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GENERALIZED TWISTED QUANTUM DOUBLES OF A FINITE GROUP AND RATIONAL ORBIFOLDS

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Dedicated to Bob Griess on the occasion of his 71^{st} *birthday.*

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Abstract

In previous work the authors introduced a new class of modular quasi-Hopf algebra $D^\omega(G,A)$, associated to a finite group G, a central subgroup A and a 3-cocycle $\omega\in Z^3(G,\mathbb{C}^\times)$. In the present paper we propose a description of the class of orbifold models of rational vertex operator algebras whose module category is tensor equivalent to $D^\omega(G,A)$-mod. The paper includes background on quasi-Hopf algebras and a discussion of some relevant orbifolds.

1. Introduction

Since its introduction by Dijkgraaf, Pasquier and Roche [4], the *twisted quantum double* has been a source of inspiration in the related areas of *quasiHopf algebras*, *modular tensor categories* (MTC), and *orbifold models* of holomorphic vertex operator algebras (VOA). It follows from the work [20] of Müger that the module category $D^\omega(G)$-mod of a twisted quantum double of a finite group G (notation and further details are provided below) is a MTC. Even before the idea of a MTC existed, DPR had more-or-less

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conjectured [4] that if V is a holomorphic VOA admitting G as a group of automorphisms then V^G-mod is equivalent to $D^\omega(G)$-mod for some 3-cocycle $\omega\in Z^3(G,\mathbb{C}^\times)$. Kirillov considered the conjecture from the perspective of fusion categories [14]. His work would imply the conjecture if one can show that all the g-twisted modules of V form a G-graded fusion category. This condition amounts to the rationality of V^G. With recent advances, this conjecture is now known for a large family of groups G. The recent work [19] of Miyamoto and Carnahan proves that V^G is rational for any solvable group G. A complete solution to the conjecture seems to be within reach.

On the other hand, much less is known in the case of *rational orbifolds*, i.e., orbifolds V^G where V is a *rational VOA*, but not necessarily holomorphic. Already in [4], the authors asked for a description of the $c=1$ ADE orbifolds V^G where $V=L(sl_2,1)$ is the affine algebra of level 1 associated to sl_2 (alternatively, the rank 1 lattice theory $V_{\sqrt{2}\mathbb{Z}}$ associated with the A_1 root lattice) and G a finite subgroup of $SO(3,\mathbb{R})$. Until recently there was no really satisfactory answer to this question. Fusion rules and S- and T-matrices for these theories have long been known (the icosahedral case proved to be particularly intractable) but a quasiHopf algebra replacing the twisted double was missing. More generally, there does not seem to be even a conjectural description of V^G-mod in the literature for any reasonably substantial class of rational orbifolds beyond the holomorphic case.

In our recent work [15] we introduced a *generalization* of the twisted quantum double, denoted by $D^\omega(G,A)$, which is a certain quasiHopf algebra quotient of $D^\omega(G)$ obtained from a central subgroup A of G. (The case $A=1$ reduces to $D^\omega(G)$.) We gave necessary and sufficient conditions that $D^\omega(G,A)$-mod is a MTC. The purpose of the present paper is to present a conjectural description of those rational orbifolds V^G whose module category is equivalent to some $D^\omega(G,A)$-mod, and to discuss a few examples. These include the ADE examples mentioned above, thus providing an answer to the question raised by DPR.

The paper is organized as follows. In Section 2 we present a general discussion of quasiHopf algebras, with emphasis on the construction and properties of $D^\omega(G,A)$. In Section 3 we state our conjecture relating $D^\omega(G,A)$ to certain rational orbifolds, and in Section 4 we consider some illustrative examples.

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2. QuasiHopf Algebras and $D^\omega(G,A)$

In this Section we provide some background, taken from [15], about the quasiHopf algebras $D^\omega(G,A)$, which we call *generalized twisted quantum doubles*.

