(L^G)$ double cosets in $O(\gamma'(A_{L^G}))$ for construction (2).
5. Determine the $O(L)$-conjugacy classes of G for each double coset.

Construction (1). Note that we only have to consider lattices of type (a) or (b), i.e., with $\alpha(K) \geq 1$. Indeed, for lattices of type (c) we have $\operatorname{rk} L^G = 23 - \operatorname{rk} L_G < 24 - \operatorname{rk} L_G = \operatorname{rk} A_{L_G}$, whence we cannot embed A_{L_G} into A_{L^G} using γ.

First consider the case $\operatorname{rk} L_G = 20$. From Table 13 we see that there are 13 lattices of type (a) corresponding to the $G \subseteq M_{23}$ with four orbits in the usual action on 24 letters, and the two lattices of type (b) corresponding to the two maximal S-lattice groups. In this case, L^G has to be positive-definite of rank 3 and the quadratic form q_{L^G} is equivalent to $q_{L_G} \oplus q_L$. Recall (proof of Lemma 4.2) that $q_L(x) = \frac{3}{2}$ for the non-zero element x in A_L. This uniquely determines the genus of L^G, and the corresponding lattices L^G can be read off from the Brandt-Intrau tables of positive definite ternary forms [9]. The result is listed in Table 9.

If $G \cong M_{10}$ and L^G is the lattice with Gramian matrix $\mathrm{Diag}(2, 4, 30)$, there are two $O(L^G)$-orbits of splittings $A_{L^G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$, whereas in all other cases there is a unique $O(L^G)$-orbit of splittings.

We assert that there is a *unique* $\overline{O}(L_G)$−S double coset in $O(A_{L_G})$. If $G \ncong S_6$ then $\overline{O}(L_G)$ coincides with $O(A_{L_G})$, and the assertion follows. A computation shows that the same result holds if $G \cong S_6$.

Finally, let $\overline{F} \subseteq \overline{O}(L_G)$ be the projection onto the first factor of the stabilizer in $\overline{O}(L_G) \times S$ of the identity element under the double coset action. Let $F \subset O(L_G)$ be the inverse image of $\overline{F}$ under the natural projection. We consider the conjugation action of F on the set of subgroups $H \subseteq O_0(L_G)$ with $L_G^H = \{0\}$. For lattices of type (a), the orbits agree with the $\overline{O}(L_G)$ classes, whereas for the two lattices of type (b), there are more orbits. We list the number of conjugacy classes H for both groups in the last two columns of Table 9.

Now assume that $\operatorname{rk} L_G < 20$. In this case, L^G has rank at least 4 and must be indefinite. We claim that the lattice L^G exists and is unique. Indeed, the 41 lattices L_G of type (a) are sublattices of the K3-lattice $N \cong E_8(-1)^2 \oplus U^3$. If N^G is an orthogonal complement of L^G in N then we let $L^G = N^G \oplus A_1(-1)$. If the uniqueness criterion of Theorem 1.7 of [33] (which follows from Eichler's theory of Spinor genera, cf. [57]) applies to N^G then it also applies to L^G. Therefore, the explicit verification in Section 6 of [33] establishes uniqueness for all 30 lattices L^G of type (a) and rank > 4. For the 11 lattices L^G of rank 4, we apply Theorem 1.7 of [33] directly. For the two lattices L_G of type (b) and rank 19 and 18, one gets for L^G the two lattices with Gramian matrix

$$\begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

respectively. Again, the uniqueness follows from Theorem 1.7 of [33].

Conjecture 7.2. *For L_G a lattice of type (a) or (b) and $\operatorname{rk} L^G \geq 4$, we have $\overline{O}(L^G) = O(A_{L^G})$.*

By applying the criterion from Theorem 1.14.2 in [57], one checks that the conjecture holds for all L_G of type (a) and (b) in Table 13, except perhaps for the cases with number

$$n = 18,\ 26,\ 31,\ 39,\ 41,\ 47,\ 48,\ 54,\ 55,\ 61.$$

Unfortunately, most lattice functions of Magma are presently implemented for definite lattices only, so we cannot verify this conjecture by a computer calculation without extra programming work. One could do this more theoretically as in [33], Section 7, by using the strong approximation theorem, but we refrain from investigating this here.

If Conjecture 7.2 holds, it is clear that there is a single $O(L^G)$-orbit of orthogonal splittings $A_{L^G} = \gamma(A_{L_G}) \oplus \langle v_\gamma \rangle$. Furthermore, we would have $S = O(A_{L_G})$ and $F = O(L_G)$. It then follows from Theorem 7.1 that for each (L_G, G), there is a *unique* $O(L)$-conjugacy class of subgroups G.

Remark. Since Conjecture 7.2 holds in particular for the three K3-lattices L_G of rank 19 (numbers 44, 46 and 52 in Table 13) for which there are two lattices in the genus of N^G, it follows that the corresponding symplectic group actions on $K3$ are *not* deformation equivalent. However, the induced symplectic group actions on $K3^{[2]}$ are so.

Our calculations have shown:

Theorem 7.3. *Let $G \subseteq \mathrm{Co}_0$ be an admissible subgroup. Then (L_G, G) can be realized as the coinvariant lattice for a subgroup $G \subseteq O(L)$ by construction* (1) *if, and only if, $\alpha(L_G) \geq 1$. If $\mathrm{rk}\, L_G = 20$, the number of corresponding conjugacy classes of $G \subseteq O(L)$ can be read off from Table* 9*. If $\mathrm{rk}\, L_G < 20$, there is a* unique *such class, provided that either L_G is not one of the cases no.* 18, 26, 31, 39, 41, 47, 48, 54, 55, 61 *in Table* 13*, or Conjecture* 7.2 *holds.*

Construction (2). We first study the $\overline{O}(L_G)$-orbits of splittings $A_{L_G} = \gamma'(A_{L^G}) \oplus \langle w_{\gamma'} \rangle$. Thus we searched for elements $w_{\gamma'} \in A_{L_G}$ of order 2 with $q_{L_G}(w_{\gamma'}) = 3/2 \pmod 2$. Such elements exist only for the eleven lattices L_G with the numbers

$$n = 9,\ 14,\ 18,\ 20,\ 27,\ 28,\ 33,\ 38,\ 40,\ 45,\ 50$$

in Table 13, and in each case there is a *unique* $\overline{O}(L_G)$-orbit.

Table 9: Conjugacy classes in $O(L)$: rank $L^G = 3$, construction (1)

No.	G	$\lvert A_{L_G}\rvert$	$\lvert O(L_G)\rvert$	$\lvert\overline{O}(L_G)\rvert$	$\lvert O(A_{L_G})\rvert$	L^G	$\lvert O(L^G)\rvert$	$\lvert\overline{O}(L^G)\rvert$	$\lvert O(A_{L^G})\rvert$	#H	#orbs
1	$\mathrm{L}_2(11)$	121	15840	24	24	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}$	8	4	24	3	3
						$\begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}$	12	12	24	3	3
2	$\mathrm{L}_3(4)$	84	483840	24	24	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 4 \\ 0 & 4 & 10 \end{pmatrix}$	8	4	24	1	1
						$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 14 \end{pmatrix}$	24	24	24	1	1
3	A_7	105	20160	8	8	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 70 \end{pmatrix}$	24	4	8	1	1
						$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 18 \end{pmatrix}$	8	4	8	1	1
						$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 12 \end{pmatrix}$	4	4	8	1	1
						$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 6 & 1 \\ 1 & 1 & 8 \end{pmatrix}$	4	4	8	1	1
4	$\mathbf{Z}_2^3{:}\mathrm{L}_2(7)$	112	21504	16	16	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 10 \end{pmatrix}$	4	4	16	3	3
5	$\mathbf{Z}_2 \times \mathrm{L}_2(7)$	196	10752	32	32	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$	16	8	32	3	3
						$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix}$	8	8	32	3	3
6	$\mathbf{Z}_2^4{:}A_6$	96	92160	16	16	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 24 \end{pmatrix}$	8	4	16	7	7
						$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$	8	8	16	7	7
7	$\mathbf{Z}_2^4{:}S_5$	160	30720	16	16	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 40 \end{pmatrix}$	8	4	16	2	2
						$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$	8	8	16	2	2
8	S_6	180	23040	32	96	$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$	24	24	96	1	1
9	M_{10}	120	5760	8	8	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$	8	4	16	1	1
							8	4	16	1	1
						$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$	8	8	16	1	1
10	$(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$	225	17280	48	48	$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$	8	8	48	5	5
						$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 5 \\ 0 & 5 & 10 \end{pmatrix}$	24	24	48	5	5
11	$Q(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$	192	36864	32	32	$\begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & -2 \\ 2 & -2 & 14 \end{pmatrix}$	8	8	64	10	10
12	$\mathbf{Z}_2^4{:}(S_3 \times S_3)$	288	36864	64	64	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 24 \end{pmatrix}$	8	8	64	2	2
13	$\mathbf{Z}_3^2{:}QD_{16}$	216	3456	24	24	$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 10 & 0 \\ 0 & 0 & 12 \end{pmatrix}$	8	8	48	2	2
14	$3^{1+4} : 2.2^2$	108	186624	96	96	$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$	48	48	288	7	8
15	$3^4 : A_6$	81	4199040	144	144	$\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$	24	24	144	40	71

