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3-DIMENSIONAL GRIESS ALGEBRAS AND MIYAMOTO INVOLUTIONS

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Abstract

We consider a series of VOAs generated by 3-dimensional Griess algebras. We will show that these VOAs can be characterized by their 3-dimensional Griess algebras and their structures are uniquely determined. As an application, we will determine the groups generated by the Miyamoto involutions associated with Virasoro vectors of these VOAs.

1. Introduction

Let $L(c_n^0,0)$ be the unitary simple Virasoro vertex operator algebra (VOA) of central charge c_n^0, where

$$c_n^0 = 1 - \frac{6}{(n+2)(n+3)}, \quad n = 1,2,3,\ldots.$$

Let V be a VOA containing a sub VOA U isomorphic to $L(c_n^0,0)$ and let e be the Virasoro vector of U. It was shown by Miyamoto [22] that one can define an automorphism of V based on the fusion rules of $L(c_n^0,0)$-modules. Namely, the zero-mode $\mathrm{o}(e)$ of e acts on V semisimply and the linear map

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$\tau_e = (-1)^{4(n+2)(n+3)o(e)}$ is well-defined on V. The map τ_e gives rise to an automorphism of V and we will call it the *Miyamoto involution* associated to e. When τ_e is trivial, one can define another Miyamoto involution σ_e on a certain sub VOA of V using a similar method. (See Theorem 2.2 for detail.) When the involution σ_e is well-defined on the whole space V, such a Virasoro vector e is said to be of *σ-type* on V. Miyamoto involutions appear naturally as symmetries of some sporadic finite simple groups in the VOA theory.

The most important and interesting case would be the first member $c_1^0 = 1/2$ of the unitary series. It is shown in [22] that there exists a one-to-one correspondence between the 2A-elements of the Monster simple group and sub VOAs of the moonshine VOA $V^\natural$ isomorphic to $L(1/2, 0)$ via the Miyamoto involutions. However, the case $c = c_1^0$ is not the only important case. It is shown in [12] that there exists a one-to-one correspondence between the 2A-elements of the Baby Monster simple group and $c = c_2^0$ Virasoro vectors of σ-type in the VOA $V\!B^\natural$. An injective correspondence between the 2C-elements of the largest Fischer 3-transposition group and $c = c_4^0$ Virasoro vectors of σ-type in the VOA $V\!F^\natural$ is also shown in [13]. Recently, in [18], the authors found new correspondences between the transpositions of the second and the third largest Fischer 3-transposition groups and $c = c_5^0$ Virasoro vectors and $c = c_6^0$ Virasoro vectors of σ-type, respectively. Those relations between involutions and Virasoro vectors are referred to as the Conway-Miyamoto correspondence in (loc. cit.).

The purpose of this paper is to study the relations between Miyamoto involutions associated to two mutually orthogonal Virasoro vectors of $c = c_n^0$ and $c = c_{n+1}^0$. Straightly speaking, there will be no special relations between their Miyamoto involutions if the Virasoro vectors are just mutually orthogonal. Therefore, we need to add extra assumptions and consider their extensions. Namely, we will consider a VOA $A(1/2, c_n^1)$ generated by a 3-dimensional Griess algebra which is spanned by two mutually orthogonal vectors and one common highest weight vector. This VOA has already been considered in [3, 17] and occurs naturally in the moonshine VOA $V^\natural$ [4]. In this paper, we will give a characterization of this VOA based on the 3-dimensional Griess algebra in Theorem 3.5. In the description, we will use commutant superalgebras studied in [26] and the $N = 1$ super Virasoro algebras. A main observation is that

$$c_n^0 + c_{n+1}^0 - \frac{1}{2} = c_n^1 = \frac{3}{2}\left(1 - \frac{8}{(n+2)(n+4)}\right), \quad \text{for } n = 1, 2, 3, \ldots,$$

while the number c_n^1 is the central charge of the unitary $N = 1$ super Virasoro algebra. (The superscript in c_n^1 means "$N = 1$", whereas that in c_n^0 means "$N = 0$", i.e., non-super case.) The VOA $A(1/2, c_n^1)$ has three Virasoro frames and we can consider associated Miyamoto involutions. Two of them consist of mutually orthogonal $c = c_n^0$ and $c = c_{n+1}^0$ Virasoro vectors. We will classify all the irreducible modules over $A(1/2, c_n^1)$ and describe their decompositions with respect to these three Virasoro frames. As an application, we will determine the group generated by Miyamoto involutions associated to these Virasoro frames of $A(1/2, c_n^1)$ in Theorem 4.6. In Theorem 4.8, we will present certain inductive relations between Miyamoto involutions associated to $c = c_n^0$ and $c = c_{n+1}^0$ Virasoro vectors of $A(1/2, c_n^1)$ when n is odd. It is worthy to mention that our characterization is very simple and easy to check in practice. Actually, this work is a part of study of Fischer 3-transposition groups in [18] and the results in this paper will be crucial in the discussion in Section 5 of (loc. cit.).

The organization of this article is as follows. We first review some basic notation and terminology about VOAs. In Section 2, we review some basic facts and results for the unitary series of Virasoro VOAs and the unitary series of the $N = 1$ super Virasoro algebras. In Section 3, we study VOAs generated by their 3-dimensional Griess algebras. In Theorem 3.5, we will give a characterization of such a VOA using its Griess algebra. In Section 4, we introduce a series of VOAs $A(1/2, c_n^1)$ which satisfy Theorem 3.5. We study three Virasoro frames of $A(1/2, c_n^1)$ and classify all its irreducible modules and their decompositions with respect to three Virasoro frames. We will then determine the actions of Miyamoto involutions on VOAs which contain $A(1/2, c_n^1)$ as a sub VOA in Theorems 4.6 and 4.8. This is the main result of this paper. In Appendix, we will prove the $\mathbb{Z}_2$-rationality of the unitary series of the Ramond algebra, one of the $N = 1$ super Virasoro algebras.

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Notation and terminology. In this paper, we will work mainly over the complex number field $\mathbb{C}$ but sometimes we also consider real forms. If V is a superspace, we denote its $\mathbb{Z}_2$-grading by $V = V^{[0]} \oplus V^{[1]}$. A VOA V is of *One-Zero type* (or *OZ-type*) if it has the grading $V = \oplus_{n\geq 0} V_n$ such that $V_0 = \mathbb{C}\mathbb{1}$ and $V_1 = 0$. We will mainly consider VOAs of OZ-type. In this case, V has a unique invariant bilinear form such that $(\mathbb{1} \,|\, \mathbb{1}) = 1$. A real form $V_{\mathbb{R}}$ of V is called *compact* if the associated bilinear form is positive definite. For a subset A of V, the subalgebra generated by A is denoted by $\langle A\rangle$. For $a \in V_n$, we define $\mathrm{wt}(a) = n$. We expand $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ for $a \in V$ and define its *zero-mode* by $\mathrm{o}(a) := a_{(\mathrm{wt}(a)-1)}$ if a is homogeneous and extend linearly. The weight two subspace V_2 of a VOA V of OZ-type carries a structure of a commutative (non-associative) algebra equipped with the product $\mathrm{o}(a)b = a_{(1)}b$ for $a,\, b \in V_2$. This algebra is called the *Griess algebra* of V. A *Virasoro vector* is a vector $e \in V_2$ such that $e_{(1)}e = 2e$. In this case the subalgebra $\langle e\rangle$ is isomorphic to a Virasoro VOA with the central charge $c = 2(e\,|\,e)$. A Virasoro vector $e \in V$ is called *simple* if it generates a simple Virasoro sub VOA. Notice that e is always simple if it is taken from a compact real form of V. A simple $c = 1/2$ Virasoro vector is called an *Ising vector.* A Virasoro vector ω is called the *conformal vector* of V if each graded subspace V_n agrees with $\mathrm{Ker}_V\,(\mathrm{o}(\omega) - n)$ and satisfies $\omega_{(0)}a = a_{(-2)}\mathbb{1}$ for all $a \in V$. When V is of OZ-type, the half of the conformal vector is the unit of the Griess algebra and hence it is uniquely determined. We write $L(n) = \omega_{(n+1)}$ for $n \in \mathbb{Z}$. A *sub VOA* (W, e) of V is a pair of a subalgebra W of V and a Virasoro vector $e \in W$ such that e is the conformal vector of W. Usually we omit to denote e and simply call W a sub VOA. A sub VOA W of V is said to be *full* if V and W shares the same conformal vector. The *commutant subalgebra* of a sub VOA (W, e) of V is defined by $\mathrm{Com}_V W = \mathrm{Ker}_V e_{(0)}$ (cf. [8]). For an automorphism σ of V

and a V-module $(M, Y_M(\cdot, z))$, we define its σ-conjugate by $(M, Y_M^\sigma(\cdot, z))$, where $Y_M^\sigma(a, z) = Y_M(\sigma a, z)$ for $a \in V$. A module M is called *σ-invariant* or *σ-stable* if M is isomorphic to its σ-conjugate.

2. Virasoro VOAs and SVOAs

2.1. Unitary series of the Virasoro algebra

Let $\mathrm{Vir} = \oplus_{n\in\mathbb{Z}}\mathbb{C}L(n) \oplus \mathbb{C}z$ be the Virasoro algebra. The irreducible highest weight module over Vir with central charge c and highest weight h will be denoted by $L(c, h)$. Let

$$\begin{aligned} c_n^0 &:= 1 - \frac{6}{(n+2)(n+3)}, \quad n = 1, 2, 3, \dots, \\ h_{r,s}^{(n)} &:= \frac{(r(n+3) - s(n+2))^2 - 1}{4(n+2)(n+3)}, \quad 1 \le r \le n+1, \quad 1 \le s \le n+2. \end{aligned} \tag{2.1}$$

Then $L(c_n^0, 0)$ is a rational C_2-cofinite VOA and $L(c_n^0, h_{r,s}^{(n)})$, $1 \le s \le r \le n+1$, exhaust the set of inequivalent irreducible $L(c_n^0, 0)$-modules (cf. [6, 25]). Note that $h_{r,s}^{(n)} = h_{n+2-r,n+3-s}^{(n)}$. It is known that all irreducible $L(c_n^0, 0)$-modules have compact real forms and $L(c_n^0, h_{r,s}^{(n)})$ are usually called the unitary series of the Virasoro algebra. The fusion rules are also known and given as follows.

$$L(c_n^0, h_{r,s}^{(n)}) \boxtimes L(c_n^0, h_{r',s'}^{(n)}) = \sum_{\substack{1\le i\le M\\ 1\le j\le N}} L(c_n^0, h_{|r-r'|+2i-1, |s-s'|+2j-1}^{(n)}), \tag{2.2}$$

where $M = \min\{r, r', n+2-r, n+2-r'\}$ and $N = \min\{s, s', n+3-s, n+3-s'\}$.