We use the following notation for a finite group G. $\widehat{G}=\mathrm{Hom}(G,\mathbb{C}^\times)$ is the group of *characters* of G, $x^g=g^{-1}xg$ $(x,g\in G)$ is right conjugation in G, the *centralizer* of x in G is $C_G(x):=\{g\in G|x^g=x\}$, $Z(G):=\cap_x C_G(x)$ is the *center* of G.

We take our base field to be the complex numbers $\mathbb{C}$. A *quasiHopf algebra* is a tuple $(H,\Delta,\epsilon,\phi,\alpha,\beta,S)$, where H is a unital algebra and $\Delta{:}H\to H\otimes H$ an algebra morphism that is *quasicoassociative* in the sense that there is a map ϕ (the *Drinfeld associator*) making the following diagram commutative:

$$\begin{array}{ccc} & H\otimes H & \\ {}^{\Delta\otimes Id}\swarrow & & \searrow^{Id\otimes\Delta} \\ (H\otimes H)\otimes H & \xrightarrow[\phi]{} & H\otimes(H\otimes H) \end{array} \tag{2.1}$$

S is the *antipode*, ϵ the *counit*, and $\alpha,\beta\in H$ are certain distinguished elements. We generally suppress all reference to these elements of a quasiHopf algebra; this will not impair the reader's understanding of what follows. One also requires ϕ to satisfy some *cocycle* conditions in the form of certain diagrams involving fourfold tensor products of H that are required to commute. Again we will generally suppress such details. A Hopf algebra is a quasiHopf algebra for which $\alpha=\beta=1$ and $\phi=1\otimes 1\otimes 1$. For further background on quasiHopf algebras, see [10], [13] and [17].

One of the great virtues of a quasiHopf algebra H is that the category H-mod of (finite-dimensional) H-modules is a finite tensor category, though not necessarily a MTC.

Fix a finite group G. The group algebra $\mathbb{C}G$ is a familiar example of a Hopf algebra, the coproduct being defined by $\Delta(g)=g\otimes g$ $(g\in G)$. Dualizing, the dual group algebra $\mathbb{C}^G$ has basis e_g $(g{\in}G)$ *dual* to the basis of group elements in G. It is a Hopf algebra with product and coproduct defined by

$$e_g e_h{=}\delta_{g,h}e_g, \qquad \Delta{:}e_g \mapsto \sum_{ab=g} e_a \otimes e_b.$$

Now fix a normalized multiplicative 3-cocycle $\omega{\in}Z^3(G,\mathbb{C}^\times)$. A basic example of a quasiHopf algebra is the *twisted dual group algebra* $\mathbb{C}^G_\omega$, obtained from $\mathbb{C}^G$ just by replacing $1\otimes 1\otimes 1$ by a more interesting Drinfeld associator defined by multiplication by

$$\phi{:=}\sum_{a,b,c}\omega(a,b,c)^{-1}e_a\otimes e_b\otimes e_c. \tag{2.2}$$

Here, the cocycle conditions amount to the 3-cocycle identity satisfied by ω.

The *twisted quantum double* $D^\omega(G){=}\mathbb{C}^G_\omega{\otimes}\mathbb{C}G$ occurs as the middle term of a sequence of morphisms of quasiHopf algebras

$$\mathbb{C}^G_\omega \xrightarrow{\quad i \quad} D^\omega(G) \xrightarrow{\quad p \quad} \mathbb{C}G$$

where $i(e_g){=}e_g\otimes 1$, $p(e_g\otimes x){=}\delta_{g,1}x$. $D^\omega(G)$ is itself a quasiHopf algebra with the following product and coproduct:

$$\begin{aligned}(e_g\otimes x)\cdot(e_h\otimes y){=}&\theta_g(x,y)\,\delta_{g^x,h}e_g\otimes xy\\ \Delta{:}e_g\otimes x\mapsto&\sum_{ab=g}\gamma_x(a,b)\,e_a\otimes x\otimes e_b\otimes x.\end{aligned}$$