Table 10: Conjugacy classes in $O(L)$: $\operatorname{rank} L^G = 3$, construction (2)

No.	G	$\lvert A_{L_G}\rvert$	$\lvert O(L_G)\rvert$	$\lvert\overline{O}(L_G)\rvert$	$\lvert O(A_{L_G})\rvert$	L^G	$\lvert O(L^G)\rvert$	$\lvert\overline{O}(L^G)\rvert$	$\lvert O(A_{L^G})\rvert$
1	M_{10}	120	5760	8	8	$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 8 \end{pmatrix}$	8	4	8
						$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 1 \\ 2 & 1 & 6 \end{pmatrix}$	8	8	8
2	$\mathbf{Z}_3^2{:}QD_{16}$	216	3456	24	24	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 1 \\ 2 & 1 & 10 \end{pmatrix}$	8	8	24
3	$32\,\Gamma_7 a_3$	256	196608	256	256	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$	16	8	32
4	$64\,\Gamma_{23} a_3$	128	36864	96	96	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 16 \end{pmatrix}$	16	4	16

The discriminant form of L^G is equal to $-q_{L_G}|_{w_{\gamma'}^{\perp}}$, so in each case the genus of L^G is uniquely determined. For the four cases $n = 58$, 45, 38 and 27 with $\operatorname{rk} L_G = 20$, Table 10 lists L^G together with additional information concerning $O(L_G)$ and $O(L^G)$. In the other seven cases L^G is indefinite, and one easily checks that L^G exists and is unique in these cases.

For reasons which will become apparent in the next section, we will not further analyze the exact number of $O(L)$-conjugacy classes of realizations of the group lattice (L_G, G) inside L. We have established:

Theorem 7.4. *Let $G \subseteq \mathrm{Co}_0$ be an admissible subgroup. Then (L_G, G) can be realized as the coinvariant lattice for a subgroup $G \subseteq O(L)$ by construction* (2) *if, and only if, L_G is one of cases no.* 9, 14, 18, 20, 27, 28, 33, 38, 40, 45, 50 *in Table* 13.

We note that there are admissible subgroups $G \subseteq \mathrm{Co}_0$ for which (L_G, G) *cannot* be realized in this way.

Remark. We also have applied this method to prove the uniqueness of the conjugacy class of G in the isometry group $O(N)$ of the K3-lattice N for the 11 K3-lattices (L_G, G) of rank 19 and the corresponding 14 rank 3 lattices N^G. This provides a somewhat more systematic approach for these cases compared to Section 8.2 of [33].

8. Symplectic Actions on $K3^{[2]}$

In this section, we will show that the groups $G \subseteq O(L)$ that can be realized as isometries of a hyperkähler manifold of type $K3^{[2]}$ are exactly those for which L_G is a lattice of type (a) or (b).

Regarding birational maps of a hyperkähler manifold of type $K3^{[2]}$, there is the following characterization by Mongardi ([51], Thm. 3.6):

Theorem 8.1. *Let $G \subseteq O(L)$ be a finite subgroup, and suppose that the coinvariant lattice L_G is negative-definite and contains no roots of norm -2. Then G is induced by a group of birational transformations of some hyperkähler manifold of type $K3^{[2]}$.*

Thus we have:

Corollary 8.2. *The conjugacy classes of groups $G \subseteq O(L)$ found in Section 7 arise as finite group of birational transformations of some hyperkähler manifold of type $K3^{[2]}$.*

We note that there are additional subgroups of $G \subseteq O(L)$ arising from birational transformations. The groups $G \subseteq O(L)$ in Section 7 have only been constructed from admissible subgroups $G \subseteq \mathrm{Co}_0$.

Based on work on the global Torelli theorem for hyperkähler manifolds, Mongardi gave the following criterion regarding symplectic automorphisms of type $K3^{[n]}$ ([55], Thm. 4.1) which we state here for the case $K3^{[2]}$:

Theorem 8.3. *Let $G \subseteq O(L)$ be a finite group. Then G is induced by a group of symplectic automorphisms for some hyperkähler manifold X of type $K3^{[2]}$ if, and only if, the following holds:*

- *L_G is negative-definite;*
- *L_G contains no numerical wall divisor.*

The numerical wall divisors for $K3^{[n]}$ have been discussed in [3] and [54] based on work in [4]. For manifolds of type $K3^{[2]}$ one has ([54], Prop. 2.12):

Theorem 8.4. *Let X be a hyperkähler manifold of type $K3^{[2]}$. Then the numerical wall divisors are the following vectors in the Picard sublattice of L:*

- *vectors v of norm $v^2 = -2$;*
- *vectors v of norm $v^2 = -10$ and $v/2 \in L^*$.*

Since L_G is a sublattice of the Picard lattice, we have to check for which of the groups $G \subseteq O(L)$ from Section 7 the lattice L_G contains no vectors v of norm -10 such that $v/2 \in L^*$.

Proposition 8.5. *If L is obtained from $L_G \oplus L^G$ by glueing construction (1) in Section 7, then L_G contains* no *numerical wall divisor. If L is obtained by glueing construction (2) then L_G contains a numerical wall divisor if, and only if, it contains a vector v of norm -10 such that $\overline{v/2} = w_{\gamma'}$ in A_{L_G}.*

Proof. By Theorem 8.4, we have to check if L_G contains a vector v of norm -10 such that $v/2 \in L^*$.

In case (1), L contains vectors of the form $(a, b) \in L_G \oplus L^G$ and the cosets $(x, \gamma(x))$, $x \in A_{L_G}$. Thus a vector $(v/2, 0) \in L_G^* \oplus (L^G)^*$ is contained in L^* if and only if $(v/2, L_G^*) \subset \mathbf{Z}$, i.e. $v/2 \in L_G$. But this is impossible since the norm of $v/2$ is $-5/2$ if v has norm -10.

In case (2), $A_{L_G} = \gamma'(A_{L^G}) \oplus \langle w_{\gamma'} \rangle$, $L/(L_G \oplus L^G) = \{(\gamma'(y), y) \mid y \in A_{L^G}\}$ so that $(w_{\gamma'}, 0)$ generates $A_L = L^*/L$. Thus a vector $(v/2, 0) \in L_G^* \oplus (L^G)^*$ is contained in L^* if and only if $(\overline{v/2}, \gamma'(A_{L^G})) = 0$, i.e. $\overline{v/2} \in \langle w_{\gamma'} \rangle$ in A_{L_G}. The case $v/2 \in L_G$ is again impossible. □

Theorem 8.6. *Let $G \subseteq \mathrm{Co}_0$ be an admissible subgroup. Then G is induced by a group of symplectic automorphisms for some hyperkähler manifold of type $K3^{[2]}$ such that $(\Lambda_G(-1), G) \cong (L_G, G)$ if, and only if, $\alpha(L_G) \geq 1$.*

Proof. If there exists a realization of (L_G, G) as coinvariant lattice by a group $G \subseteq O(L)$ using construction (1), then by Proposition 8.5 L_G contains *no* numerical wall divisor.

We will show that any realization of (L_G, G) using construction (2) contains a numerical wall divisor. With this in mind, for each of the eleven lattices L_G as in Theorem 7.4 we searched randomly for vectors v in the dual lattice L_G^* of norm $-5/2$ such that $2v \in L_G$. We always found such a v. As discussed in the last section, in each case there is a *unique* $O(L_G)$-orbit of norm $3/2$ vectors $w_{\gamma'}$ in A_{L_G}, i.e. we can assume that $\overline{v/2} = w_{\gamma'}$. Thus by Proposition 8.5, L_G contains a numerical wall divisor.

It therefore follows from Theorem 8.3 that G is induced by a group of symplectic automorphisms if, and only if, there is a realization of (L_G, G) using construction (1). By Theorem 7.4, these are exactly the admissible groups for which $\alpha(L_G) \geq 1$. □

Combining Theorem 8.6 with Theorem 6.1 we have:

Theorem 8.7. *The finite groups G arising as symplectic automorphisms of a hyperkähler manifold of type $K3^{[2]}$ are:*

(a) *subgroups of M_{23} with at least four orbits in the natural action on* 24 *elements,*
(b) *subgroups of* $3^{1+4}{:}2.2^2$ *and* $3^4{:}A_6$ *associated to the corresponding $\mathcal{S}$-lattices.*

At this juncture, we have established Theorem A and Theorem B.