Let V be a VOA and e a Virasoro vector of V. Suppose e generates a simple $c = c_n^0$ Virasoro sub VOA of V. Then V is an $\langle e\rangle$-module and one has an isotypical decomposition

$$V = \bigoplus_{1\le s\le r\le n+1} V[h_{r,s}^{(n)}]_e, \tag{2.3}$$

where $V[h]_e$ is the sum of all irreducible $\langle e\rangle$-submodules isomorphic to $L(c_n^0, h)$. Moreover, the zero-mode $\mathrm{o}(e) = e_{(1)}$ of e acts semisimply on V .

Theorem 2.1 ([22]). *Let e be a simple $c = c_n^0$ Virasoro vector of V. Then the linear map $\tau_e = (-1)^{4(n+2)(n+3)o(e)}$ defines an automorphism in* $\mathrm{Aut}(V)$. *By* (2.1), *τ_e acts on $V[h_{r,s}^{(n)}]_e$ as $(-1)^{r+1}$ if n is even and as $(-1)^{s+1}$ if n is odd.*

Set

$$P_n := \begin{cases} \{h_{1,s}^{(n)} \mid 1 \le s \le n+2\} & \text{if } n \text{ is even}, \\ \{h_{r,1}^{(n)} \mid 1 \le r \le n+1\} & \text{if } n \text{ is odd}. \end{cases} \tag{2.4}$$

It follows from the fusion rules in (2.2) that the subspace $V[P_n]_e = \bigoplus_{h\in P_n} V[h]_e$ forms a subalgebra of V. We say e is *of σ-type on V* if $V = V[P_n]_e$. Again by the fusion rules in (2.2), we have the following $\mathbb{Z}_2$-symmetry.

Theorem 2.2 ([22]). *The linear map*

$$\sigma_e := \begin{cases} (-1)^{s+1} & on \quad V[h_{1,s}^{(n)}]_e \quad if \quad n \quad is\ even, \\ (-1)^{r+1} & on \quad V[h_{r,1}^{(n)}]_e \quad if \quad n \quad is\ odd, \end{cases}$$

defines an automorphism of $V[P_n]_e$.

2.2. Unitary series of the $N = 1$ super Virasoro algebra

There are two extensions of the Virasoro algebra to Lie superalgebras called the $N = 1$ super Virasoro algebras: $\mathrm{NS} = \mathrm{NS}^{[0]} \oplus \mathrm{NS}^{[1]}$ and $\mathrm{R} = \mathrm{R}^{[0]} \oplus \mathrm{R}^{[1]}$, where

$$\mathrm{NS}^{[0]} = \mathrm{R}^{[0]} = \mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L(n) \oplus \mathbb{C}\mathsf{c},\ \mathrm{NS}^{[1]} = \bigoplus_{r\in\mathbb{Z}+1/2} \mathbb{C}G(r),\ \mathrm{R}^{[1]} = \bigoplus_{r\in\mathbb{Z}} \mathbb{C}G(r).$$

In addition to the Virasoro algebra relations, they also satisfy the following relations:

$$\begin{aligned} &[L(m), G(r)] = \left(\tfrac{1}{2}m - r\right) G(m+r), \quad [\mathsf{c}, G(r)] = 0, \text{ and} \\ &[G(r), G(s)] = 2L(r+s) + \delta_{r+s,0}\left(r^2 - \tfrac{1}{4}\right)\tfrac{\mathsf{c}}{3}. \end{aligned} \tag{2.5}$$

NS is called the *Neveu-Schwarz algebra* and R is called the *Ramond algebra*.

Neveu-Schwarz sectors. The Neveu–Schwarz algebra has a standard triangular decomposition $\mathrm{NS} = \mathrm{NS}_- \oplus \mathrm{NS}_0 \oplus \mathrm{NS}_+$ such that

$$\mathrm{NS}_\pm = \bigoplus_{\pm n>0} \mathbb{C}L(n) \oplus \bigoplus_{\pm r>0} \mathbb{C}G(r), \quad \mathrm{NS}_0 = \mathbb{C}L(0) \oplus \mathbb{C}\mathsf{c}.$$

For $c, h \in \mathbb{C}$, let $\mathbb{C}v_{c,h}$ be a one-dimensional module over $\mathrm{NS}_0 \oplus \mathrm{NS}_+$ defined by

$$\mathsf{c}v_{c,h} = cv_{c,h}, \quad L(0)v_{c,h} = hv_{c,h}, \quad \mathrm{NS}_+ v_{c,h} = 0,$$

and define the Verma module over the Neveu-Schwarz algebra $M_{\mathrm{NS}}(c,h)$ with central charge c and highest weight h by the induced module. We choose the $\mathbb{Z}_2$-grading of $M_{\mathrm{NS}}(c,h) = M_{\mathrm{NS}}(c,h)^{[0]} \oplus M_{\mathrm{NS}}(c,h)^{[1]}$ so that $v_{c,h} \in M_{\mathrm{NS}}(c,h)^{[0]}$. We denote the unique simple quotient of $M_{\mathrm{NS}}(c,h)$ by $L_{\mathrm{NS}}(c,h)$ which is called the NS-sector. Set

$$L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}, \quad \text{and} \quad G_{\mathrm{NS}}(z) = \sum_{r\in\mathbb{Z}+1/2} G(r)z^{-r-3/2}. \tag{2.6}$$

Then $L(z)$ and $G_{\mathrm{NS}}(z)$ are local fields on $M_{\mathrm{NS}}(c,h)$. Namely, they are elements of $\mathrm{End}(M_{\mathrm{NS}}(c,h))[\![z,z^{-1}]\!]$ and one has the following OPEs:

$$\begin{aligned} L(z)L(w) &\sim \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}, \\ L(z)G_{\mathrm{NS}}(w) &\sim \frac{3G_{\mathrm{NS}}(w)}{2(z-w)^2} + \frac{\partial G_{\mathrm{NS}}(w)}{z-w}, \\ G_{\mathrm{NS}}(z)G_{\mathrm{NS}}(w) &\sim \frac{2c}{3(z-w)^3} + \frac{2L(w)}{z-w}. \end{aligned} \tag{2.7}$$

We also have the derivation relations

$$[L(-1), L(z)] = \partial_z L(z) \quad \text{and} \quad [L(-1), G_{\mathrm{NS}}(z)] = \partial_z G_{\mathrm{NS}}(z).$$

Therefore, $L(z)$ and $G_{\mathrm{NS}}(z)$ generate a vertex superalgebra inside $\mathrm{End}(M_{\mathrm{NS}}(c,h))[\![z,z^{-1}]\!]$. Set $\overline{M}_{\mathrm{NS}}(c,h) := M_{\mathrm{NS}}(c,0)/\langle L(-1)v_{c,0}\rangle$. We denote the images of $v_{c,0}$, $L(-2)v_{c,0}$ and $G(-3/2)v_{c,0}$ in the quotient $\overline{M}_{\mathrm{NS}}(c,0)$ by $\mathbb{1}$, ω and τ, respectively. Then $\overline{M}_{\mathrm{NS}}(c,0)$ carries a unique structure of a vertex operator superalgebra such that $\mathbb{1}$ is the vacuum vector, $\omega = L(-2)\mathbb{1} = \frac{1}{2}G(-1/2)G(-3/2)\mathbb{1}$ is the conformal vector satisfying $Y(\omega,z) = L(z)$ and $Y(\tau,z) = G_{\mathrm{NS}}(z)$ by Theorem 4.5 of [14]. Note that $\langle G(-1/2)v_{c,0}\rangle =$

$\langle L(-1)v_{c,0}\rangle$ in $M_{\rm NS}(c,0)$ as $G(-1/2)^2 = L(-1)$ and $[G(1/2), L(-1)] = G(-1/2)$. It is clear that $M_{\rm NS}(c,h)$ is an $\overline{M}_{\rm NS}(c,0)$-module such that $Y(\tau,z) = G_{\rm NS}(z)$ on $M_{\rm NS}(c,h)$ and its simple quotient $L_{\rm NS}(c,h)$ is an irreducible $\overline{M}_{\rm NS}(c,0)$-module. In particular, the simple quotient $L_{\rm NS}(c,0)$ is a simple SVOA.

Ramond sectors. The Ramond algebra also has a standard triangular decomposition $\mathrm{R} = \mathrm{R}_- \oplus \mathrm{R}_0 \oplus \mathrm{R}_+$ such that

$$\mathrm{R}_\pm = \bigoplus_{\pm n>0} \mathbb{C}L(n) \oplus \bigoplus_{\pm r>0} \mathbb{C}G(r), \quad \mathrm{R}_0 = \mathbb{C}L(0) \oplus \mathbb{C}\mathsf{c} \oplus \mathbb{C}G(0).$$

Since $G(0)^2 = L(0) - \mathsf{c}/24$, the subalgebra R_0 is not $\mathbb{Z}_2$-homogeneous in this case. For c, $d \in \mathbb{C}$, let $\mathbb{C}v_{c,d}$ be a one-dimensional module over $\mathrm{R}_0 \oplus \mathrm{R}_+$ defined by

$$\mathsf{c}v_{c,d} = cv_{c,d}, \quad G(0)v_{c,d} = dv_{c,d}, \quad \mathrm{R}_+ v_{c,d} = 0,$$

and define the Verma module over the Ramond algebra $M_{\rm R}(c,d)$ with central charge c and top weight d by the induced module. Note that $L(0)v_{c,d} = (d^2 + c/24)v_{c,d}$. We call the eigenvalue $d^2 + c/24$ the *highest weight* of $M_{\rm R}(c,d)$. Our notion of Verma modules is unusual in the sense that $G(0)$ always acts semisimply on highest weight vectors and there is no canonical superspace structure. We denote the unique simple quotient of $M_{\rm R}(c,d)$ by $L_{\rm R}(c,d)$ which is called the Ramond sector. Set

$$L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}, \quad \text{and} \quad G_{\rm R}(z) = \sum_{n\in\mathbb{Z}} G(n)z^{-n-3/2}. \tag{2.8}$$

Then $L(z)$ and $G_{\rm R}(z)$ are local $\mathbb{Z}_2$-twisted fields on $M_{\rm R}(c,d)$. Namely, one has the same OPEs as in (2.7). We also have the derivation relations $[L(-1), L(z)] = \partial_z L(z)$ and $[L(-1), G_{\rm R}(z)] = \partial_z G_{\rm R}(z)$. Therefore, $L(z)$ and $G_{\rm R}(z)$ generate a vertex superalgebra inside $\mathrm{End}(M_{\rm R})[\![z^{1/2}, z^{-1/2}]\!]$ (cf. [19]). Let $\theta = (-1)^{2L(0)}$ be the canonical $\mathbb{Z}_2$-symmetry (supersymmetry) of $\overline{M}_{\rm NS}(c,0)$. One can directly verify that $M_{\rm R}(c,d)$ is a θ-twisted $\overline{M}_{\rm NS}(c,0)$-module and its simple quotient $L_{\rm R}(c,d)$ is an irreducible θ-twisted $\overline{M}_{\rm NS}(c,0)$-module. It follows that $L_{\rm R}(c,d)$ and $L_{\rm R}(c,-d)$ are θ-conjugate to each other.