The scalars $\theta_g(x,y),\gamma_x(a,b)$ are determined by ω as follows:

$$\begin{aligned}\theta_g(x,y){:=}&\frac{\omega(g,x,y)\omega(x,y,g^{xy})}{\omega(x,g^x,y)},\\ \gamma_x(a,b){:=}&\frac{\omega(a,b,x)\omega(x,a^x,b^x)}{\omega(a,x,b^x)}.\end{aligned}$$

We remark that the 2-cochains defined by the θ's and γ's have subtle properties which govern much of the behaviour of $D^\omega(G)$ – for example, the fact that $D^\omega(G)$ really is a quasiHopf algebra, which is not obvious. As another example, if we restrict x,y to $C_G(g)$ then the 2-cochains θ_g,γ_g

coincide and become 2-cocycles. We informally record this as

$$\gamma_g = \theta_g \in Z^2(C_G(g), \mathbb{C}^\times), \tag{2.3}$$

where it is understood that the domains of γ_g, θ_g are here restricted to $C_G(g)$.

Since $D^\omega(G)$-mod is the Drinfeld center of $\mathrm{Vec}(G,\omega)$, the fusion category of G-graded vector space with the associativity given by ω, it follows from [20] that $D^\omega(G)$-mod is a MTC.

We now fix another piece of data, namely a central subgroup $A \subseteq Z(G)$, and introduce

$$D^\omega(G, A) := \mathbb{C}^G_\omega \otimes \mathbb{C}(G/A).$$

Notice that because the conjugation action of A on G is *trivial* then the *identical* formulas used to define the operations in $D^\omega(G)$ still make sense in $D^\omega(G,A)$. When ω is *compatible* with A, one can equip $D^\omega(G,A)$ with a product and coproduct similar to the case of $D^\omega(G)$ so that $D^\omega(G,A)$ is a quasiHopf algebra.

Now it is natural to ask if, for suitable π', there is a commuting diagram of quasiHopf algebras and morphisms

$$\begin{array}{ccccc} \mathbb{C}^G_\omega & \xrightarrow{\ i\ } & D^\omega(G) & \xrightarrow{\ p\ } & \mathbb{C}G \\ {\scriptstyle =}\big\downarrow & & {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \mathbb{C}^G_\omega & \xrightarrow{\ i\ } & D^\omega(G,A) & \xrightarrow{\ \bar{p}\ } & \mathbb{C}(G/A) \end{array} \tag{2.4}$$

where $\pi(x) = xA$ and $\bar{p}(e_g \otimes xA) = \delta_{g,1} xA$ for $x \in G$. In case π' exists, it will satisfy $\pi'(e_g \otimes x) = \lambda e_g \otimes xA$ for a scalar λ that depends on g and x.

Generally there will be no such π', but we can give necessary and sufficient conditions for its existence. To explain this we first consider the *group-like elements* of $D^\omega(G)$. These are the (nonzero) elements $u \in D^\omega(G)$ such that $\Delta(u) = u \otimes u$. As in the case of Hopf algebras, the group-like elements form a multiplicative subgroup $\Gamma^\omega(G) \subseteq D^\omega(G)^\times$ of the group of units. We are more interested in the *central group-like elements*, which is the subgroup $\Gamma^\omega_0(G) \subseteq \Gamma^\omega(G)$ consisting of elements that commute with all elements of $D^{\omega}(G)$. One can show (cf. [17]) that there is a diagram of short exact sequences of groups (actually central extensions)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{G} & \xrightarrow{\ i\ } & \Gamma^{\omega}(G) & \xrightarrow{\ p\ } & B^{\omega}(G) & \longrightarrow & 1 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{\ i\ } & \Gamma_0^{\omega}(G) & \xrightarrow{\ p\ } & Z^{\omega}(G) & \longrightarrow & 1 \end{array}$$

where

$$B^{\omega}(G){:=}\{g{\in}G|\gamma_g{\in}B^2(G,\mathbb{C}^{\times})\}, \quad Z^{\omega}(G){:=}B^{\omega}(G)\cap Z(G),$$

and vertical arrows are containments.