Let X_i be hyperkähler manifolds of type $K3^{[2]}$ with finite groups of symplectic automorphism $G_i \subseteq \operatorname{Aut} X_i$ $(i = 1, 2)$. We say that (X_1, G_1) is *deformation equivalent* to (X_2, G_2) if there exists a flat family $\chi \longrightarrow B$ over a connected base B with hyperkähler manifolds of type $K3^{[2]}$ as fibers, together with a fiberwise symplectic action of a group G such that (X_1, G_1) and (X_2, G_2) are *isomorphic* to the action of G at certain fibers.

The following result follows from [53]; cf. the proof of Corollary 5.2 in [55].

Theorem 8.8. *Two hyperkähler manifolds X_i of type $K3^{[2]}$ with symplectic automorphism groups G_i $(i = 1, 2)$, are equivariantly deformation equivalent if, and only if, the associated group lattices (L_i, G_i) are isomorphic.*

Together with Theorem 8.6 and the enumeration results from the previous section, this proves Theorem C. Indeed, there are $198 - 13 = 185$ group lattices (L_G, G) that can be realized. For the 88 groups G with rk $L_G = 20$, there are 146 conjugacy classes in $O(L)$ (see the last column in Table 9), giving at least $185 - 88 + 146 = 243$ conjugacy classes in all. If Conjecture 7.2 holds, there are also exactly 243 deformation classes.

We finally note that Theorem D is equivalent to Theorem 8.6 by Theorem 8.3, Theorem 8.4 and Theorem 8.8.

Remarks. The admissible conjugacy classes can also be determined with the help of Theorem 8.3 and Theorem 8.4 instead of the method we used in

Section 6. the exact fixed-point configuration (described in Table 2) in the final section. Also, Theorem 8.3 became only available after main parts of the paper had been written. Theorem 8.6 was conjectured in [55].

Explicit examples of group actions. It is known that certain maximal groups G can be realized as symplectic actions on a $K3^{[2]}$ through induced actions on Fano schemes of lines of certain cubic fourfolds $S \subset \mathbf{C}P^5$, cf. [24], [50] Ch. 4 and the references therein. We collect these examples in Table 11. In addition, we consider two apparently new examples, with $G \cong 3^{1+4}{:}2.2^2$ and M_{10}. These are discussed in the next few paragraphs.

Concerning $G \cong 3^{1+4}{:}2.2^2$, first note that the cubic polynomial f in Table 11 is invariant under an obvious action of $H = (3^2.S_3 \times 3^2.S_3).\mathbf{Z}_2 \subseteq \mathrm{GL}(6, \mathbf{C})$ given by permutations and multiplication of the coordinates by cube roots of unity. In addition, f is invariant under the unimodular matrix

$$\alpha := \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & 1 & & & \\ 1 & 1 & 1 & & & \\ \omega^2 & \omega & 1 & & & \\ & & & \omega^2 & \omega & 1 \\ & & & \omega^2 & \omega^2 & \omega^2 \\ & & & \omega^2 & 1 & \omega \end{pmatrix} \qquad (\omega = e^{2\pi i/3}).$$

An element in $\mathrm{GL}(6, \mathbf{C})$ leaving f invariant acts symplectically on the Fano scheme of S if, and only if, it is in $\mathrm{SL}(6, \mathbf{C})$ (cf. [24]). Calculations show that the projections of $H \cap \mathrm{SL}(6, \mathbf{C})$ and α into $\mathrm{PSL}(6, \mathbf{C})$ generate a group isomorphic to G. Finally, it can be verified that the resulting cubic fourfold S is smooth by solving the equations

$$f = \partial f/\partial x_0 = \cdots = \partial f/\partial x_5 = 0,$$

confirming that 0 is an isolated singularity of f.

For the case $G \cong M_{10}$, we start with the containment $3.A_6 \subseteq \mathrm{SL}(6, \mathbf{C})$ given by the generators

$$\begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}, \qquad \begin{pmatrix} & 1 & & & & \\ & & \omega & & & \\ & & & 1 & & \\ \omega^2 & & & & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}$$

Table 11: Fano schemes of cubic fourfolds.

Group	Equation for S
$L_2(11)$	$x_0^3 + x_1^2x_5 + x_2^2x_4 + x_3^2x_2 + x_4^2x_1 + x_5^2x_3$
A_7	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^3$
$(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_0 + x_4^3 + x_5^3$
$3^4{:}A_6$	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$
$3^{1+4}{:}2.2^2$	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + 3(i - 2e^{\pi i/6} - 1)\,(x_0x_1x_2 + x_3x_4x_5)$
M_{10}	$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + \lambda \cdot \tilde{\sigma}_3(x_0, \ldots, x_5)$.

from the Atlas of finite group representations [61]. In addition, we choose the unimodular matrix

$$\beta := \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \omega & \omega^2 & \omega & 1 & \omega \\ \omega^2 & 1 & 1 & \omega & \omega & \omega \\ \omega & 1 & \omega^2 & \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega^2 & \omega & \omega & 1 & \omega^2 \\ 1 & \omega^2 & 1 & \omega & \omega^2 & 1 \\ \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega \end{pmatrix}.$$

which normalizes $3.A_6$. We found the following cubic polynomial f invariant under the action of $3.A_6$ and β:

$$\begin{aligned} f \;=\; & (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3) + \frac{1}{5}(-3\zeta^7 - 3\zeta^5 + 3\zeta^4 - 3\zeta^3 + 6\zeta - 3) \\ & \times \Big[x_1x_2x_3 + x_1x_2x_4 + (\zeta^4 - 1)x_1x_2x_5 + x_1x_2x_6 + (\zeta^4 - 1)x_1x_3x_4 \\ & + x_1x_3x_5 + x_1x_3x_6 + (\zeta^4 - 1)x_1x_4x_5 - \zeta^4x_1x_4x_6 - \zeta^4x_1x_5x_6 \\ & + (\zeta^4 - 1)x_2x_3x_4 + (\zeta^4 - 1)x_2x_3x_5 - \zeta^4x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 \\ & - \zeta^4x_2x_5x_6 + x_3x_4x_5 - \zeta^4x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6 \Big] \end{aligned}$$

with $\zeta = e^{2\pi i/24}$. The projection of $3.A_6\langle\beta\rangle$ into $\mathrm{PSL}(6, \mathbf{C})$ is isomorphic to M_{10}. Again, we verified that S is smooth.

It would be of interest to find explicit realizations for the nine remaining maximal groups of Theorem A not treated here.

9. Connections with Mathieu Moonshine

Suppose that $g \in M_{24}$ belongs to one of the 11 admissible classes. In this section we compare the equivariant complex elliptic genus $\chi_y(g;q,\mathcal{L}X)$ of a hyperkähler manifold X of type $K3^{[2]}$ with the prediction of Mathieu Moonshine applied to the second quantized complex elliptic genus of a $K3$ surface. See Theorem 9.3 below for a precise statement. This connection was one of our main motivations for studying symplectic automorphisms of $K3^{[2]}$.

9.1. Mathieu moonshine

Recall [34] that a *complex genus* in the sense of Hirzebruch is a graded ring homomorphism from the complex bordism ring into some other graded ring R. For a d-dimensional complex manifold X and a holomorphic vector bundle E on X, the χ_y-genus twisted by E is

$$\chi_y(X,E) := \sum_{p=0}^{d} \chi(X, \Lambda^p T^* \otimes E)\, y^p,$$

where $\chi(X,E) = \sum_{q=0}^{d}(-1)^q \dim H^q(X,\mathcal{O}(E))$.

The complex elliptic genus can be formally defined as the S^1-equivariant χ_y-genus of the loop space of a manifold:

$$\chi_y(q,\mathcal{L}X) := (-y)^{-d/2}\chi_y\big(X, \bigotimes_{n=1}^{\infty}\Lambda_{yq^n}T^* \otimes \bigotimes_{n=1}^{\infty}\Lambda_{y^{-1}q^n}T \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T^*\oplus T)\big)$$

taking values in $\mathbf{Q}[y^{1/2},y^{-1/2}][[q]]$. Here, we let $S_tE = \bigoplus_{i=0}^{\infty} S^iE\cdot t^i$ and $\Lambda_tE = \bigoplus_{i=0}^{\infty}\Lambda^iE\cdot t^i$. If the first Chern class *vanishes*, this is the Fourier expansion of a Jacobi form of weight 0 and index equal to one half of the complex dimension of X [37]. For automorphisms g of X one has the corresponding equivariant elliptic genus $\chi_y(g;q,\mathcal{L}X)$ where we let $\chi(g;X,E) = \sum_{q=0}^{d}(-1)^q\,\mathrm{Tr}(g|H^q(X,\mathcal{O}(E)))$.

For a K3 surface Y, the elliptic genus $\chi_{-y}(q,\mathcal{L}Y)$ is the Jacobi form

$$2\,\phi_{0,1}(z;\tau) = 8\left(\left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2 + \left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right) \qquad (6)$$

of weight 0 and index 1.