Unitary series. For $n = 1, 2, 3, \ldots$, let

$$\begin{aligned} c_n^1 &= \frac{3}{2}\left(1 - \frac{8}{(n+2)(n+4)}\right), \\ \Delta_{r,s}^{(n)} &= \frac{s(n+2) - r(n+4)}{\sqrt{8(n+2)(n+4)}}, \\ h_{r,s,p}^{(n)} &= \left(\Delta_{r,s}^{(n)}\right)^2 + \frac{c_n^1}{24} - \frac{1}{16}\delta_{p,0} = \frac{(r(n+4) - s(n+2))^2 - 4}{8(n+2)(n+4)} + \frac{p}{8}, \end{aligned} \tag{2.9}$$

where $1 \le r \le n+1$, $1 \le s \le n+3$, $p = 0$ or $1/2$ and $r - s \equiv 2p \bmod 2$. Note that $\Delta^{(n)}_{n+2-r,n+4-s} = -\Delta^{(n)}_{r,s}$ and $h^{(n)}_{n+2-r,n+4-s,p} = h^{(n)}_{r,s,p}$.

Theorem 2.3 ([2, 10, 16, 21])**.** *The $N = 1$ Virasoro SVOA $L_{\mathrm{NS}}(c_n^1, 0)$ is rational and $\mathbb{Z}_2$-rational. The irreducible representations are as follows.*

(1) *The NS-sectors $L_{\mathrm{NS}}(c_n^1, h^{(n)}_{r,s,0})$, $1 \le r \le m+1$, $1 \le s \le m+3$, $r \equiv s \bmod 2$, are all the irreducible untwisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules.*

(2) *The R-sectors $L_{\mathrm{R}}(c_n^1, \Delta^{(n)}_{r,s})$, $1 \le r \le m+1$, $1 \le s \le m+3$, $r \equiv s + 1 \bmod 2$, are all the irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules.*

Proof. The irreducible untwisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules are classified in [16, 2] and the rationality is established in [2]. The irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules are classified in [21]. The $\mathbb{Z}_2$-rationality will be given in Appendix. □

Remark 2.4. The NS-sectors $L_{\mathrm{NS}}(c_n^1, h^{(n)}_{r,s,0})$ and the R-sectors $L_{\mathrm{R}}(c_n^1, \Delta^{(n)}_{r,s})$ have compact real forms and are called the *unitary series* of the $N = 1$ super Virasoro algebra.

Remark 2.5. If n is even, the top weight $\Delta^{(n)}_{(n+2)/2,(n+4)/2} = 0$ is the fixed point of the $\mathbb{Z}_2$-symmetry $\Delta^{(n)}_{n+2-r,n+4-s,} = -\Delta^{(n)}_{r,s}$ and the corresponding representation $L_{\mathrm{R}}(c_n^1, 0)$ is θ-stable. Therefore, among the irreducible R-sectors in (2) of Theorem 2.3, the representation $L_{\mathrm{R}}(c_n^1, 0)$ has a distinguished property such that it is not irreducible as an $L_{\mathrm{NS}}(c_n^1, 0)^{[0]}$-module while others are still irreducible over $L_{\mathrm{NS}}(c_n^1, 0)^{[0]}$.

3. VOAs with 3-dimensional Griess Algebras

3.1. Free Majorana fermion and $c = 1/2$ Virasoro SVOA

Let C be the associative algebra generated by ψ_r, $r \in \mathbb{Z} + 1/2$, subject to the relations

$$\psi_r \psi_s + \psi_s \psi_r = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + 1/2. \tag{3.1}$$

The Fock representation F of C is a cyclic C-module generated by $\mathbb{1}$ with relations $\psi_r \mathbb{1} = 0$ for $r > 0$. Then F has a natural $\mathbb{Z}_2$-grading $F = F^{[0]} \oplus F^{[1]}$ with

$$F^{[i]} = \mathrm{Span}_{\mathbb{C}}\{\psi_{-r_1} \cdots \psi_{-r_k} \mathbb{1} \mid r_1 > \cdots > r_k > 0, \ \ k \equiv i \bmod 2\}. \tag{3.2}$$

The generating series $\psi(z) := \sum_{r \in \mathbb{Z}+1/2} \psi_r z^{-r-1/2}$ is an odd field on F and satisfies the following locality

$$(z_1 - z_2)[\psi(z_1), \psi(z_2)]_+ = 0. \tag{3.3}$$

Therefore, $\psi(z)$ generates a vertex superalgebra inside $\mathrm{End}(F)[\![z, z^{-1}]\!]$ and F can be equipped with a unique structure of a vertex superalgebra such that $\mathbb{1}$ is the vacuum vector and $Y(\psi_{-1/2}\mathbb{1}, z) = \psi(z)$ (cf. [14]). The vector $\omega = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}\mathbb{1}$ provides the conformal vector of central charge $1/2$ and we have the isomorphisms

$$F^{[0]} \cong L(1/2, 0), \quad F^{[1]} \cong L(1/2, 1/2) \tag{3.4}$$

as $\langle \omega \rangle$-modules (cf. [15]).

Let C_{tw} be the associative algebra generated by ϕ_n, $n \in \mathbb{Z}$, subject to the relations

$$\phi_m \phi_n + \phi_n \phi_n = \delta_{m+n,0}, \quad m, n \in \mathbb{Z}. \tag{3.5}$$

The Fock representation F_{tw} of C_{tw} is a cyclic C_{tw}-module generated by $v_{1/16}$ with relations $\phi_n v_{1/16} = 0$ for $n > 0$. Set $F_{\mathrm{tw}}^{\pm} := \langle v_{1/16} \pm \sqrt{2}\, \phi_0 v_{1/16} \rangle$. Then we have a decomposition $F_{\mathrm{tw}} = F_{\mathrm{tw}}^{+} \oplus F_{\mathrm{tw}}^{-}$ as a C_{tw}-module. The generating series $\phi(z) := \sum_{n \in \mathbb{Z}} \phi_n z^{-n-1/2}$ is an odd $\mathbb{Z}_2$-twisted field on F_{tw} and satisfies the following locality

$$(z_1 - z_2)[\phi(z_1), \phi(z_2)]_+ = 0. \tag{3.6}$$

Therefore, $\phi(z)$ generates a vertex superalgebra inside $\mathrm{End}(F_{\mathrm{tw}})[\![z^{1/2}, z^{-1/2}]\!]$ and F_{tw} can be equipped with a unique structure of a $(-1)^{2L(0)}$-twisted F-module such that $Y(\psi_{-1/2}\mathbb{1}, z) = \phi(z)$ (cf. [19]). As $F^{[0]}$-modules, we have the isomorphisms (cf. [15])

$$F_{\mathrm{tw}}^{+} \cong F_{\mathrm{tw}}^{-} \cong L(1/2, 1/16), \tag{3.7}$$

whereas F_{tw}^{+} and F_{tw}^{-} are inequivalent $(-1)^{2L(0)}$-twisted F-modules since the zero-mode $o(\psi_{-1/2}\mathbb{1}) = \phi_0$ acts as $\pm 2^{-1/2}$ on the top levels of $F_{\mathrm{tw}}^{\pm}$. Indeed, $F_{\mathrm{tw}}^{\pm}$ are mutually $(-1)^{2L(0)}$-conjugate F-modules.

Let $V = V^{[0]} \oplus V^{[1]}$ be an SVOA with a non-trivial odd part. A tensor product $F \otimes V$ is an SVOA with $\mathbb{Z}_2$-grading

$$(F \otimes V)^{[0]} = F^{[0]} \otimes V^{[0]} \oplus F^{[1]} \otimes V^{[1]}, \quad (F \otimes V)^{[1]} = F^{[0]} \otimes V^{[1]} \oplus F^{[1]} \otimes V^{[0]}. \tag{3.8}$$

Let $M = M^{[0]} \oplus M^{[1]}$ be an untwisted V-module. Then $F \otimes M$ is an $F \otimes V$-module and its $\mathbb{Z}_2$-homogeneous parts

$$(F \otimes M)^{[0]} = F^{[0]} \otimes M^{[0]} \oplus F^{[1]} \otimes M^{[1]}, \quad (F \otimes M)^{[1]} = F^{[0]} \otimes M^{[1]} \oplus F^{[1]} \otimes M^{[0]}, \tag{3.9}$$

are $(F \otimes V)^{[0]}$-submodules. Let N be a $\mathbb{Z}_2$-twisted V-module. Then tensor products

$$F_{\mathrm{tw}}^{\pm} \otimes N \tag{3.10}$$

are untwisted $(F \otimes V)^{[0]}$-modules. In this way, given an untwisted or $\mathbb{Z}_2$-twisted V-module, we can construct an $(F \otimes V)^{[0]}$-module. In the next subsection, we will show that there is a canonical reverse construction of V-modules from $(F \otimes V)^{[0]}$-modules.

3.2. Commutant SVOAs

Let V be a VOA and e an Ising vector of V. Let M be a V-module. For $h = 0,\ 1/2,\ 1/16$, we set

$$T_{e,M}(h) := \{a \in M \mid e_{(1)}a = ha\}. \tag{3.11}$$

Then we have the isotypical decomposition

$$M = L(1/2,0)\otimes T_{e,M}(0)\oplus L(1/2,1/2)\otimes T_{e,M}(1/2)\oplus L(1/2,1/16)\otimes T_{e,M}(1/16) \tag{3.12}$$

of M as $\langle e\rangle$-module. Since $\langle e\rangle \cong L(1/2,0)$ is rational, its zero-mode $\mathrm{o}(e)$ acts on M semisimply. Therefore, the Miyamoto involution $\tau_e = (-1)^{48\mathrm{o}(e)}$ is also well-defined on M and M is τ_e-stable. Set

$$\begin{aligned} M^{\langle\tau_e\rangle} &= L(1/2,0)\otimes T_{e,M}(0)\oplus L(1/2,1/2)\otimes T_{e,M}(1/2),\\ M^{\langle-\tau_e\rangle} &= L(1/2,1/16)\otimes T_{e,M}(1/16). \end{aligned} \tag{3.13}$$

Then $M^{\langle\pm\tau_e\rangle}$ are $V^{\langle\tau_e\rangle}$-submodules. It is known that $T_{e,V}(0)$ is the commutant subalgebra of $\langle e\rangle$ in V and $T_{e,M}(h)$, $h = 1/2,\ 1/16$, are $T_{e,V}(0)$-modules (cf. [8, 26]). As $L(1/2,0)$ can be extended to an SVOA $L(1/2,0)\oplus L(1/2,1/2)$, the commutant $T_{e,V}(0)$ can also be extended to an SVOA.