We emphasize that here, unlike the situation of (2.3), in order for g to lie in $B^{\omega}(G)$ it is necessary that the 2-cochain γ_g be a 2-coboundary on G rather than just the centralizer $C_G(g)$. On the other hand, if $g{\in}Z(G)$ then the context of (2.3) pertains, so that $\gamma_g{=}\theta_g$ is always a 2-cocycle on G, and the requirement to belong to $Z^{\omega}(G)$ is that this 2-cocycle is in fact a 2-coboundary.

It is shown in [15] that the existence of the morphism π' in (2.4) is equivalent to the existence of an enlarged diagram of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{G} & \xrightarrow{\ i\ } & \Gamma^{\omega}(G) & \xrightarrow{\ p\ } & B^{\omega}(G) & \longrightarrow & 1 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{\ i\ } & \Gamma_0^{\omega}(G) & \xrightarrow{\ p\ } & Z^{\omega}(G) & \longrightarrow & 1 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \widehat{G} & \xrightarrow{\ i\ } & p^{-1}(A) & \xrightarrow{\ p\ } & A & \longrightarrow & 1 \end{array} \tag{2.5}$$

where vertical maps are again containments and the *lower s.e.s splits.*

What is being asserted here is the following: the central subgroup $A \subseteq Z(G)$ is required to also lie in $B^{\omega}(G)$, i.e., the 2-cocycles θ_g for $g{\in}A$ are 2-coboundaries on G. Moreover, the s.e.s obtained by pulling-back A along p must split.

Once π' is available, it follows that $D^{\omega}(G,A)$-mod is a subcategory of $D^{\omega}(G)$-mod. However, Müger's theorem will generally not hold for $D^{\omega}(G,A)$, that is to say $D^{\omega}(G,A)$-mod is generally *not* a MTC. We will describe necessary and sufficient conditions, established in [15], that make this so.

First we say a bit more about the middle s.e.s in (2.5). Given $g\in Z^{\omega}(G)$ we have $\gamma_g=\theta_g\in B^2(G,\mathbb{C}^{\times})$, so that there is $\tau_g\in C^1(G,\mathbb{C}^{\times})$ satisfying $\delta\tau_g=\theta_g$, i.e.,

$$\tau_g(x)\tau_g(y)=\theta_g(x,y)\tau_g(xy) \qquad (x,y\in G).$$

Because θ_g is a 2-coboundary, the twisted group algebra $\mathbb{C}^{\theta_g}G$ that it defines is isomorphic to the group algebra $\mathbb{C}G$, and τ_g defines a choice of isomorphism

$$\mathbb{C}^{\theta_g}G \xrightarrow{\cong} \mathbb{C}G, \quad x\mapsto \tau_g(x)x.$$

There is no canonical choice of τ_g, but any two of them differ by a character $\chi\in\widehat{G}$. Indeed, we have

$$\Gamma_0^{\omega}(G)=\left\{\sum_{x\in G}\tau_g(x)\chi(x)e_x\otimes g|\chi\in\widehat{G}, g\in Z^{\omega}(G)\right\}.$$

A 2-cocycle $\beta\in Z^2(Z^{\omega}(G),\widehat{G})$ that defines the central extension that is the middle s.e.s in (2.5) is given by the formula

$$\beta(g,h)(k)=\frac{\tau_g(k)\tau_h(k)}{\tau_{gh}(k)}\theta_k(g,h) \qquad (g,h\in Z^{\omega}(G), k\in G).$$

Because we are assuming that the lower s.e.s in (2.5) splits, the restriction of β to A is a 2-coboundary on A. That is, there is a 1-cochain $\nu\in C^1(A,\widehat{G})$ such that

$$\beta(a,b)=\frac{\nu(a)\nu(b)}{\nu(ab)} \qquad (a,b\in A).$$

Now we can show [15] that the formula

$$(a|b)_{\nu}:=\frac{\tau_a(b)\tau_b(a)}{\nu(a)(b)\nu(b)(a)} \tag{2.6}$$

defines a *symmetric bicharacter* $(\ |\ )_{\nu}:A\times A\to\mathbb{C}^{\times}$. Using results of Müger [20], [21], it follows that $D^{\omega}(G,A)$-mod is a MTC if, and only if, $(\ |\ )_{\nu}$

is *nondegenerate*. Actually, $(\ |\ )_\nu$ is the restriction of a natural bicharacter defined on $\Gamma_0^\omega(G)$, however we will not go into this here.