Eguchi, Ooguri and Tachikawa observed [23] that the decomposition of $2\,\phi_{0,1}(z;\tau)$ into characters of an $\mathcal{N}=4$ super algebra at central charge $c=6$ has multiplicities which are sums of dimensions of irreducible representations of M_{24}.

The observation of Eguchi, Ooguri and Tachikawa suggests the existence of a graded M_{24}-module $K = \bigoplus_{n=0}^{\infty} K_n\, q^{n-1/8}$ whose graded character is given by the decomposition of the elliptic genus into characters of the $\mathcal{N}=4$ super algebra. Subsequently, analogues of McKay-Thompson series in monstrous moonshine were proposed in several works [12, 21, 26, 27, 13]. The corresponding McKay-Thompson series for $g \in M_{24}$ are of the form

$$\Sigma_g(q) \;=\; q^{-1/8} \sum_{n=0}^{\infty} \mathrm{Tr}(g|K_n)\, q^n \;=\; \frac{e(g)}{24}\,\Sigma(q) - \frac{f_g(q)}{\eta(q)^3}. \tag{7}$$

Here, $\Sigma = \Sigma_e$ is the graded dimension of K (an explicit mock modular form), $e(g)$ is the character of the 24-dimensional permutation representation of M_{24}, f_g is a certain explicit modular form of weight 2 on a congruence subgroup $\Gamma_0(N_g)$, and η is the Dedekind eta function. Gannon has shown [25] that these McKay-Thompson series indeed determine a graded M_{24}-module.

In [17], Creutzig and the first author have shown that for symplectic automorphisms of a K3 surface, the McKay-Thompson series of Mathieu Moonshine determines the equivariant elliptic genus:

Theorem 9.1. *Let g be a finite symplectic automorphism of a K3 surface Y. Then the equivariant elliptic genus and the character determined by the McKay-Thompson series of Mathieu moonshine agree, i.e. one has*

$$\chi_{-y}(g;q,\mathcal{L}Y) = \frac{e(g)}{12}\,\phi_{0,1} + f_g\,\phi_{-2,1},$$

where g is considered on the right-hand-side as an element in M_{24}.

Here, $\phi_{-2,1}$ is the Jacobi form

$$\phi_{-2,1} = y^{-1}(1-y)^2 \prod_{n=1}^{\infty} \frac{(1-q^n y)^2(1-q^n y^{-1})^2}{(1-q^n)^4} \tag{8}$$

of weight -2 and index 1.

9.2. The second quantized elliptic genus and its relation to Hilbert schemes of K3

The elliptic genus of an orbifold X/G for a finite group G acting on a complex manifold X is defined by

$$\chi_y(q,\mathcal{L}(X/G)) := \frac{1}{|G|}\sum_{\substack{g,h\in G\\ [g,h]=1}} \chi_y(g;q,\mathcal{L}_hX),$$

where $\mathcal{L}_h(X)$ is the h-twisted loop space and $\chi_y(g;q,\mathcal{L}_hX)$ is determined by formally applying the equivariant Atiyah-Singer index theorem.

For a space X, let $\exp(pX) := \sum_{n=0}^{\infty} X^n/S_n \cdot p^n$ be the generating series of its symmetric powers. It follows from calculations by Verlinde, Verlinde, Dijkgraaf and Moore [19] that the second quantized elliptic genus $\chi_{-y}(q,\mathcal{L}\exp(pX))$ is, up to an automorphic correction factor, the Borcherds lift of $\chi_{-y}(q,\mathcal{L}X)$. Explicitly one has

$$\chi_{-y}(q,\mathcal{L}\exp(pX)) = \prod_{n>0,\, m\geq 0,\, \ell} \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}c(4nm-\ell^2)\,(p^nq^my^\ell)^k\right),$$

where $\chi_{-y}(q,\mathcal{L}X) = \sum_{n,\,\ell\in\mathbf{Z}} c(4n-\ell^2)\,q^ny^\ell$.

For K3 surfaces, there is the following connection between the orbifold elliptic genus of symmetric powers and the Hilbert schemes conjectured by [19].

Theorem 9.2 (Borisov and Libgober [8])**.** *Let Y be a K3 surface. Then*

$$\chi_y(q,\mathcal{L}\exp(pY)) = \sum_{n=0}^{\infty}\chi_y(q,\mathcal{L}Y^{[n]})\,p^n.$$

If g acts on X then there is an induced action of g on X^n/S_n since the diagonal action of g on X^n commutes with the S_n-action. There is then the following equivariant generalization (cf. [38, 12, 22]):

$$\chi_{-y}(g;q,\mathcal{L}\exp(pX)) = \prod_{n>0,\, m\geq 0,\, \ell} \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}c_{g^k}(4nm-\ell^2)\,(p^nq^my^\ell)^k\right),$$

where $\chi_{-y}(g;q,\mathcal{L}X) = \sum_{n,\,\ell\in\mathbf{Z}} c_g(4n-\ell^2)q^ny^\ell$.

Alternatively, we may take this directly as the definition of the equivariant second quantized elliptic genus.

For a K3 surface Y, we can use the McKay-Thompson series of Mathieu Moonshine to define $\chi_{-y}(g;q,\mathcal{L}Y)$ for all $g\in M_{24}$ by the formula in Theorem 9.1. Then the previous formula allows us to also define the equivariant second quantized elliptic genus for $g\in M_{24}$. We prove:

Theorem 9.3. *Let $g\in M_{24}$ be a finite symplectic automorphism of order 1, 3, 4, 5, 7, 8 or 11 acting on a hyperkähler manifold X of type $K3^{[2]}$. Then the equivariant elliptic genus $\chi_{-y}(g;q,\mathcal{L}X)$ and the coefficient of p^2 in the equivariant second quantized elliptic genus determined by the McKay-Thompson series of Mathieu moonshine agree.*

There are 11 classes $g\in M_{24}$ acting symplectically on a manifold of type $K3^{[2]}$. If $g=1$, the theorem follows from Theorem 9.2. For g acting by symplectic automorphisms on a K3 surface, there is probably an equivariant generalization, however this currently seems to be unknown. This would not, in any case, apply to the three classes of order 11, 14 and 15 which do not correspond to symplectic automorphisms of K3. We have verified the result also for the four cases 2, 6, 14 or 15 for the first coefficients (up to the order 4 in q).

To prove Theorem 9.3, we start by showing that both the equivariant elliptic genus and the p^2 coefficient of the equivariant second quantized elliptic genus are weak Jacobi forms.

Proposition 9.4. *Let $N=\operatorname{ord}(g)$ and assume $N>2$. Then $\chi_{-y}(g;q,\mathcal{L}X)$ is a weak Jacobi form on $\Gamma_0(N)$ of weight 0 and index 2.*

Proof. The proof is similar to that of Lemma 4.2 in [17]. Because g has order ≥ 3, it follows from Table 2 that X^g consists only of isolated fixed-points $\{p_i\}$. Set

$$\varphi(u;\,\tau):=\vartheta_1(u;\,\tau)\eta(\tau)^{-3}=-i(y^{1/2}-y^{-1/2})\prod_{n=1}^{\infty}(1-yq^n)(1-y^{-1}q^n)(1-q^n)^{-2}.$$

The fixed-point formula gives (cf. also equation (1) in the case of $\chi_{-y}(g;X)$):

$$\chi_{-y}(g;q,\mathcal{L}X)=\sum_{p_i}\frac{\varphi\big(u+\frac{n_i}{N};\,\tau\big)\varphi\big(u-\frac{n_i}{N};\,\tau\big)}{\varphi\big(\frac{n_i}{N};\,\tau\big)\varphi\big(-\frac{n_i}{N};\,\tau\big)}\cdot\frac{\varphi\big(u+\frac{m_i}{N};\,\tau\big)\varphi\big(u-\frac{m_i}{N};\,\tau\big)}{\varphi\big(\frac{m_i}{N};\,\tau\big)\varphi\big(-\frac{m_i}{N};\,\tau\big)},$$

where the pair $\{n_i, m_i\}$ for a given p_i can be read off from Table 2. Namely, $\{\zeta^{n_i}, \zeta^{-n_i}, \zeta^{m_i}, \zeta^{-m_i}\}$ with ζ a primitive N-th root of unity are the eigenvalues of g acting at the tangent space of p_i.