Proposition 3.1 ([11], [26]). *Let V be a VOA and e an Ising vector of V and suppose $T_{e,V}(1/2) \neq 0$. Then there exists an SVOA structure on $T_{e,V}(0)\oplus T_{e,V}(1/2)$ which is an extension of the commutant sub VOA $T_{e,V}(0)$ such that the even part*

$$((L(1/2,0)\oplus L(1/2,1/2))\otimes (T_{e,V}(0)\oplus T_{e,V}(1/2)))^{[0]}$$

of a tensor product of SVOAs is isomorphic to the sub VOA $V^{\langle\tau_e\rangle}$. If V is simple, then $T_{e,V}(0)\oplus T_{e,V}(1/2)$ is a simple SVOA and $T_{e,V}(0)$ is a simple sub VOA.

Next theorem provides a construction of $T_{e,V}(0)\oplus T_{e,V}(1/2)$-modules from $V^{\langle\tau_e\rangle}$-modules, which is a sort of reverse of (3.9) and (3.10).

Proposition 3.2. *Let V be a VOA and e an Ising vector of V such that $T_e(1/2)\neq 0$. Let M be a $V^{\langle\tau_e\rangle}$-module. Decompose M as in* (3.12) *and define $M^{\langle\pm\tau_e\rangle}$ as in* (3.13)*.*

(1) *The space* $T_{e,M}(0) \oplus T_{e,M}(1/2)$ *forms an untwisted* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module such that* $M^{\langle \tau_e \rangle}$ *as a* $V^{\langle \tau_e \rangle}$*-module is isomorphic to one of the* $\mathbb{Z}_2$*-homogeneous parts of the tensor product of the adjoint module of* $L(1/2, 0) \oplus L(1/2, 1/2)$ *and the* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module* $T_{e,M}(0) \oplus T_{e,M}(1/2)$*. If* $M^{\langle \tau_e \rangle}$ *is an irreducible* $V^{\langle \tau_e \rangle}$*-module, then* $T_{e,M}(0) \oplus T_{e,M}(1/2)$ *is also irreducible as a* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module.*

(2) *The space* $T_{e,M}(1/16)$ *forms a* $\mathbb{Z}_2$*-twisted* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module such that* $M^{\langle -\tau_e \rangle}$ *as a* $V^{\langle \tau_e \rangle}$*-module is isomorphic to a tensor product of a* $\mathbb{Z}_2$*-twisted* $L(1/2, 0) \oplus L(1/2, 1/2)$*-module* $L(1/2, 1/16)^+$ *and the* $\mathbb{Z}_2$*-twisted* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module* $T_{e,M}(1/16)$*. If* $M^{\langle -\tau_e \rangle}$ *is an irreducible* $V^{\langle \tau_e \rangle}$*-module, then* $T_{e,M}(1/16)$ *is also irreducible as a* $\mathbb{Z}_2$*-twisted* $T_{e,V}(0) \oplus T_{e,V}(1/2)$*-module.*

Proof. The proof for the existences of structures of modules is similar to that of Proposition 3.1 (see Theorem 2.2 of [26]). The irreducibility is clear and follows from the fusion rules of $L(1/2, 0)$-modules. □

Remark 3.3. If V is simple and both $V[1/2]_e$ and $V[1/16]_e$ are non-zero, then the fusion rules of $L(1/2, 0)$-modules guarantee that $T_{e,M}(h)$ is also non-zero for all non-zero V-modules and for $h = 0, 1/2, 1/16$. On the other hand, if V is simple, $V[1/2]_e \neq 0$ and $V[1/16]_e = 0$, then again by the fusion rules of $L(1/2, 0)$-modules one of $T_{e,M}(0) \oplus T_{e,M}(1/2)$ or $T_{e,M}(1/16)$ is zero for an irreducible V-module M.

Suppose e is an Ising vector of V such that $T_{e,V}(1/2) \neq 0$. By Proposition 3.2, there is a correspondence between $V^{\langle \tau_e \rangle}$-modules and untwisted and $\mathbb{Z}_2$-twisted $T_{e,V}(0) \oplus T_{e,V}(1/2)$-modules via (3.9) and (3.10). It is shown in [24] that V is C_2-cofinite if and only if $T_{e,V}(0)$ is. Therefore, we have the following theorem.

Theorem 3.4 (cf. [24])**.** *Suppose* e *is an Ising vector of* V *such that* $T_{e,V}(1/2) \neq 0$.

(1) V *is* C_2*-cofinite if and only if* $T_{e,V}(0)$ *is* C_2*-cofinite.*

(2) $V^{\langle \tau_e \rangle}$ *is rational if and only if* $T_{e,V}(0) \oplus T_{e,V}(1/2)$ *is rational and* $\mathbb{Z}_2$*-rational.*

3.3. A characterization by 3-dimensional Griess algebras

In this subsection, we will prove the following theorem.

Theorem 3.5. *Let (V, ω) be a VOA of OZ-type. Suppose the following.*

(1) *The central charge c_V of V is not equal to $1/2$.*

(2) *V has an Ising vector e.*

(3) *There exists a 3-dimensional subalgebra $B = \mathbb{C}\omega + \mathbb{C}e + \mathbb{C}x$ of the Griess algebra of V such that $2e_{(1)}x = x$ and $(x \,|\, x)$ is non-zero.*

Suppose that $W = \langle B \rangle$ is a full sub VOA of V. Then the commutant superalgebra

$$T_{e,W}(0) \oplus T_{e,W}(1/2)$$

is isomorphic to an $N = 1$ Virasoro SVOA with the conformal vector $\omega - e$.

Proof. Set $f := \omega - e$. Since V is of OZ-type, $e_{(2)}\omega = 0$ so that e and f are mutually orthogonal Virasoro vectors by Theorem 5.1 of [8]. The central charge of f is $c_f := 2(f \,|\, f) = c_V - 1/2 \neq 0$. By a normalization, we may assume that $(x \,|\, x) = 2c_f/3$. Since x is a highest weight vector for $\langle e \rangle \otimes \langle f \rangle$ with highest weight $(1/2, 3/2)$, we have

$$\begin{aligned} \big(x_{(1)}x \,|\, e\big) &= \big(x \,|\, x_{(1)}e\big) = \frac{(x \,|\, x)}{2} = \frac{c_f}{3}, \\ \big(x_{(1)}x \,|\, f\big) &= \big(x \,|\, x_{(1)}f\big) = \frac{3(x \,|\, x)}{2} = c_f. \end{aligned} \tag{3.14}$$

By assumption, $\mathrm{o}(e)$ acts on B semisimply with eigenvalues 0, 1/2 and 2. Thus e is of σ-type on $W = \langle B \rangle$ and defines the involution σ_e by Theorem 2.2. Since σ_e negates x, we have $x_{(1)}x \in B^{\langle \sigma_e \rangle} = \mathbb{C}e + \mathbb{C}f$ and so we can write $x_{(1)}x = \alpha e + \beta f$ with α, $\beta \in \mathbb{C}$. Then

$$\begin{aligned} \big(x_{(1)}x \,|\, e\big) &= (\alpha e + \beta f \,|\, e) = \alpha(e \,|\, e) = \frac{\alpha}{4}, \\ \big(x_{(1)}x \,|\, f\big) &= (\alpha e + \beta f \,|\, f) = \beta(f \,|\, f) = \frac{\beta c_f}{2}. \end{aligned} \tag{3.15}$$

By (3.14) and (3.15) we have

$$x_{(1)}x = \frac{4c_f}{3}e + 2f. \tag{3.16}$$

Decompose W as

$$W = L(1/2, 0) \otimes T_{e,W}(0) \oplus L(1/2, 1/2) \otimes T_{e,W}(1/2).$$

Then $T := T_{e,W}(0) \oplus T_{e,W}(1/2)$ is an SVOA with the conformal vector f by Proposition 3.1. Since x is a highest weight vector for $\langle e\rangle$ and $\langle f\rangle$ with highest weight $1/2$ and $3/2$, respectively, there exists a highest weight vector $a \in T_{e,W}(1/2)$ for $\langle f\rangle$ with highest weight $3/2$ such that $x = \psi_{-1/2}\mathbb{1} \otimes a$. Let $Y_V(\cdot, z)$ and $Y_T(\cdot, z)$ be vertex operator maps of V and T, respectively. Then $Y_V(x, z) = \psi(z) \otimes Y_T(a, z)$ by Proposition 3.1 and we have

$$x_{(n)}x = \mathbb{1} \otimes a_{(n-1)}a + \sum_{j\geq 0} \psi_{-j-3/2}\psi_{-1/2}\mathbb{1} \otimes a_{(n+j+1)}a. \tag{3.17}$$

Since V is of OZ-type, so is $T_{e,V}(0)$ and we have[1] $a_{(1)}a = 0$, $a_{(2)}a \in \mathbb{C}\mathbb{1}$ and $a_{(n)}a = 0$ for $n > 2$. Then by (3.17), we have

$$x_{(1)}x = \mathbb{1} \otimes a_{(0)}a + \psi_{-3/2}\psi_{-1/2}\mathbb{1} \otimes a_{(2)}a. \tag{3.18}$$

Since $\psi_{-3/2}\psi_{-1/2}\mathbb{1} = 2e$, comparing (3.16) and (3.18) we obtain

$$a_{(2)}a = \frac{2c_f}{3}\mathbb{1}, \quad a_{(1)}a = 0, \quad a_{(0)}a = 2f. \tag{3.19}$$

Now set $L^f(m) = f_{(m+1)}$ and $G^a(r) = a_{(r+1/2)}$ for $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + 1/2$. By (3.19) their commutators are as follows.

$$\begin{aligned}
\left[L^f(m), G^a(r)\right] &= \left[f_{(m+1)}, a_{(r+1/2)}\right] = \sum_{i=0}^{\infty} \binom{m+1}{i} \left(f_{(i)}a\right)_{(m+r+3/2-i)} \\
&= \left(f_{(0)}a\right)_{(m+r+3/2)} + (m+1)\left(f_{(1)}a\right)_{(m+r+1/2)} \\
&= -\left(m + r + \tfrac{3}{2}\right) a_{(m+r+1/2)} + \tfrac{3}{2}(m+1)a_{(m+r+1/2)} \\
&= \left(\tfrac{1}{2}m - r\right) a_{(m+r+1/2)} = \left(\tfrac{1}{2}m - r\right) G^a(m+r),
\end{aligned}$$

[1] That $a_{(1)}a = 0$ also follows from the skew-symmetry.

$$\begin{aligned}
[G^a(r), G^a(s)] &= \left[a_{(r+1/2)}, a_{(s+1/2)}\right] = \sum_{i=0}^{\infty} \binom{r+1/2}{i} \left(a_{(i)}a\right)_{(r+s+1-i)} \\
&= \left(a_{(0)}a\right)_{(r+s+1)} + \binom{r+1/2}{2} \left(a_{(2)}a\right)_{(r+s-1)} \\
&= 2f_{(r+s+1)} + \binom{r+1/2}{2} \cdot \frac{2c_f}{3} \cdot \mathbb{1}_{(r+s-1)} \\
&= 2L^f(r+s) + \delta_{r+s,0} \left(r^2 - \tfrac{1}{4}\right) \tfrac{c_f}{3}.
\end{aligned}$$

Therefore, $Y_T(f,z)$ and $Y_T(a,z)$ generate a representation of the Neveu-Schwarz algebra NS on T. Since W is generated by e, f and $x = \psi_{-1/2}\mathbb{1} \otimes a$, it follows that $T = T_{e,W}(0) \oplus T_{e,W}(1/2)$ is generated by f and a. Therefore, T is isomorphic to an $N = 1$ super Virasoro VOA. □

4. Extension of a Pair of Unitary Virasoro VOAs

Consider the even part

$$A(1/2, c_n^1) := L(1/2, 0) \otimes L_{\mathrm{NS}}(c_n^1, 0)^{[0]} \oplus L(1/2, 1/2) \otimes L_{\mathrm{NS}}(c_n^1, 0)^{[1]} \tag{4.1}$$

of the tensor product of SVOAs $L(1/2,0) \oplus L(1/2,1/2)$ and $L_{\mathrm{NS}}(c_n^1, 0)$ where c_n^1 is defined as in (2.9). This VOA is also considered in [3, 17]. The VOA $A(1/2, c_n^1)$ inherits the invariant bilinear forms of $L(1/2,0) \oplus L(1/2,1/2)$ and $L_{\mathrm{NS}}(c_n^1, 0)$ and has a compact real form.