3. Simple Current Orbifolds

In the spirit of the proposal in [4] that the module categories of holomorphic orbifolds coincide with the module categories $D^\omega(G)$-mod, in this Section we describe those rational orbifolds V^G expected to have the property that V^G-mod is equivalent to some $D^\omega(G, A)$. In this setting, the case A=1 will reduce to the holomorphic orbifold case of DPR.

We will generally be lapse about the detailed properties of the VOAs which we consider, but we will be concerned with *rational* VOAs V, which at the very least means that V is a *simple* VOA and V-mod is a fusion category, i.e. a semisimple rigid tensor category with only finitely many simple objects. (See [7] for further background.) In fact we will only need a small subset of such theories, and to explain which ones these are we will review the theory of *simple currents*.

A simple current is a simple (or irreducible) V-module, call it M, with the property that for all simple V-modules N, the tensor product $M \boxtimes N$ is again a simple V-module. Otherwise stated, M represents an element in the Grothendieck group of V-mod, and an object of Frobenius-Perron dimension one. Thanks to the associativity of $\boxtimes$, the distinct (isomorphism classes of) simple currents form a group with respect to tensor product of modules, called the *group of simple currents*. It is an abelian group because V-mod is also braided. The identity element is, of course, the vacuum space V. We are concerned here with rational VOAs V with the property that *every simple V-module is a simple current* i.e., that V-mod is pointed. We call such a V a *simple current VOA*. There are many examples of such theories. In addition to holomorphic VOAs, where V is the *only* simple module, it is well-known that *lattice theories* V_L associated to a positive-definite even lattice L are also simple current VOAs. In this case, the group of simple currents is isomorphic to L^*/L where L^* is the *dual lattice* of L [5].

Given a simple current VOA V, we may form the sum of *all* simple V-modules M to obtain a larger space

$$\widetilde{V} := \oplus_M M$$

$\widetilde{V}$ can be equipped with the structure of an *abelian intertwining algebra* in the sense of Dong-Leowsky [6]. See [8] for further details.

Conjecture. Suppose that V is a simple current VOA with group of simple currents A. Let $F \subseteq \mathrm{Aut}(V)$ be a finite group of automorphisms of V (the *orbifold group*) such that F leaves invariant every simple V-module. Then there is *central extension*

$$1 \longrightarrow A \longrightarrow G \longrightarrow F \longrightarrow 1$$

and a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ such that V^F-mod is equivalent to $D^\omega(G,A)$-mod as modular tensor categories. Conversely, if $D^\omega(G,A)$ exists and $D^\omega(G,A)$-mod is a MTC, then there is a simple current VOA V with group of simple currents A and a group of automorphisms $F=G/A$ such that V^F-mod is tensor equivalent to $D^\omega(G,A)$-mod.

4. Examples

We illustrate the Conjecture by discussing some examples in greater detail. Let the notation be as before.

Example 1. The holomorphic case.
Here, $A=1$ means that V is a holomorphic VOA and $D^\omega(G,A)=D^\omega(G)$. The Conjecture thus reduces that of DPR.

Example 2. The case $F=1$, i.e., $G=A$ and $D^\omega(G,A)=\mathbb{C}_\omega^G$ as quasiHopf algebras.