In order to check the Jacobi transformation property, we consider the action of $(\mathbf{Z}/N\mathbf{Z})^*$ (see the proof of Lemma 4.2 in [17]). The fixed-point of type $\{\zeta^{n_i}, \zeta^{m_i}\}$ is mapped by $d \in (\mathbf{Z}/N\mathbf{Z})^*$ to $\{(\zeta^{n_i})^d, (\zeta^{m_i})^d\}$, and it is clear from Table 2 that this induces a permutation action of $(\mathbf{Z}/N\mathbf{Z})^*$ on $\{p_i\}$. Therefore, the above expression for $\chi_{-y}(g; q, \mathcal{L}X)$ is left invariant. □

Proposition 9.5. *Let* $N = \mathrm{ord}(g)$ *and assume* $N \neq 6,\ 14,\ 15$. *Then the coefficient of* p^2 *in the equivariant second quantized elliptic genus* $\chi_{-y}(g; q, \mathcal{L}\exp(pX))$ *is a weak Jacobi form for* $\Gamma_0(N)$ *of weight* 0 *and index* 2.

Proof. According to M. Raum [59] (Theorem 1.2),

$$\Phi_g := pqy \prod_{(n,m,\ell)>0} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} c_{g^k}(4nm-\ell^2)\,(p^n q^m y^\ell)^k\right)$$

is a Siegel modular form of degree 2 of a certain weight k_g on the subgroup $\Gamma_0^{(2)}(N) \subseteq Sp(4, \mathbf{Z})$. Here, the product runs over triples of integers (n, m, ℓ) and $(n, m, \ell) > 0$ means that $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $\ell < 0$. This implies that the coefficient $\psi_{g,n}$ of p^n in the Fourier-Jacobi expansion

$$\Phi_g^{-1} = \sum_{n=0}^{\infty} \psi_{g,n}\, p^n$$

is a Jacobi form of weight $-k_g$ and index m for $\Gamma_0(N)$.

To compare Φ_g^{-1} with the second quantized elliptic genus, we write

$$\Phi_g^{-1} = \chi_{-y}(g; q, \mathcal{L}\exp(pY)) \cdot \alpha_g^{-1}$$

so that

$$\begin{aligned}\alpha_g \;=\; & \prod_{(m,\ell)\geq 0} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} c_{g^k}(-\ell^2)\,(q^m y^\ell)^k\right) \\ & \prod_{(m,\ell)\geq 0} \prod_{d|N} \left(1-(q^m y^\ell)^d\right)^{d^{-1}\sum_{e|d}\mu(d/e)c_{g^e}(-\ell^2)}.\end{aligned}$$

The only contributions come from terms $c_h(-\ell^2)$ for $\ell \in \{0, \pm 1\}$, which are determined by the Hodge structure of the K3 surface. One has $c_h(-1) = 2$ and $c_h(0) = e(h) - 4$ where $e(h)$ is the number of fixed-points of the element $h \in M_{24}$ acting on 24 elements.

We claim that

$$\alpha_g = p\,\eta_g\,\phi_{-2,1}$$

(cf. [12], Section 4) where η_g is the usual twisted eta-product for g. (If g has cycle shape $a_1^{b_1} a_2^{b_2} \ldots$, $\eta_g(q) := \eta(q^{a_1})^{b_1} \eta(q^{a_2})^{b_2} \ldots$.) Indeed, the contribution for $m > 0$, $\ell = \pm 1$ and the constant term -4 of $c_h(0)$ gives the product formula for $\phi_{-2,1}$ as in equation (8) up to the leading y-factors. The contribution for $m > 0$, $\ell = 0$ is the twisted eta-product η_g. Note that $d^{-1} \sum_{e|d} \mu(d/e) e(g^e)$ counts the number of d-cycles of g. The contribution for $m = 0$ and $\ell = -1$ gives the missing y-factors, and the factor p comes from the factor qyp in front of Φ_g.

Since the weight of η_g is $k_g + 2$ [47] and $\phi_{-2,1}$ has weight -2 and index 1, it follows that the coefficient $\psi_{g,n-1} \cdot \alpha_g$ of p^n in $\chi_{-y}(g; q, \mathcal{L}\exp(pY))$ is a Jacobi form of weight 0 and index n for $\Gamma_0(N)$. ☐

Proof of Theorem 9.3. Let $J_{k,m}(\Gamma_0(N))$ be the space of weak holomorphic Jacobi forms for $\Gamma_0(N)$ of weight k and index m. It follows from [2] Prop. 6.1 that

$$J_{0,2}(\Gamma_0(N)) \cong M_0(\Gamma_0(N)) \times M_2(\Gamma_0(N)) \times M_4(\Gamma_0(N)),$$

where $M_l(\Gamma_0(N))$ is the space of holomorphic modular forms of weight l for $\Gamma_0(N)$. The isomorphism is defined by

$$(f_0, f_2, f_4) \mapsto f_0\,\phi_{0,1}^2 + f_2\,\phi_{0,1}\phi_{-2,1} + f_4\,\phi_{-2,1}^2$$

for $(f_0, f_2, f_4) \in M_0(\Gamma_0(N)) \times M_2(\Gamma_0(N)) \times M_4(\Gamma_0(N))$.

By using explicit bases for the spaces of modular forms of weights 0, 2 and 4 on $\Gamma_0(N)$, the result follows from Proposition 9.4 and Proposition 9.5 by checking the equality for sufficiently many coefficients. This was carried through using Magma. ☐

Our calculation shows that Theorem 9.3 will also hold for the remaining four classes g of order 2, 6, 14 and 15, once the Jacobi form property has been verified. See [58] for work in this direction.

Remark. In general, a finite symplectic automorphism g of a hyperkähler manifold X of type $K3^{[n]}$ defines a conjugacy class $[g]$ in Co_0. We conjecture that for g in M_{24} the corresponding equivariant elliptic genus $\chi_{-y}(g;q,\mathcal{L}X)$ equals the coefficient of p^n in the equivariant second quantized elliptic genus determined by the corresponding McKay-Thompson series of Mathieu moonshine. There may even be a generalization of the McKay-Thompson series of Mathieu moonshine for g *not* in M_{24}.

Appendix. Tables of admissible groups and their lattices

The appendix contains three tables, listing all admissible subgroups of Co_0 and information about the coinvariant lattices L_G.

Table 12 lists the 198 classes of admissible subgroups G of Co_0 sorted by size. The entry of the first two columns is clear. The third column lists the abstract isomorphism type of the group either by name or the number of the small group library [6]. The fourth column has an entry "K3" if (L_G, G) is one of the 82 group lattices of Table 10.2 of [33], arising from a symplectic action on a K3 surface; an "M" if the group is inside M_{23} and realized; an "S" if it is only realized by an $\mathcal{S}$-lattice group; and "$-$" if the group is not realized. The fifth column records the dimension of L_G, the sixth column gives the number of the unique largest group with the same lattice L_G, and the last column lists the numbers of all admissible groups containing G as a maximal subgroup.

Table 13 lists the 69 isomorphism types of lattices L_G for admissible groups G. The second column gives the number (from Table 12) of the unique largest group with the same lattice L_G, while the next three columns are self explanatory. Column Det gives the determinant of L_G, and column A_{L_G} the structure of the discriminant group. In the Genus column we provide the genus symbol of L_G as defined in [16] (we omit the signature since it is directly determined by the rank). The last column provides information about the realization: if L_G is one of the 41 lattices of the table in Section 10.3 of [33], the entry is "K3 #n" where n is the K3 group number; if L_G is one of the 13 lattices belonging to the groups in Theorem 5.1, the entry is "max #n" where n is the n-th group of that theorem; if L_G belongs to (maximal) $\mathcal{S}$-groups, the entry is "(max) $\mathcal{S}$-lattice"; if G can be realized in $O(L)$ the entry is "$O(L)$"; in the remaining cases there is a "–".

Finally, in Table 14 we list all lattices L_G for which there are several admissible groups with the same lattice L_G. The first column gives the number of the lattice as in Table 13. The second column gives the number of groups with the same lattice L_G. The last column lists all such groups by their number as in Table 12.

Table 12: Conjugacy classes of admissible groups.