4.1. Griess algebra

Let $e = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}\mathbb{1}$ and $f = \frac{1}{2}G(-1/2)G(-3/2)\mathbb{1}$ be the conformal vectors of $L(1/2,0)$ and $L_{\mathrm{NS}}(c_m^1, 0)^{[0]}$, respectively, and let $x = \sqrt{(n+2)(n+4)}\,\psi_{-1/2}\mathbb{1} \otimes G(-3/2)\mathbb{1}$ be the highest weight vector of $L(1/2, 1/2)$ $\otimes L_{\mathrm{NS}}(c_n^1, 0)^{[1]}$. Then $A(1/2, c_n^1)$ is of OZ-type and its Griess algebra is 3-dimensional with an orthogonal basis e, f and x such that

$$\begin{gathered}
e_{(1)}e = 2e,\ e_{(1)}f = 0,\ e_{(1)}x = \frac{1}{2}x,\ f_{(1)}f = 2f,\ f_{(1)}x = \frac{3}{2}x,\ (e\,|\,e) = \frac{1}{4}, \\
x_{(1)}x = 2n(n+6)e + 2(n+2)(n+4)f, \quad (f\,|\,f) = \frac{c_n^1}{2}, \quad (x\,|\,x) = n(n+6).
\end{gathered} \tag{4.2}$$

By (4.1) and (4.2), we can apply Theorem 3.5 to $A(1/2, c_n^1)$ and obtain a characterization of it. By a direct calculation, we can classify the Virasoro vectors in $A(1/2, c_n^1)$.

Proposition 4.1 ([3, 17]). *Let*

$$u := \frac{1}{2(n+3)}\left(ne + (n+4)f + x\right), \qquad \textit{and} \qquad v := e + f - u. \tag{4.3}$$

(1) *u and v are mutually orthogonal Virasoro vectors with central charges c_n^0 and c_{n+1}^0.*
(2) *The set of Virasoro vectors of $A(1/2, c_n^1)$ is given by $\{\omega, e, f, u, v, \sigma_e u, \sigma_e v\}$.*
(3) *$A(1/2, c_n^1)$ is generated by its Griess algebra.*
(4) $\mathrm{Aut}(A(1/2, c_n^1)) = \langle \sigma_e \rangle$ *if $n > 1$ and* $\mathrm{Aut}(A(1/2, c_1^1)) = \langle \sigma_e, \sigma_u \rangle \cong \mathrm{S}_3$.

Proof.

(1): It is straightforward to verify that u and v are mutually orthogonal Virasoro vectors and $\omega = u + v$ is a Virasoro frame of $A(1/2, c_n^1)$ (cf. [3]).

(2): The solutions of the quadratic equation $y^2 = 2y$ in the Griess algebra provide a complete list of Virasoro vectors in $A(1/2, c_n^1)$ which is as in the assertion.

(3): Let V be a subalgebra of $A(1/2, c_n^1)$ generated by the Griess algebra. Then V satisfies the conditions in Theorem 3.5 and it follows that $V = A(1/2, c_n^1)$.

(4): If $n > 1$ then e is the unique Ising vector of $A(1/2, c_n^1)$ and σ_e is the unique non-trivial automorphism of the Griess algebra. Since $A(1/2, c_n^1)$ is generated by its Griess algebra, we have $\mathrm{Aut}(A(1/2, c_n^1)) = \langle \sigma_e \rangle$. If $n = 1$, then $A(1/2, c_n^1)$ has three Ising vectors e, u and $\sigma_e u$ of σ-types and σ-involutions associated to these Ising vectors generate S_3. □

Since $A(1/2, c_n^1)$ has a compact real form, it contains a full sub VOA $\langle u \rangle \otimes \langle v \rangle \cong L(c_n^0, 0) \otimes L(c_{n+1}^0, 0)$. By (4.2) and (4.3), the Griess algebra is spanned by e, u and v where their multiplications are as follows.

$$\begin{aligned} e_{(1)}u &= \frac{n+1}{n+3}e + \frac{n+2}{4(n+3)}u - \frac{n+4}{4(n+3)}v, \\ e_{(1)}v &= \frac{n+5}{n+3}e - \frac{n+2}{4(n+3)}u + \frac{n+4}{4(n+3)}v. \end{aligned} \tag{4.4}$$

Clearly, the relations above uniquely determine the Griess algebra so that $A(1/2, c_n^1)$ is also characterized by the structure in (4.4) thanks to Theorem 3.5.

Proposition 4.2. *Let $V_{\mathbb{R}}$ be a compact VOA of OZ-type. Suppose e, u and v are simple $c = c_1^0$, $c = c_n^0$ and $c = c_{n+1}^0$ Virasoro vectors of $V_{\mathbb{R}}$, respectively, such that u and v are mutually orthogonal and satisfy* (4.4)*. Then the sub VOA generated by e, u, and v is isomorphic to $A(1/2, c_n^1)$.*

Proof. Suppose $u_{(1)}v = 0$ and (4.4) holds. The inner products $(e\,|\,u)$ and $(e\,|\,v)$ are uniquely determined by the invariance property $(e_{(1)}u|u) = (e|u_{(1)}u) = 2(e|u)$ and $(e_{(1)}v|v) = (e|v_{(1)}v) = 2(e|v)$. Then by change of basis, we recover the relations (4.2) and hence the subalgebra generated by u, v and e is isomorphic to $A(1/2, c_n^1)$ by Theorem 3.5. □

Set

$$w = 3n(n+6)e - (n+2)(n+4)f + 3x. \tag{4.5}$$

Then w is a highest weight vector for $\langle u\rangle \otimes \langle v\rangle$ with the highest weight

$$(h_{1,3}^{(n)}, h_{3,1}^{(n+1)}) = \left(\frac{n+1}{n+3}, \frac{n+5}{n+3}\right).$$

Therefore, $A(1/2, c_n^1)$ contains $L(c_n^0, h_{1,3}^{(n)}) \otimes L(c_{n+1}^0, h_{3,1}^{(n+1)})$ as a $\langle u\rangle \otimes \langle v\rangle$-submodule. A complete decomposition will be given in the next subsection.

Theorem 4.3 (cf. [3])**.**

(1) *$A(1/2, c_n^1)$ is rational and C_2-cofinite.*

(2) *The even part $L_{\mathrm{NS}}(c_n^1, 0)^{[0]}$ is C_2-cofinite.*

Proof. The rationality of $A(1/2, c_n^1)$ follows from Theorem 2.3 and (2) of Theorem 3.4. Since $A(1/2, c_n^1)$ has a C_2-cofinite full sub VOA $L(c_n^0, 0) \otimes L(c_{n+1}^0, 0)$, it is also C_2-cofinite (cf. [1, 3]), and the C_2-cofiniteness of $L_{\mathrm{NS}}(c_n^1, 0)^{[0]}$ follows from (1) of Theorem 3.4. □

4.2. Modules

Let n be a positive integer and let $L_{\hat{\mathfrak{sl}}_2}(n, j) = L_{\hat{\mathfrak{sl}}_2}((n-j)\Lambda_0 + j\Lambda_1)$ be the level n integrable highest weight $\hat{\mathfrak{sl}}_2$-module with highest weight $(n -$

$j)\Lambda_0 + j\Lambda_1$, $0 \le j \le n$. By [9], $L_{\hat{\mathfrak{sl}}_2}(1,0) \otimes L_{\hat{\mathfrak{sl}}_2}(n,0)$ contains a full sub VOA $L(c_n^0, 0) \otimes L_{\hat{\mathfrak{sl}}_2}(n+1,0)$ and we have the following decompositions for $i = 0,1$ and $0 \le j \le n$.

$$L_{\hat{\mathfrak{sl}}_2}(1,i) \otimes L_{\hat{\mathfrak{sl}}_2}(n,j) = \bigoplus_{\substack{0\le k\le n+1\\ k\equiv i+j (2)}} L(c_n^0, h_{j+1,k+1}^{(n)}) \otimes L_{\hat{\mathfrak{sl}}_2}(n+1,k). \tag{4.6}$$

By [20], the affine VOA $L_{\hat{\mathfrak{sl}}_2}(2,0)$ admits an extension to a simple SVOA $L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L_{\hat{\mathfrak{sl}}_2}(2,2)$ by a simple current module $L_{\hat{\mathfrak{sl}}_2}(2,2)$. The classifications of irreducible untwisted and $\mathbb{Z}_2$-twisted $L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L_{\hat{\mathfrak{sl}}_2}(2,2)$-modules are established in (loc. cit.). The adjoint module is the unique irreducible untwisted $L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L_{\hat{\mathfrak{sl}}_2}(2,2)$-module and there exist two inequivalent structures $L_{\hat{\mathfrak{sl}}_2}(2,1)^\pm$ of irreducible $\mathbb{Z}_2$-twisted $L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L_{\hat{\mathfrak{sl}}_2}(2,2)$-modules on $L_{\hat{\mathfrak{sl}}_2}(2,1)$ which are mutually $\mathbb{Z}_2$-conjugate to each other. It is shown in [9] that a tensor product

$$(L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L_{\hat{\mathfrak{sl}}_2}(2,2)) \otimes L_{\hat{\mathfrak{sl}}_2}(n,0)$$

contains a full sub SVOA $L_{\mathrm{NS}}(c_n^1, 0) \otimes L_{\hat{\mathfrak{sl}}_2}(n+2,0)$ and we have the following decompositions for $i = 0$, 1 and $0 \le j \le n$.