We saw before that in order for π' to exist (i.e., $D^\omega(G,A)$ is a quasiHopf algebra quotient of $D^\omega(G)$) it is necessary that $A \subseteq Z^\omega(G)$, meaning that each $\theta_g \in B^2(G, \mathbb{C}^\times)$ is a 2-coboundary for all $g \in A$. That is, $\omega \in Z^3(A, \mathbb{C}^\times)_{ab}$ is an *abelian* 3-cocycle on A in the sense of [17], where such cocycles are studied extensively. They are closely related to the abelian cohomology groups introduced by Eilenberg and Maclane [11].

Thus in the case at hand, the Conjecture asserts that V-mod is tensor equivalent to $\mathbb{C}_\omega^A$ for an abelian 3-cocycle ω that is nondegenerate in a suitable sense. Rather than explain what degeneracy means here, we consider a special case in more detail.

Let L be an even lattice with bilinear form $\langle\ ,\ \rangle{:}L{\times}L{\to}\mathbb{Z}$, and let V_L be the corresponding lattice VOA. Then $A{=}L^*/L$. The conjecture says that V_L-mod is tensor equivalent to the dual group algebra of L^*/L twisted by an abelian 3-cocycle $\omega{\in}Z^3(L^*/L,\mathbb{C}^\times)$. The origin of ω is well-known in this case (cf. [6], Chapter 12 and [17], Section 11). Let $s{:}A{\to}L^*$ be a normalized section of the canonical s.e.s

$$1 \longrightarrow L \longrightarrow L^* \longrightarrow A \longrightarrow 1$$

Let $c_0{:}L^*{\times}L^*{\to}\mathbb{C}^\times$ be an *alternating bicharacter* with $c_0(\alpha,\beta){=}(-1)^{\langle\alpha,\beta\rangle}$ $(\alpha,\beta{\in}L)$. Set

$$\omega(a,b,c){=}(-1)^{\langle s(c),s(a)+s(b)-s(a+b)\rangle}c_0(s(c),s(a){+}s(b)-s(a+b)).$$

ω is indeed an abelian 3-cocycle (cf. [17], Proposition 11.1) whose cohomology class is *independent of the choice of bicharacter and section.*

Example 3. The case $|A|{=}2$.
Here, we are discussing simple VOAs V with just two simple modules. One of them is the adjoint module V, the other we denote by M. Roughly speaking, we may think of $\widetilde{V}{=}V{\oplus}M$ as a holomorphic *super* VOA, though this may not conform to some definitions. (This will not matter in the following discussion.)

Next we discuss results of [18] concerning the existence of $D^\omega(G,A)$ in the special case that G is a finite group with *exactly one* subgroup of order 2. We take A to be the unique subgroup of order 2. Let T be a 2-Sylow subgroup of G. It is well-known that T is either cyclic or generalized quaternion. Groups with a unique involution have 2-periodic cohomology by the Artin-Tate theory [1], and for such groups the 2-Sylow subgroup of $H^3(G,\mathbb{C}^\times)$ is *cyclic* of order $|T|$. We call a 3-cocycle $\omega{\in}Z^3(G,\mathbb{C}^\times)$ a 2-*generator* if the corresponding cohomology class $[\omega]{\in}H^3(G,\mathbb{C}^\times)$ has order *divisible by* $|T|$.

In the setting of the previous paragraph, we can prove [18] that π' always exists, so that $D^\omega(G,A)$ is a quasiHopf algebra quotient of $D^\omega(G)$. Moreover, $D^\omega(G,A)$-mod is a MTC if, and only if, ω is a 2-generator.

There are a number of interesting classes of finite groups with a unique subgroup A of order 2. The following lists some of them.

$$\begin{aligned} & SL_2(q) \quad (q \geq 3 \text{ an odd prime power}) \\ & 2.A_6, 2.A_7, 6.A_6, 6.A_7 \\ & \text{binary polyhedral groups} = \text{finite subgroups of } SU_2(\mathbb{C}) \end{aligned} \tag{4.1}$$

Here, $SL_2(q)$ is the group of 2×2 matrices of determinant 1 over the finite field of cardinality q, and $2.A_n$ is the 2-fold perfect central extension of the alternating group A_n. For $n{=}6,7$ Schur discovered that there are exceptional 6-fold perfect central extensions of A_n. There are a few overlaps among these groups: $SL_2(3)$ and $SL_2(5)$ are the binary octahedral and icosahedral groups respectively, and $2.A_6 \cong SL_2(9)$.