No.	order	G	Type	dim	fix	minimal overgroups
1	1	#1	K3	0	1	{ 2, 3, 4, 7, 11, 23 }
2	2	#1	K3	8	2	{ 5, 6, 8, 9, 10, 22, 31 }
3	3	#1	K3	12	3	{ 8, 9, 18, 19, 21, 25, 32, 54 }
4	3	#1	S	18	170	{ 10, 18, 20, 21 }
5	4	#1	K3	14	5	{ 12, 13, 14, 15, 17, 26, 28, 30, 53, 79 }
6	4	#2	K3	12	6	{ 13, 14, 16, 24, 25, 27, 29 }
7	5	#1	K3	16	22	{ 22, 32, 99, 114, 169 }
8	6	#2	K3	16	24	{ 24, 27, 29, 30, 48, 51, 52, 55, 57, 58, 67, 83 }
9	6	#1	K3	14	9	{ 24, 48, 49, 50, 51, 56, 102 }
10	6	#2	S	18	170	{ 26, 28, 49, 52 }
11	7	#1	K3	18	54	{ 31, 54, 100 }
12	8	#1	K3	18	40	{ 33, 40, 44, 45, 109 }
13	8	#3	K3	15	13	{ 40, 46, 47, 56, 60, 111 }
14	8	#2	K3	16	47	{ 33, 34, 36, 38, 39, 41, 42, 43, 44, 46, 47 }
15	8	#4	K3	17	68	{ 42, 43, 45, 46, 57 }
16	8	#5	K3	14	16	{ 35, 36, 37, 41, 47, 58, 100 }
17	8	#4	K3	17	17	{ 39, 40, 45, 55, 59, 110 }
18	9	#2	S	18	170	{ 49, 61, 62, 63, 64, 65, 66 }
19	9	#2	K3	16	50	{ 48, 50, 62, 66, 78 }
20	9	#1	S	20	198	{ 63, 64, 65 }
21	9	#2	S	20	198	{ 51, 52, 62 }
22	10	#1	K3	16	22	{ 53, 67, 102, 136, 181 }
23	11	#1	M	20	177	{ 99 }
24	12	#4	K3	16	24	{ 60, 80, 81, 86, 88, 132, 177 }
25	12	#3	K3	16	25	{ 56, 58, 78, 85, 87, 102, 160 }
26	12	#1	S	19	185	{ 59, 82, 130 }
27	12	#5	–	18	27	{ 90, 93 }
28	12	#2	S	20	191	{ 59, 127, 131 }
29	12	#5	K3	18	112	{ 60, 78, 80, 89, 92 }
30	12	#1	K3	18	112	{ 60, 82, 84, 91, 94, 101, 129 }
31	14	#2	M	20	162	{ 83 }
32	15	#1	M	20	164	{ 67 }
33	16	#6	K3	19	168	{ 69, 70, 71, 75 }
34	16	#2	–	18	34	{}

Table 12: Conjugacy classes of admissible groups.

No.	order	G	Type	dim	fix	minimal overgroups
35	16	#14	–	15	35	{ 93 }
36	16	#10	–	17	36	{}
37	16	#14	K3	15	37	{ 77, 85, 92, 114 }
38	16	#2	K3	18	150	{ 69, 72, 87 }
39	16	#12	K3	18	150	{ 71, 72, 73, 89 }
40	16	#8	K3	18	40	{ 71, 86, 135, 178 }
41	16	#3	K3	17	77	{ 72, 76, 77, 84 }
42	16	#12	–	18	42	{ 90 }
43	16	#12	–	18	74	{ 70, 74 }
44	16	#6	–	19	44	{ 73 }
45	16	#9	K3	19	168	{ 71, 91, 94 }
46	16	#13	K3	17	68	{ 68, 69, 71, 74 }
47	16	#11	K3	16	47	{ 68, 72, 75, 76, 77, 88 }
48	18	#3	K3	18	81	{ 81, 95, 98, 142, 156 }
49	18	#3	S	18	170	{ 95, 96, 97 }
50	18	#4	K3	16	50	{ 79, 81, 96, 98, 112 }
51	18	#3	S	20	198	{ 80, 95, 98 }
52	18	#5	S	20	198	{ 80, 82, 95 }
53	20	#3	K3	18	53	{ 101, 132, 158, 178 }
54	21	#1	K3	18	54	{ 83, 140, 141 }
55	24	#3	K3	19	86	{ 86, 89, 94 }
56	24	#12	K3	17	56	{ 88, 112, 120, 124, 132, 140, 163, 176 }
57	24	#3	K3	19	143	{ 90, 91, 122 }
58	24	#13	K3	18	88	{ 84, 88, 92, 93, 122, 125, 141 }
59	24	#4	S	20	191	{ 151 }
60	24	#8	K3	18	112	{ 112, 113, 121, 123, 152 }
61	27	#3	S	18	170	{ 97, 116, 117, 119 }
62	27	#5	S	20	198	{ 95, 98, 117, 118 }
63	27	#4	S	20	198	{ 115, 116 }
64	27	#4	S	20	198	{ 115, 117 }
65	27	#2	S	20	198	{ 115, 116 }
66	27	#5	S	18	170	{ 96, 115, 118, 119 }
67	30	#2	M	20	164	{ 101, 142 }
68	32	#49	K3	17	68	{ 106, 108, 122 }
69	32	#11	K3	19	168	{ 106, 124 }
70	32	#8	–	20	70	{}
71	32	#44	K3	19	168	{ 106, 123 }
72	32	#31	K3	18	150	{ 103, 105, 106, 107 }
73	32	#8	–	20	107	{ 107 }
74	32	#50	–	18	74	{}
75	32	#7	K3	19	168	{ 104, 106 }
76	32	#6	K3	18	108	{ 104, 105, 108 }
77	32	#27	K3	17	77	{ 103, 104, 108, 120, 121, 125, 136 }
78	36	#11	K3	18	112	{ 112, 134, 142, 184 }
79	36	#9	K3	18	79	{ 109, 110, 111, 129, 130, 131, 163, 172 }
80	36	#12	S	20	198	{ 113, 126 }
81	36	#10	K3	18	81	{ 111, 126, 128, 164, 173 }

Table 12: Conjugacy classes of admissible groups.

No.	order	G	Type	dim	fix	minimal overgroups
82	36	#7	S	20	198	{ 113, 161 }
83	42	#2	M	20	162	{ 162 }
84	48	#30	K3	19	157	{ 121, 147, 149 }
85	48	#50	K3	17	85	{ 120, 125, 134, 150 }
86	48	#29	K3	19	86	{ 123 }
87	48	#3	K3	18	150	{ 124, 150 }
88	48	#48	K3	18	88	{ 121, 143, 148, 162, 179 }
89	48	#32	M	20	187	{ 123, 144 }
90	48	#32	–	20	90	{}
91	48	#28	M	20	196	{ 149 }
92	48	#49	K3	19	157	{ 121, 134, 144, 145, 146 }
93	48	#49	–	19	93	{}
94	48	#28	M	20	187	{ 123 }
95	54	#12	S	20	198	{ 126, 137, 139 }
96	54	#13	S	18	170	{ 130, 131, 138, 139 }
97	54	#8	S	18	170	{ 127, 137, 138 }
98	54	#13	S	20	198	{ 126, 128, 129, 139 }
99	55	#1	M	20	177	{ 177 }
100	56	#11	M	20	188	{ 141 }
101	60	#7	M	20	164	{ 164 }
102	60	#5	K3	18	102	{ 132, 142, 163, 177, 182, 183, 195 }
103	64	#242	K3	18	150	{ 133, 144, 146, 150 }
104	64	#32	K3	19	168	{ 133, 147, 158, 172 }
105	64	#35	K3	19	168	{ 133 }
106	64	#136	K3	19	168	{ 133, 149 }
107	64	#36	–	20	107	{}
108	64	#138	K3	18	108	{ 133, 143, 145, 148 }
109	72	#39	M	20	135	{ 135 }
110	72	#41	K3	19	110	{ 135, 151, 178, 197 }
111	72	#40	K3	19	111	{ 135, 152, 179 }
112	72	#43	K3	18	112	{ 157, 164, 192, 193 }
113	72	#22	S	20	198	{ 175 }
114	80	#49	K3	19	183	{ 136 }
115	81	#13	S	20	198	{ 153, 155 }
116	81	#8	S	20	198	{ 153 }
117	81	#7	S	20	198	{ 137, 155, 160 }
118	81	#15	S	20	198	{ 139, 155, 169 }
119	81	#12	S	18	170	{ 138, 153, 154, 155 }
120	96	#227	K3	18	120	{ 148, 156, 157, 168, 183 }
121	96	#195	K3	19	157	{ 157, 165, 166, 167, 182 }
122	96	#204	K3	19	143	{ 143, 145, 149 }
123	96	#190	M	20	187	{ 167 }
124	96	#64	K3	19	168	{ 168 }
125	96	#70	K3	19	148	{ 146, 147, 148, 156 }
126	108	#38	S	20	198	{ 152, 159 }
127	108	#15	S	20	191	{ 186 }
128	108	#40	S	20	198	{ 152, 159, 160 }

Table 12: Conjugacy classes of admissible groups.