$$\begin{aligned}
&L_{\hat{\mathfrak{sl}}_2}(2,2i) \otimes L_{\hat{\mathfrak{sl}}_2}(n,j) = \bigoplus_{\substack{0\le k\le n+2\\ k\equiv j\ (2)}} L_{\mathrm{NS}}(c_n^1, h_{j+1,k+1,0}^{(n)})^{[i+\frac{j-k}{2}]} \otimes L_{\hat{\mathfrak{sl}}_2}(n+2,k),\\
&\Big(L_{\hat{\mathfrak{sl}}_2}(2,1)^+ \oplus L_{\hat{\mathfrak{sl}}_2}(2,1)^-\Big) \otimes L_{\hat{\mathfrak{sl}}_2}(n,j)\\
&\quad = \bigoplus_{\substack{0\le k\le n+2\\ k\equiv j+1\ (2)}} \Big(L_{\mathrm{R}}(c_n^1, \Delta_{j+1,k+1}^{(n)}) \oplus L_{\mathrm{R}}(c_n^1, -\Delta_{j+1,k+1}^{(n)})\Big) \otimes L_{\hat{\mathfrak{sl}}_2}(n+2,k).
\end{aligned} \tag{4.7}$$

As we have seen, $A(1/2, c_n^1) = L(1/2,0) \otimes L_{\mathrm{NS}}(c_n^1,0)^{[0]} \oplus L(1/2,1/2) \otimes L_{\mathrm{NS}}(c_n^1,0)^{[1]}$ contains a full sub VOA $L(c_n^0,0) \otimes L(c_{n+1}^0, 0)$. We consider the decompositions of irreducible $A(1/2, c_n^1)$-modules as $L(c_n^0,0) \otimes L(c_{n+1}^0,0)$-modules.

First, we label the irreducible $A(1/2, c_n^1)$-modules as follows.

$$\begin{aligned}
M(0, h_{r,s,0}^{(n)}) &= L(1/2,0) \otimes L_{\mathrm{NS}}(c_n^1, h_{r,s,0}^{(n)})^{[0]} \oplus L(1/2,1/2) \otimes L_{\mathrm{NS}}(c_n^1, h_{r,s,0}^{(n)})^{[1]},\\
M(1/2, h_{r,s,0}^{(n)}) &= L(1/2,1/2) \otimes L_{\mathrm{NS}}(c_n^1, h_{r,s,0}^{(n)})^{[0]} \oplus L(1/2,0) \otimes L_{\mathrm{NS}}(c_n^1, h_{r,s,0}^{(n)})^{[1]},\\
M(1/16, \Delta_{r,s}^{(n)}) &= L(1/2,1/16)^+ \otimes L_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)}),
\end{aligned} \tag{4.8}$$

where $h^{(n)}_{r,s,p}$ and $\Delta^{(n)}_{r,s}$ with $1 \le r \le n+1$, $1 \le s \le n+3$, $p = 0, 1$ and $r - s \equiv 2p \bmod 2$, are as in (2.9). The zero-mode $o(\psi_{-1/2}\mathbb{1} \otimes G(-3/2)\mathbb{1}) = \phi_0 \otimes G(0)$ acts on the top level of $M(1/16, \Delta^{(n)}_{r,s})$ by

$$\frac{1}{\sqrt{2}} \cdot \Delta^{(n)}_{r,s} = \frac{s(n+2) - r(n+4)}{4\sqrt{(n+2)(n+4)}}. \tag{4.9}$$

Note that $\Delta^{(n)}_{n+2-r,n+4-s} = -\Delta^{(n)}_{r,s}$ so that

$$L(1/16, 0)^- \otimes L_{\mathrm{R}}(c^1_n, \Delta^{(n)}_{r,s}) \cong L(1/16, 0)^+ \otimes L_{\mathrm{R}}(c^1_n, \Delta^{(n)}_{n+2-r,n+4-s})$$

as $A(1/2, c^1_n)$-modules and $M(1/16, \pm\Delta^{(n)}_{r,s})$ are mutually σ_e-conjugate while $M(1/2, h^{(n)}_{r,s,0})$ are σ_e-invariant. The next theorem follows from Proposition 3.2.

Theorem 4.4. *The set of irreducible $A(1/2, c^1_n)$-modules is given by the list* (4.8).

We decompose an irreducible $A(1/2, c^1_n)$-module into a direct sum of irreducible modules over $L(c^0_n, 0) \otimes L(c^0_{n+1}, 0)$. By (4.6), we have

$$\begin{aligned}
&L_{\hat{\mathfrak{sl}}_2}(1,0) \otimes L_{\hat{\mathfrak{sl}}_2}(1,0) = L(1/2,0) \otimes L_{\hat{\mathfrak{sl}}_2}(2,0) \oplus L(1/2,1/2) \otimes L_{\hat{\mathfrak{sl}}_2}(2,2),\\
&L_{\hat{\mathfrak{sl}}_2}(1,1) \otimes L_{\hat{\mathfrak{sl}}_2}(1,1) = L(1/2,0) \otimes L_{\hat{\mathfrak{sl}}_2}(2,2) \oplus L(1/2,1/2) \otimes L_{\hat{\mathfrak{sl}}_2}(2,0),\\
&L_{\hat{\mathfrak{sl}}_2}(1,0) \otimes L_{\hat{\mathfrak{sl}}_2}(1,1) \oplus L_{\hat{\mathfrak{sl}}_2}(1,1) \otimes L_{\hat{\mathfrak{sl}}_2}(1,0)\\
&= \Big(L(1/2,1/16) \otimes L_{\hat{\mathfrak{sl}}_2}(2,1)\Big)^+ \oplus \Big(L(1/2,1/16) \otimes L_{\hat{\mathfrak{sl}}_2}(2,1)\Big)^- .
\end{aligned} \tag{4.10}$$

Plugging (4.10) into (4.7), we obtain the following.

Proposition 4.5 ([3, 17])**.** *As $L(c^0_n, 0) \otimes L(c^0_{n+1}, 0)$-modules, we have the following decompositions.*

$$M(\varepsilon, h^{(n)}_{r,s,0}) = \bigoplus_{\substack{1 \le j \le n+2 \\ j \equiv (r+s)/2 + 2\varepsilon\,(2)}} L(c^0_n, h^{(n)}_{r,j}) \otimes L(c^0_{n+1}, h^{(n+1)}_{j,s}), \quad \varepsilon = 0, \frac{1}{2},$$

$$M(1/16, \pm\Delta^{(n)}_{r,s}) = \bigoplus_{\substack{1 \le j \le n+2 \\ j \equiv (r+s\pm 1)/2\,(2)}} L(c^0_n, h^{(n)}_{r,j}) \otimes L(c^0_{n+1}, h^{(n+1)}_{j,s}).$$

Proof. The decomposition of $M(\varepsilon, h^{(n)}_{r,s,0})$ with $\varepsilon = 0, 1/2$ is straightforward (cf. [3]). By (4.6) and (4.7) we obtain

$$M(1/16, \Delta^{(n)}_{r,s}) \oplus M(1/16, -\Delta^{(n)}_{r,s}) = \bigoplus_{1\le j\le n+2} L(c^0_n, h^{(n)}_{r,j}) \otimes L(c^0_{n+1}, h^{(n+1)}_{j,s}).$$

By (4.3) and (4.9), the top levels of $M(1/16, \pm\Delta^{(n)}_{r,s})$ contain those of

$$L(c^0_n, h^{(n)}_{r,(r+s\pm1)/2}) \otimes L(c^0_{n+1}, h^{(n+1)}_{(r+s\pm1)/2,s}),$$

respectively. Since

$$A(1/2, c^1_n) = M(0, h^{(n)}_{1,1,0}) = \bigoplus_{\substack{1\le j\le n+2\\ j\equiv 1\,(2)}} L(c^0_n, h^{(n)}_{1,j}) \otimes L(c^0_n, h^{(n+1)}_{j,1}),$$

the decompositions of $M(1/16, \pm\Delta^{(n)}_{r,s})$ are determined by the fusion rules of $L(c^0_n, 0)$ and $L(c^0_{n+1}, 0)$-modules in (2.2) as in the assertion. □

4.3. Automorphisms

We will determine the group generated by Miyamoto involutions of $c = c^0_n$ and $c = c^0_{n+1}$ Virasoro vectors in $A(1/2, c^1_n)$.

Theorem 4.6. *Suppose a VOA V contains a sub VOA U isomorphic to $A(1/2, c^1_n)$. Let e, u and v be $c = c^0_1$, $c = c^0_n$ and $c = c^0_{n+1}$ Virasoro vectors of U given by* (4.3)*, respectively. Then the following hold.*

(1) $[\tau_u, \tau_v] = [\tau_{\sigma_e u}, \tau_{\sigma_e v}] = 1$ *in* $\mathrm{Aut}(V)$ *and* τ_e *centralizes* $\langle \tau_u, \tau_v, \tau_{\sigma_e u}, \tau_{\sigma_e v}\rangle$.

(2) *If n is even then* $\tau_u\tau_v = \tau_{\sigma_e u}\tau_{\sigma_e v} = \tau_e$, $\tau_u = \tau_{\sigma_e u}$ *and* $\tau_v = \tau_{\sigma_e v}$ *in* $\mathrm{Aut}(V)$.

(3) *If n is odd then* $\tau_u = \tau_v$, $\tau_{\sigma_e u} = \tau_{\sigma_e v}$ *and* $\tau_u\tau_{\sigma_e u} = \tau_v\tau_{\sigma_e v} = \tau_e$ *in* $\mathrm{Aut}(V)$.

(4) $\langle \tau_e, \tau_u, \tau_v, \tau_{\sigma_e u}, \tau_{\sigma_e v}\rangle$ *is an elementary abelian* 2*-group of rank at most* 2.