Our Conjecture says that for each pair (G,ω) such that G has a unique subgroup of order 2 and $[\omega]$ is a 2-generator of $H^3(G,\mathbb{C}^\times)$, there is a holomorphic super VOA $\widetilde{V}{=}V \oplus M$ (in the sense described before) such that $G{\subseteq}\mathrm{Aut}(\widetilde{V}), G/A{\subseteq}\mathrm{Aut}(V)$ and $V^{G/A}$-mod is tensor equivalent to $D^\omega(G,A)$-mod.

As far as we know, the existence of a $\widetilde{V}$ for most groups G on the list (4.1) is unknown. However, it is well-known that the binary polyhedral groups, and indeed $SU_2(\mathbb{C})$ itself, act on the $c{=}1$ holomorphic super VOA defined by the A_1 root lattice theory $V_{\sqrt{2}\mathbb{Z}}$, also known as the affine algebra $L_{sl_2,1}$ of level 1 associated to the Lie algebra $sl_2(\mathbb{C})$. We review some of the details.

Adopting the lattice perspective, the irreducible modules for $V_{\sqrt{2}\mathbb{Z}}$ consist of the adjoint module $V_{\sqrt{2}\mathbb{Z}}$ and the module $M{=}V_{1/\sqrt{2}+\sqrt{2}\mathbb{Z}}$ corresponding to the two cosets of $\sqrt{2}\mathbb{Z}$ in its dual lattice $\frac{1}{\sqrt{2}}\mathbb{Z}$. Thus $\widetilde{V}{=}V_{\sqrt{2}\mathbb{Z}}{\oplus}V_{1/\sqrt{2}+\sqrt{2}\mathbb{Z}}$ $=V_{\frac{1}{\sqrt{2}}\mathbb{Z}}$. We have $\mathrm{Aut}(V){=}SO_3(\mathbb{R})$, obtained by exponentiating the weight 1 states of V (which form the Lie algebra sl_2). The projective action of this group on M lifts to a linear action of its universal cover $SU_2(\mathbb{C})$, which is the full automorphism group of $\widetilde{V}$.

The binary polyhedral groups G of even order are the finite subgroups of $SU_2(\mathbb{C})$ that contain the center $\{\pm 1\}$. Setting $A{=}\{\pm 1\}$, the preceding discussion shows that $G{\subseteq}\mathrm{Aut}(\widetilde{V})$ and $G/A{\subseteq}\mathrm{Aut}(V)$, so we have rational orbifolds $V^{G/A}$. These theories have been studied extensively in both the physical and mathematical literature, and the S- and T-matrices are known. See, for example, [2] and [3] for type D. Dong and Nagatomo [9] computed the fusion rules for $V^{G/A}$ on the basis of the VOA axioms in all cases except the icosahedral example.

Now we can compute the S- and T-matrices for all $D^{\omega}(G,A)$ whenever G is a binary polyhedral group and ω is a 2-generator. Note that $H^3(G,\mathbb{C}^\times)$ is cyclic of order $|G|$ in this case, so that the number of 2-generators is exactly $|G|/2$. Consider, for example the icosahedral group $G{=}SL_2(5)$. This is a group of order 120, so there are 60 2-generators $[\omega]$ of order $8, 24, 40$ or 120. In the case of $V^{G/A}$-mod discussed above, the S-matrix appears to correspond to an $[\omega]$ of maximal order 120.

In this way we get a large number of MTCs and accompanying modular data, and the Conjecture says that among them we should find the categories $V^{G/A}$-mod. This appears to indeed be the case, since in particular we can find matching S-matrices in all cases.

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