No.	order	G	Type	dim	fix	minimal overgroups
129	108	#37	S	20	198	{ 152, 161 }
130	108	#37	S	19	185	{ 151, 161, 185 }
131	108	#36	S	20	191	{ 151, 186 }
132	120	#34	K3	19	132	{ 164, 179, 190, 193 }
133	128	#931	K3	19	168	{ 165, 166, 167, 168 }
134	144	#184	K3	19	157	{ 156, 157, 174 }
135	144	$\mathbf{Z}_3^2{:}QD_{16}$	M	20	135	{}
136	160	#234	K3	19	183	{ 158, 182, 183 }
137	162	#10	S	20	198	{ 171, 176 }
138	162	#46	S	18	170	{ 170, 171 }
139	162	#52	S	20	198	{ 159, 161, 171, 181 }
140	168	#42	K3	19	140	{ 162, 188, 193, 197 }
141	168	#43	M	20	188	{ 188 }
142	180	#19	M	20	164	{ 164 }
143	192	#1493	K3	19	143	{ 165, 188 }
144	192	#1024	M	20	187	{ 167, 174 }
145	192	#201	M	20	196	{ 165, 182 }
146	192	#1009	M	20	187	{ 166, 174 }
147	192	#184	M	20	187	{ 166 }
148	192	#955	K3	19	148	{ 166, 173, 188, 190 }
149	192	#1492	M	20	196	{ 165 }
150	192	#1023	K3	18	150	{ 168, 174, 183 }
151	216	#161	S	20	191	{ 191 }
152	216	#158	S	20	198	{ 175, 176 }
153	243	#57	S	20	198	{ 180 }
154	243	#65	S	18	170	{ 170, 180 }
155	243	#51	S	20	198	{ 171, 180, 184 }
156	288	#1025	M	20	173	{ 173 }
157	288	#1026	K3	19	157	{ 172, 173, 187 }
158	320	#1635	M	20	190	{ 190 }
159	324	#167	S	20	198	{ 175, 184 }
160	324	#160	S	20	198	{ 176, 184 }
161	324	#163	S	20	198	{ 175, 194 }
162	336	$\mathbf{Z}_2 \times L_2(7)$	M	20	162	{}
163	360	#118	K3	19	163	{ 178, 179, 193, 196, 197, 198 }
164	360	$(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$	M	20	164	{}
165	384	#5603	M	20	196	{ 196 }
166	384	#5678	M	20	187	{ 187 }
167	384	#18133	M	20	187	{ 187 }
168	384	#18135	K3	19	168	{ 187, 190, 196 }
169	405	#15	S	20	198	{ 181 }
170	486	#249	S	18	170	{ 185, 186, 189 }
171	486	#166	S	20	198	{ 189, 192, 195 }
172	576	#8652	M	20	196	{ 196 }
173	576	$\mathbf{Z}_2^4{:}(S_3 \times S_3)$	M	20	173	{}
174	576	#5129	M	20	187	{ 187 }
175	648	#722	S	20	198	{ 192 }

Table 12: Conjugacy classes of admissible groups.

No.	order	G	Type	dim	fix	minimal overgroups
176	648	#704	S	20	198	{ 192 }
177	660	$L_2(11)$	M	20	177	{}
178	720	M_{10}	M	20	178	{}
179	720	S_6	M	20	179	{}
180	729	#321	S	20	198	{ 189 }
181	810	#101	S	20	198	{ 195 }
182	960	#11358	M	20	196	{ 196 }
183	960	#11357	K3	19	183	{ 190, 196, 197 }
184	972	#877	S	20	198	{ 192, 195 }
185	972	#776	S	19	185	{ 191, 194 }
186	972	#777	S	20	191	{ 191 }
187	1152	$2^6(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$	M	20	187	{}
188	1344	$\mathbf{Z}_2^3{:}L_2(7)$	M	20	188	{}
189	1458	#1229	S	20	198	{ 194 }
190	1920	$\mathbf{Z}_2^4{:}S_5$	M	20	190	{}
191	1944	$3^{1+4}{:}2.2^2$	S	20	191	{}
192	1944	#3877	S	20	198	{ 198 }
193	2520	A_7	M	20	193	{}
194	2916	$3^4{:}(3^2{:}\mathbf{Z}_4)$	S	20	198	{ 198 }
195	4860	$3^4{:}A_5$	S	20	198	{ 198 }
196	5760	$\mathbf{Z}_2^4{:}A_6$	M	20	196	{}
197	20160	$L_3(4)$	M	20	197	{}
198	29160	$3^4{:}A_6$	S	20	198	{}

Table 13: Isometry types of coinvariant lattices L_G.

No.	G-No.	$\lvert G\rvert$	Symbol	Rank	Det	A_{L_G}	Genus	Type
1	1	1	1	0	1	1	1	K3 # 0
2	2	2	$\mathbf{Z}_2$	8	256	2^8	2_{II}^{+8}	K3 # 1
3	3	3	$\mathbf{Z}_3$	12	729	3^6	3^{+6}	K3 # 2
4	5	4	$\mathbf{Z}_4$	14	1024	$2^2 4^4$	$2_2^{+2} 4_{II}^{+4}$	K3 # 4
5	6	4	$\mathbf{Z}_2^2$	12	1024	$2^6 4^2$	$2_{II}^{-6} 4_{II}^{-2}$	K3 # 3
6	9	6	$\mathbf{Z}_6$	14	972	$3^3 6^2$	$2_{II}^{-2} 3^{+5}$	K3 # 6
7	13	8	D_8	15	-1024	4^5	4_1^{+5}	K3 # 15
8	16	8	$\mathbf{Z}_2^3$	14	1024	$2^6 4^2$	$2_{II}^{+6} 4_2^{+2}$	K3 # 14
9	17	8	Q_8	17	-512	$2^3 8^2$	$2_3^{+3} 8_{II}^{-2}$	K3 # 12
10	22	10	D_{10}	16	625	5^4	5^{+4}	K3 # 16
11	24	12	D_{12}	16	1296	6^4	$2_{II}^{+4} 3^{+4}$	K3 # 18
12	25	12	A_4	16	576	$2^2 12^2$	$2_{II}^{-2} 4_{II}^{-2} 3^{+2}$	K3 # 17
13	27	12	$\mathbf{Z}_2 \times \mathbf{Z}_6$	18	1728	$2^3 6^3$	$2_{II}^{-6} 3^{+3}$	–
14	34	16	$\mathbf{Z}_4^2$	18	1024	$2^2 4^4$	$2_6^{+2} 4_{II}^{+4}$	$O(L)$
15	35	16	$\mathbf{Z}_2^4$	15	-1024	$2^8 4^1$	$2_{II}^{+8} 4_1^{+1}$	–
16	36	16	$\mathbf{Z}_2^2 \times \mathbf{Z}_4$	17	-1024	$2^4 4^3$	$2_{II}^{+4} 4_7^{+3}$	–
17	37	16	$\mathbf{Z}_2^4$	15	-512	$2^6 8^1$	$2_{II}^{+6} 8_1^{+1}$	K3 # 21
18	40	16	$\Gamma_3 a_2$	18	512	$2^1 4^1 8^2$	$2_5^{+1} 4_1^{+1} 8_{II}^{+2}$	K3 # 26
19	42	16	$\Gamma_2 a_2$	18	1024	$2^2 4^4$	$2_{II}^{+2} 4_6^{+4}$	–
20	44	16	$\Gamma_2 d$	19	-512	$2^3 8^2$	$2_5^{+3} 8_{II}^{+2}$	$O(L)$
21	47	16	$\Gamma_2 a_1$	16	1024	$2^2 4^4$	$2_{II}^{+2} 4_0^{+4}$	K3 # 22
22	50	18	$A_{3,3}$	16	729	$3^4 9^1$	$3^{+4} 9^{-1}$	K3 # 30
23	53	20	$\mathrm{Hol}(\mathbf{Z}_4)$	18	500	$5^1 10^2$	$2_2^{+2} 5^{+3}$	K3 # 32
24	54	21	$\mathbf{Z}_7{:}\mathbf{Z}_3$	18	343	7^3	7^{+3}	K3 # 33
25	56	24	S_4	17	-576	$4^1 12^2$	$4_3^{+3} 3^{+2}$	K3 # 34
26	68	32	$\Gamma_5 a_1$	17	-1024	4^5	4_7^{+5}	K3 # 40
27	70	32	$\Gamma_7 a_3$	20	256	$2^2 8^2$	$2_6^{+2} 8_6^{+2}$	$O(L)$
28	74	32	$\Gamma_5 a_2$	18	1024	$2^2 4^4$	$2_6^{+2} 4_{II}^{+4}$	$O(L)$
29	77	32	$\Gamma_4 a_1$	17	-512	$2^2 4^2 8^1$	$2_{II}^{+2} 4_6^{+2} 8_1^{+1}$	K3 # 39
30	79	36	$3^2.\mathbf{Z}_4$	18	324	$3^1 6^1 18^1$	$2_2^{+2} 3^{+2} 9^{-1}$	K3 # 46
31	81	36	$S_{3,3}$	18	972	$3^2 6^1 18^1$	$2_{II}^{-2} 3^{+3} 9^{-1}$	K3 # 48
32	85	48	$2^4{:}\mathbf{Z}_3$	17	-384	$2^4 24^1$	$2_{II}^{-4} 8_1^{+1} 3^{-1}$	K3 # 49
33	86	48	T_{48}	19	-384	$2^1 8^1 24^1$	$2_7^{+1} 8_{II}^{-2} 3^{-1}$	K3 # 54
34	88	48	$\mathbf{Z}_2 \times S_4$	18	576	$2^2 12^2$	$2_{II}^{+2} 4_2^{+2} 3^{+2}$	K3 # 51
35	90	48	$\mathbf{Z}_2 \times \mathrm{SL}_2(3)$	20	192	$2^2 4^1 12^1$	$2_{II}^{-2} 4_2^{+2} 3^{+1}$	–
36	93	48	$\mathbf{Z}_2^2 \times A_4$	19	-576	$2^3 6^1 12^1$	$2_{II}^{+4} 4_1^{+1} 3^{+2}$	–
37	102	60	A_5	18	300	$10^1 30^1$	$2_{II}^{-2} 3^{+1} 5^{-2}$	K3 # 55
38	107	64	$\Gamma_{23} a_3$	20	128	$2^3 16^1$	$2_5^{+3} 16_7^{+1}$	$O(L)$
39	108	64	$\Gamma_{25} a_1$	18	512	$4^3 8^1$	$4_5^{+3} 8_1^{+1}$	K3 # 56
40	110	72	M_9	19	-216	$2^1 6^1 18^1$	$2_3^{+3} 3^{-1} 9^{-1}$	K3 # 63
41	111	72	N_{72}	19	-324	$3^2 36^1$	$4_1^{+1} 3^{+2} 9^{-1}$	K3 # 62