Proof. Since $\omega = u + v = \sigma_e u + \sigma_e v$ are Virasoro frames of U, we have $[\tau_u, \tau_v] = [\tau_{\sigma_e u}, \tau_{\sigma_e v}] = 1$ in $\mathrm{Aut}(V)$. Since e is of σ-type on U, τ_e is trivial on U. Then it follows from $\tau_y = \tau_{\tau_e y} = \tau_e\tau_y\tau_e$ for $y \in \{u, v, \sigma_e u, \sigma_e v\}$ that τ_e centralizes $\langle \tau_u, \tau_v, \tau_{\sigma_e u}, \tau_{\sigma_e v}\rangle$. By (1) of Theorem 4.3, V is a direct sum of irreducible $A(1/2, c^1_n)$-submodules. By definition, Miyamoto involutions preserve each irreducible $A(1/2, c_n^1)$-modules. If n is even, it follows from the decompositions in Proposition 4.5 that τ_u and τ_v act on $M(\varepsilon, h_{r,s,0}^{(n)})$, $\varepsilon = 0$, $1/2$ and $M(1/16, \Delta_{r,s}^{(n)})$ as $(-1)^{r+1}$ and $(-1)^{s+1}$, respectively. Then the product $\tau_u\tau_v = (-1)^{r+s}$ is trivial on $M(\varepsilon, h_{r,s,0}^{(n)})$, $\varepsilon = 0$, $1/2$, and is equal to -1 on $M(1/16, \Delta_{r,s}^{(n)})$ since $r \equiv s \bmod 2$ for the NS-sectors and $r \not\equiv s \bmod 2$ for the R-sectors. On the other hand, τ_e is trivial on $M(\varepsilon, h_{r,s,0}^{(n)})$, $\varepsilon = 0$, $1/2$ and acts as -1 on $M(1/16, \Delta_{r,s}^{(n)})$. Thus $\tau_u\tau_v = \tau_e$ in $\mathrm{Aut}(V)$. Since $M(\varepsilon, h_{r,s,0}^{(n)})$ is σ_e-invariant and $M(1/16, \pm\Delta_{r,s}^{(n)})$ are mutually σ_e-conjugate, we have $\tau_u = \tau_{\sigma_e u}$ and $\tau_v = \tau_{\sigma_e v}$ in $\mathrm{Aut}(V)$.

If n is odd then both τ_u and τ_v act on $L(c_n^0, h_{r,j}^{(n)}) \otimes L(c_{n+1}^0, h_{j,s}^{(n+1)})$ by $(-1)^{j+1}$ and it follows from the decompositions in Proposition 4.5 that $\tau_u = \tau_v$ in $\mathrm{Aut}(V)$. Each summand $L(c_n^0, h_{r,j}^{(n)}) \otimes L(c_{n+1}^0, h_{j,s}^{(n+1)})$ of $M(\varepsilon, h_{r,s,0}^{(n)})$ with $\varepsilon = 0$, $1/2$ satisfies $j \equiv (r+s)/2 + 2\varepsilon \bmod 2$ and hence $\tau_u = \tau_v$ acts as $(-1)^{(r+s)/2+2\varepsilon}$ on $M(\varepsilon, h_{r,s,0}^{(n)})$. Since $M(\varepsilon, h_{r,s,0}^{(n)})$ is σ_e-stable, its decomposition with respect to $\langle\sigma_e u\rangle \otimes \langle\sigma_e v\rangle$ is isomorphic to that with respect to $\langle u\rangle \otimes \langle v\rangle$. Therefore $\tau_{\sigma_e u} = \tau_{\sigma_e v}$ also satisfies $\tau_{\sigma_e u} = \tau_{\sigma_e v} = (-1)^{(r+s)/2+2\varepsilon}$ on $M(\varepsilon, h_{r,s,0}^{(n)})$. Thus $\tau_u\tau_{\sigma_e u} = \tau_v\tau_{\sigma_e v} = 1 = \tau_e$ on $M(\varepsilon, h_{r,s,0}^{(n)})$. On the other hand, each summand $L(c_n^0, h_{r,j}^{(n)}) \otimes L(c_{n+1}^0, h_{j,s}^{(n+1)})$ of $M(1/16, \Delta_{r,s}^{(n)})$ satisfies $j \equiv (r+s+1)/2 \bmod 2$ and hence $\tau_u = \tau_v$ acts as $(-1)^{(r+s+1)/2}$ on $M(1/16, \Delta_{r,s}^{(n)})$. Since the decomposition of $M(1/16, \Delta_{r,s}^{(n)})$ as a $\langle\sigma_e u\rangle \otimes \langle\sigma_e v\rangle$-module is isomorphic to that of $M(1/16, -\Delta_{r,s}^{(n)})$ as a $\langle u\rangle \otimes \langle v\rangle$-module, we have $\tau_{\sigma_e u} = \tau_{\sigma_e v} = (-1)^{(r+s-1)/2}$ on $M(1/16, \Delta_{r,s}^{(n)})$. Therefore $\tau_u\tau_{\sigma_e u} = \tau_v\tau_{\sigma_e v} = (-1)^{r+s} = -1 = \tau_e$ on $M(1/16, \Delta_{r,s}^{(n)})$. This completes the proof. $\square$

Remark 4.7. (3) of Theorem 4.6 is also proved in [12] in the case of $n = 1$.

Theorem 4.8. *Suppose n is odd and a VOA V contains a sub VOA U isomorphic to $A(1/2, c_n^1)$. Let e, u and v be $c = c_1^0$, $c = c_n^0$ and $c = c_{n+1}^0$ Virasoro vectors of U given by* (4.3)*, respectively. Then v is of σ-type on the commutant* $\mathrm{Com}_V\langle u\rangle$*. Moreover, σ_v and τ_e define the same automorphism of* $\mathrm{Com}_V\langle u\rangle$.

Proof. Let X be an irreducible $\langle v\rangle$-submodule of V. Then $X \subset \mathrm{Com}_V\langle u\rangle$ if and only if there exists an irreducible $A(1/2, c_n^1)$-submodule M of V containing a $\langle u\rangle \otimes \langle v\rangle$-submodule isomorphic to $L(c_n^0, 0) \otimes X$. By Theorem 4.4 and Proposition 4.5, $X \cong L(c_{n+1}^0, h_{1,s}^{(n+1)})$ and $M \cong M(\varepsilon, h_{1,s,0}^{(n)})$ with $s \equiv 4\varepsilon + 1 \bmod 4$ for $\varepsilon = 0$, $1/2$ or $M \cong M(1/16, \pm\Delta_{1,s}^{(n)})$ with $s \equiv 3 \pm 1 \bmod 4$.

Therefore, v is of σ-type on $\mathrm{Com}_V\langle u\rangle$. From the above possibility of X and M, both σ_v and τ_e define the same automorphism on the commutant $\mathrm{Com}_V\langle u\rangle$. This completes the proof. □

Remark 4.9. Theorem 4.8 is also proved in [12] in the case of $n = 1$ and in [13] in the case of $n = 3$.

Appendix

In this appendix we prove the $\mathbb{Z}_2$-rationality of $L_{\mathrm{NS}}(c_n^1, 0)$. The classification of irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules is accomplished in [21] in the category of superspaces. Our argument is almost the same as in (loc. cit.) but we do not assume the superspace structure on $\mathbb{Z}_2$-twisted modules. First we recall the $\mathbb{Z}_2$-twisted Zhu algebra of an SVOA. Let $V = V^{[0]} \oplus V^{[1]}$ be an SVOA such that $V^{[i]}$ has $(\mathbb{Z} + i/2)$-grading. For homogeneous a and $b \in V$, we define

$$\begin{aligned} a \underset{\mathrm{tw}}{*} b &:= \mathrm{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}(a)}}{z} = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(a)}{i} a_{(i-1)}b, \\ a \underset{\mathrm{tw}}{\circ} b &:= \mathrm{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(a)}{i} a_{(i-2)}b, \end{aligned} \tag{A.1}$$

and extend bilinearly. We then set

$$A_{\mathrm{tw}}(V) := V/O_{\mathrm{tw}}(V), \qquad O_{\mathrm{tw}}(V) := \mathrm{Span}_{\mathbb{C}}\{a \underset{\mathrm{tw}}{\circ} b \mid a, b \in V\}. \tag{A.2}$$

We denote the class $a + O_{\mathrm{tw}}(V)$ of $a \in V$ in $A_{\mathrm{tw}}(V)$ by $[a]$. It is shown in [7] that $A_{\mathrm{tw}}(V)$ equipped with the product $\underset{\mathrm{tw}}{*}$ in (A.1) forms a unital associative algebra such that $[\mathbb{1}]$ is the unit and $[\omega]$ is in the center. The twisted Zhu algebra $A_{\mathrm{tw}}(V)$ determines irreducible $\mathbb{Z}_2$-twisted representations.

Theorem A.1 ([27, 7]). *Let $V = V^{[0]} \oplus V^{[1]}$ be an SVOA such that $V^{[i]}$ has $(\mathbb{Z} + i/2)$-grading.*

(1) *Let M be a $\mathbb{Z}_2$-twisted V-module and $\Omega(M)$ its top level. Then the zero-mode $\mathrm{o}(a) = a_{(\mathrm{wt}(a)-1)}$ defines a representation of $A_{\mathrm{tw}}(V)$ on $\Omega(M)$.*

(2) *Let N be an irreducible $A_{\mathrm{tw}}(V)$-module. Then there exists the unique irreducible $\mathbb{Z}_2$-twisted V-module $\tilde{N}$ such that its top level is isomorphic to N as $A_{\mathrm{tw}}(V)$-modules.*

(3) *There is a one-to-one correspondence between irreducible* $\mathbb{Z}_2$*-twisted* V*-modules and irreducible* $A_{\rm tw}(V)$*-modules.*

We classify irreducible $\mathbb{Z}_2$-twisted modules over the $N=1$ Virasoro SVOA $L_{\rm NS}(c_n^1,0)$ based on Theorem A.1. We first consider the $\mathbb{Z}_2$-twisted Zhu algebra $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ of the universal $N=1$ Virasoro SVOA $\overline{M}_{\rm NS}(c,0) = M_{\rm NS}(c,0)/\langle G(-1/2)\mathbb{1}\rangle$. Note that $\overline{M}_{\rm NS}(c,0)$ has a linear basis

$$L(-n_1)\cdots L(-n_i)G(-r_1)\cdots G(-r_j)\mathbb{1},\ n_1\geq\cdots\geq n_i\geq 2,\ r_1>\cdots>r_j\geq\frac{3}{2}. \tag{A.3}$$

Images of Virasoro descendants in $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ are easy to compute.

Lemma A.2. $[L(-n)a]=(-1)^n(n-1)[a\underset{\rm tw}{*}\omega]+(-1)^n[L(0)a]$ *for* $n\geq 1$.

Proof. See (4.2) of [25]. □

For odd elements we have the following recursion.

Lemma A.3. $[G(-r)a]=-\sum_{s<r}\binom{3/2}{r-s}[G(-s)a]$ *for* $r\geq 5/2$.

Proof. By Lemma 2.1.2 of [27], for any $n\geq 0$ one has

$$\mathrm{Res}_z Y(\tau,z)a\frac{(1+z)^{3/2}}{z^{2+n}}=\sum_{i\geq 0}\binom{3/2}{i}\tau_{(-2-n+i)}a\in O_{\rm tw}\left(\overline{M}_{\rm NS}(c,0)\right).$$

Noting $\tau_{(-2-n+i)}=G(-5/2-n+i)$, we obtain the lemma. □

Lemma A.4. $[\tau]^2=[\omega]-\frac{c}{24}[\mathbb{1}]$.

Proof. By definition, one has

$$\begin{aligned}\tau\underset{\rm tw}{*}\tau &= \sum_{i\geq 0}\binom{3/2}{i}\tau_{(i-1)}\tau=\sum_{i\geq 0}\binom{3/2}{i}G(i-3/2)G(-3/2)\mathbb{1}\\ &= G(-3/2)^2\mathbb{1}+\binom{3/2}{i}G(-1/2)G(-3/2)\mathbb{1}+\binom{3/2}{3}G(3/2)G(-3/2)\mathbb{1}\\ &= L(-3)\mathbb{1}+3L(-2)\mathbb{1}-\frac{c}{24}\mathbb{1}.\end{aligned}$$

Then by Lemma A.2, one obtains

$$[\tau]^2=[\tau\underset{\rm tw}{*}\tau]=[L(-3)\mathbb{1}]+3[L(-2)\mathbb{1}]-\frac{c}{24}[\mathbb{1}]=[\omega]-\frac{c}{24}[\mathbb{1}].$$

This completes the proof. □

Proposition A.5. *The $\mathbb{Z}_2$-twisted Zhu algebra $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ is isomorphic to the polynomial algebra in $[\tau]$. More precisely, the class*

$$[L(-n_1)\cdots L(-n_i)G(-r_1)\cdots G(-r_j)\mathbb{1}],\ n_1\geq\cdots\geq n_i\geq 2,\ r_1>\cdots>r_j\geq\frac{3}{2},$$

corresponds to a polynomial in $[\tau]$ of degree at most $2i+j$.

Proof. We prove that $[L(-n_1)\cdots L(-n_i)G(-r_1)\cdots G(-r_j)\mathbb{1}]$ is equivalent to a polynomial in $[\tau]$ of degree at most $2i+j$ by induction on the length $i+j$. By Lemma A.2 there exists a polynomial $f(X)\in\mathbb{C}[X]$ of degree at most i such that

$$[L(-n_1)\cdots L(-n_i)G(-r_1)\cdots G(-r_j)\mathbb{1}] = f([\omega])\cdot[G(-r_1)\cdots G(-r_j)\mathbb{1}].$$

It follows from Lemma A.4 that $f([\omega])$ is equivalent to a polynomial in $[\tau]$ of degree at most $2i$. By Lemma A.3 we can rewrite the class $[G(-r_1)\cdots G(-r_j)\mathbb{1}]$ into a sum of shorter monomials in $L(-n)$ and $G(-r)$. Note that an even element $L(-n)$ appears when we rewrite a pair of two odd elements $G(-r)G(-s)$ by taking a commutator. In the rewriting procedure $L(-n)$ increases the degree of $[\tau]$ in the terminal form at most two whereas $G(-r)$ does at most one. So we can apply the induction and $[G(-r_1)\cdots G(-r_j)\mathbb{1}]$ is equivalent to a polynomial in $[\tau]$ of degree at most j. Therefore, every element of $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ is equivalent to a polynomial in $[\tau]$ of degree as described in the assertion. This shows that there exists an epimorphism from $\mathbb{C}[X]$ to $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ defined by $X\longmapsto[\tau]$. We prove that this is the isomorphism. For any $d\in\mathbb{C}$, the zero-mode $\mathrm{o}(\tau)=G(0)$ has a minimal polynomial $X-d$ on the top level of the Verma module $M_{\rm R}(c,d)$ over the Ramond algebra. Hence, $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ has an irreducible representation on which $[\tau]$ acts by an arbitrary scalar. This implies $A_{\rm tw}\big(\overline{M}_{\rm NS}(c,0)\big)$ is indeed isomorphic to a polynomial algebra in $[\tau]$. □

Now we describe the twisted Zhu algebra of $L_{\rm NS}(c^1_n,0)$.

Theorem A.6 ([21])**.** *The $\mathbb{Z}_2$-twisted Zhu algebra of $L_{\mathrm{NS}}(c_n^1, 0)$ is isomorphic to a quotient of a polynomial ring $\mathbb{C}[X]$ modulo the following polynomial.*

$$\prod_{\substack{1\le r\le n+1\\ 1\le s\le n+3\\ r-s\equiv 1(2)}} \left(X - \Delta_{r,s}^{(n)}\right).$$

The isomorphism is given by $[\tau] \longmapsto X$.

Proof. By Proposition A.5, there is a polynomial $f(X)$ such that

$$A_{\mathrm{tw}}(L_{\mathrm{NS}}(c_n^1, 0)) \cong \mathbb{C}[X]/\langle f(X)\rangle.$$

It follows from the structure of the Verma modules over the Neveu-Schwarz algebra that the maximal ideal of $M_{\mathrm{NS}}(c_n^1, 0)$ is generated by two singular vectors, $G(-1/2)v_{c_n^1,0}$ and the one of weight $(n+1)(n+3)/2$ (cf. [10]). Let x be the singular vector of $\overline{M}_{\mathrm{NS}}(c_n^1, 0)$ of weight $(n+1)(n+3)/2$. By Proposition A.5 there exists a polynomial $g(X)$ such that $g([\tau]) = [x]$ in $A_{\mathrm{tw}}\big(\overline{M}_{\mathrm{NS}}(c_n^1, 0)\big)$. It is clear that $f(X)$ divides $g(X)$ since $[x] = g([\tau]) = 0$ in $A_{\mathrm{tw}}(L_{\mathrm{NS}}(c_n^1, 0))$. If n is odd then the weight $(n+1)(n+3)/2$ is an even integer and x is an even element, whereas if n is even then $(n+1)(n+3)/2$ is a half-integer and x is an odd element. By this we see that the possible longest monomial of the form (A.3) in x is $L(-2)^{\frac{1}{4}(n+1)(n+3)}\mathbb{1}$ if n is odd and $L(-2)^{\frac{1}{4}n(n+4)}G(-3/2)\mathbb{1}$ if n is even, respectively. Therefore, by Proposition A.5, the degree of $g(X)$ is at most $(n+1)(n+3)/2$ if n is odd and $n(n+4)/2+1$ if n is even, respectively. On the other hand, by the GKO construction [9] (cf. Eq. (4.7)), we have irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules $L_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)})$ for $1 \le r \le n+1$, $1 \le s \le n+3$ and $r - s \equiv 1$ (2). The zero-mode $\mathrm{o}(\tau) = G(0)$ acts on the top level of $L_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)})$ by $\Delta_{r,s}^{(n)}$. It is straightforward to see that $\Delta_{r,s}^{(n)}$ with $1 \le r \le n+1$, $1 \le s \le n+3$, $r - s \equiv 1$ (2) are mutually distinct. (Note that $h_{r,s,1/2}^{(n)} = h_{n+2-r,n+4-s,1/2}^{(n)}$ but $\Delta_{n+2-r,n+4-s}^{(n)} = -\Delta_{r,s}^{(n)}$.) Therefore, $f(X)$ is divisible by the polynomial

$$\prod_{\substack{1\le r\le n+1\\ 1\le s\le n+3\\ r-s\equiv 1(2)}} (X - \Delta_{r,s}^{(n)}).$$

The degree of the polynomial above is $(n+1)(n+3)/2$ if n is odd and $n(n+4)/2+1$ if n is even. Therefore, by comparing degrees, we see that both $g(X)$ and $f(X)$ are scalar multiples of the polynomial above. This completes the proof. $\square$

As a corollary, we obtain the classification of irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules.

Theorem A.7 ([21]). *The irreducible $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-modules are $L_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)})$, $1 \le r \le n+1$, $1 \le s \le n+3$, $r-s \equiv 1 \bmod 2$.*

Theorem A.8. *$L_{\mathrm{NS}}(c_n^1, 0)$ is $\mathbb{Z}_2$-rational.*

Proof. In this proof we use the notation as in Section 2.2. Since the even part $L_{\mathrm{NS}}(c_n^1, 0)^{[0]}$ is C_2-cofinite by (2) of Theorem 4.3, every $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-module is $\mathbb{N}$-gradable by [1, 23]. Let $M = \oplus_{n\ge 0} M(n)$ be an $\mathbb{N}$-graded $\mathbb{Z}_2$-twisted $L_{\mathrm{NS}}(c_n^1, 0)$-module with non-trivial top level $M(0)$. Then by Theorems A.1 and A.6 $M(0)$ is a semisimple $A_{\mathrm{tw}}(L_{\mathrm{NS}}(c_n^1, 0))$-module and is a direct sum of eigenvectors of $\mathrm{o}(G(-3/2)\mathbb{1}) = G(0)$ with eigenvalues $\Delta_{r,s}^{(n)}$, $1 \le r \le n+1$, $1 \le s \le n+3$, $r-s \equiv 1 \bmod 2$. We shall show that every eigenvector of $G(0)$ generates an irreducible $\mathbb{Z}_2$-twisted submodule. Let x be a $G(0)$-eigenvector of M with eigenvalue $\Delta_{r,s}^{(n)}$ and let X be the submodule generated by x. Then up to linearity there is a unique epimorphism $\pi : M_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)}) \longrightarrow X$. It is shown in Theorem 4.2 of [10] that every submodule of $M_{\mathrm{R}}(c_n^1, \Delta_{r,s}^{(n)})$ is generated by singular vectors[2]. For $1 \le r \le n+1$ and $1 \le s \le n+3$, set

$$h_{r,s,1/2}^{(n)}(i) := \begin{cases} \dfrac{((i(n+2)+r)(n+4)-s(n+2))^2-4}{8(n+2)(n+4)} + \dfrac{1}{16} & \text{if } i \equiv 0 \bmod 2, \\[2ex] \dfrac{(((i-1)(n+2)+r)(n+4)+s(n+2))^2-4}{8(n+2)(n+4)} + \dfrac{1}{16} & \text{if } i \equiv 1 \bmod 2. \end{cases}$$

It is also shown in (loc. cit.) that the $L(0)$-weights of singular vectors of $M_{\mathrm{R}}(c_n^1, \Delta_{r,s})$ belong to the set

$$H_{r,s,1/2}^{(n)} = \left\{ h_{r,s,1/2}^{(n)}(i) \;\middle|\; i \in \mathbb{Z} \setminus \{0\} \right\}.$$

[2] In the Ramond case the Verma module of central charge c and highest weight h in [10] corresponds to $M_{\mathrm{R}}(c, \sqrt{h-c/24}) \oplus M_{\mathrm{R}}(c, -\sqrt{h-c/24})$ if $h \ne c/24$, and to an extension of $M_{\mathrm{R}}(c, 0)$ by itself if $h = c/24$.

Suppose X is reducible and we have an $L(0)$-homogeneous singular vector y of X. Then by Theorem A.6 the $L(0)$-weight of y is equal to $h^{(n)}_{r',s',1/2} = (\Delta^{(n)}_{r',s'})^2 + c^1_n/24$ for some $1 \le r' \le n+1$, $1 \le s' \le n+3$ and $r'-s' \equiv 1 \bmod 2$. However, for $1 \le r, r' \le n+1$ and $1 \le s, s' \le n+3$, it is directly verified that $h^{(n)}_{r',s',1/2} \notin H^{(n)}_{r,s,1/2}$. Thus X has no singular vector and is irreducible. Now the theorem follows from Proposition 5.11 of [5]. □

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