Table 13: Isometry types of coinvariant lattices L_G

No.	G-No.	$\lvert G\rvert$	Symbol	Rank	Det	A_{L_G}	Genus	Type
42	112	72	$A_{4,3}$	18	432	$3^1 12^2$	$4_{\mathrm{II}}^{-2}3^{-3}$	K3 # 61
43	120	96	$2^4.D_6$	18	384	$2^2 4^1 24^1$	$2_{\mathrm{II}}^{-2}4_7^{+1}8_1^{+1}3^{-1}$	K3 # 65
44	132	120	S_5	19	-300	$5^1 60^1$	$4_3^{-1}3^{+1}5^{-2}$	K3 # 70
45	135	144	$\mathbf{Z}_3^2{:}QD_{16}$	20	216	$6^1 36^1$	$2_1^{+1}4_1^{+1}3^{-1}9^{-1}$	max # 13
46	140	168	$L_2(7)$	19	-196	$7^1 28^1$	$4_1^{+1}7^{+2}$	K3 # 74
47	143	192	T_{192}	19	-192	$4^2 12^1$	$4_7^{-3}3^{+1}$	K3 # 77
48	148	192	H_{192}	19	-384	$4^2 24^1$	$4_2^{-2}8_1^{+1}3^{-1}$	K3 # 76
49	150	192	$4^2.A_4$	18	256	$2^2 8^2$	$2_{\mathrm{II}}^{-2}8_6^{-2}$	K3 # 75
50	157	288	$A_{4,4}$	19	-288	$2^1 6^1 24^1$	$2_{\mathrm{II}}^{+2}8_1^{+1}3^{+2}$	K3 # 78
51	162	336	$\mathbf{Z}_2 \times \mathrm{L}_2(7)$	20	196	14^2	$2_{\mathrm{II}}^{+2}7^{+2}$	max # 5
52	163	360	A_6	19	-180	$3^1 60^1$	$4_5^{-1}3^{+2}5^{+1}$	K3 # 79
53	164	360	$(\mathbf{Z}_3 \times A_5){:}\mathbf{Z}_2$	20	225	15^2	$3^{-2}5^{-2}$	max # 10
54	168	384	F_{384}	19	-256	$4^1 8^2$	$4_3^{+1}8_2^{+2}$	K3 # 80
55	170	486	$3^{1+4}{:}2$	18	243	3^5	3^{+5}	$\mathcal{S}$-lattice
56	173	576	$\mathbf{Z}_2^4{:}(S_3 \times S_3)$	20	288	$12^1 24^1$	$4_7^{+1}8_1^{+1}3^{+2}$	max # 12
57	177	660	$\mathrm{L}_2(11)$	20	121	11^2	11^{+2}	max # 1
58	178	720	M_{10}	20	120	$2^1 60^1$	$2_5^{+1}4_1^{+1}3^{-1}5^{+1}$	max # 9
59	179	720	S_6	20	180	$6^1 30^1$	$2_{\mathrm{II}}^{-2}3^{+2}5^{+1}$	max # 8
60	183	960	M_{20}	19	-160	$2^2 40^1$	$2_{\mathrm{II}}^{-2}8_1^{+1}5^{-1}$	K3 # 81
61	185	972	$3^{1+4}{:}2.2$	19	-162	$3^3 6^1$	$2_1^{+1}3^{-4}$	$\mathcal{S}$-lattice
62	187	1152	$Q(\mathbf{Z}_3^2{:}\mathbf{Z}_2)$	20	192	$8^1 24^1$	$8_6^{-2}3^{-1}$	max # 11
63	188	1344	$\mathbf{Z}_2^3{:}\mathrm{L}_2(7)$	20	112	$4^1 28^1$	$4_2^{+2}7^{+1}$	max # 4
64	190	1920	$\mathbf{Z}_2^4{:}S_5$	20	160	$4^1 40^1$	$4_3^{-1}8_1^{+1}5^{-1}$	max # 7
65	191	1944	$3^{1+4}{:}2.2^2$	20	108	$3^1 6^2$	$2_2^{+2}3^{+3}$	max $\mathcal{S}$-lattice
66	193	2520	A_7	20	105	105^1	$3^{+1}5^{+1}7^{+1}$	max # 3
67	196	5760	$\mathbf{Z}_2^4{:}A_6$	20	96	$4^1 24^1$	$4_5^{-1}8_1^{+1}3^{+1}$	max # 6
68	197	20160	$\mathrm{L}_3(4)$	20	84	$2^1 42^1$	$2_{\mathrm{II}}^{-2}3^{-1}7^{-1}$	max # 2
69	198	29160	$3^4{:}A_6$	20	81	$3^2 9^1$	$3^{+2}9^{+1}$	max $\mathcal{S}$-lattice

Table 14: The 34 coinvariant lattices L_G with several groups.

No.	#{G}	G-No.
69	40	{198, 195, 194, 192, 189, 184, 181, 180, 176, 175, 171, 169, 161, 160, 159, 155, 153, 152, 139, 137, 129, 128, 126, 118, 117, 116, 115, 113, 98, 95, 82, 80, 65, 64, 63, 62, 52, 51, 21, 20}
67	7	{196, 182, 172, 165, 149, 145, 91}
65	7	{191, 186, 151, 131, 127, 59, 28}
64	2	{190, 158}
63	3	{188, 141, 100}
62	10	{187, 174, 167, 166, 147, 146, 144, 123, 94, 89}
61	3	{185, 130, 26}
60	3	{183, 136, 114}
57	3	{177, 99, 23}
56	2	{173, 156}
55	12	{170, 154, 138, 119, 97, 96, 66, 61, 49, 18, 10, 4}
54	11	{168, 133, 124, 106, 105, 104, 75, 71, 69, 45, 33}
53	5	{164, 142, 101, 67, 32}
51	3	{162, 83, 31}
50	5	{157, 134, 121, 92, 84}
49	6	{150, 103, 87, 72, 39, 38}
48	2	{148, 125}
47	3	{143, 122, 57}
45	2	{135, 109}
42	5	{112, 78, 60, 30, 29}
39	2	{108, 76}
38	2	{107, 73}
34	2	{88, 58}
33	2	{86, 55}
31	2	{81, 48}
29	2	{77, 41}
28	2	{74, 43}
26	3	{68, 46, 15}
24	2	{54, 11}
22	2	{50, 19}
21	2	{47, 14}
18	2	{40, 12}
11	2	{24, 8}
10	2	{22, 7}

Postscript

In our paper [31] we classified all orbits of fixed-point sublattices of the Leech lattice and their respective stabilizers inside Co_0. There are 290 different cases, listed in Table 1 of [31] . This allows us to obtain the 22 classes of maximal admissible groups H as described in Theorem 6.1 (a), (b), (c) by selecting those fixed-point lattices Λ^G from Table 1 (loc. cit) for which H is a subgroup of the stabilizer G with $\Lambda^H = \Lambda^G$ containing only admissible elements. The thirteen groups of part (a) with $\alpha(\Lambda^G) \geq 2$ correspond to the entries #102, #106, #108, #110, #111, #112, #118, #119, #120, #121, #128, #129, #134, the two groups of part (b) with $\alpha(\Lambda^G) = 1$ correspond to the entries #101 and #109 and the seven groups in part (c) with $\alpha(\Lambda^G) = 0$ correspond to the entries #83, #126, #27, #40, #41, #116, #124. For the groups of parts (a) and (b), the group H is the full stabilizer of Λ^H, whereas for the groups in part (c), H is strictly smaller than the full stabilizer G, cf. Theorem 7.1.

This approach will not lead to much shorter calculations in the present paper, furthermore the results of [31] also use the list of 279,343 conjugacy classes of non 2-subgroups of $2^{12}{:}M_{24}$ determined in the present paper.

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