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REVISITING CHARACTER THEORY OF FINITE GROUPS

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Abstract

Two conjectures proposed (old and somewhat new) by the author elsewhere are discussed in this article. One is concerned with a modular version of the regular character of a finite group G, and the second one is concerned with the ratio of the product of the sizes of all conjugacy classes of G and the product of the degrees of all irreducible characters.

1. Conjectures I and II

Let G be a finite group and $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_s\}$ be the set of all inequivalent irreducible characters of G. Furthermore, let $\mathrm{Conj}(G) = \{K_1, K_2, \ldots, K_s\}$ be the set of all conjugacy classes of G. Choose a representative $x_j \in K_j$ for each $j = 1, \ldots, s$ and choose once and for all, $\chi_1 = 1$ and $K_1 = \{1\}$.

The group G acts on the set G by (left) multiplication

$$\rho(g) : G \ni x \to gx \in G.$$

The corresponding (permutation) character ρ_G is called the regular character of G and it satisfies

$$\rho_G = \sum_{i=1}^{s} \chi_i(1)\chi_i.$$

It is trivial to see that $\rho_G(1) = \deg(\rho_G) = |G|$ and $\rho_G(x) = 0$ for all $x \neq 1$. Conversely, if a character χ of G satisfies $\chi(x) = 0$ for all $x \neq 1$, then χ is an integer multiple of the regular character ρ_G.

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Let us consider a modular version of the property of ρ_G. We choose and fix a prime divisor p of the order $|G|$ of G. If an element $x \in G$ has order not divisible by p, then x is called p-regular and in the contrary case x is p-singular. G^0 denotes the set of all p-regular elements of G. By modular representation theory, we have a block decomposition (direct union):

$$\mathrm{Irr}(G) = \mathrm{Irr}(B_1) \cup \mathrm{Irr}(B_2) \cup \cdots \cup \mathrm{Irr}(B_t)$$

where $B_1, B_2, \ldots, B_t$ are the p-blocks of G and $\mathrm{Irr}(B_i)$'s are the subsets of the irreducible characters of G belonging to the respective blocks. Likewise there exists a block decomposition of conjugacy classes of G (direct union, not uniquely determined):

$$\mathrm{Conj}(G) = \mathrm{Conj}(B_1) \cup \mathrm{Conj}(B_2) \cup \cdots \cup \mathrm{Conj}(B_t).$$

In 1981, we proposed the following conjecture:

Conjecture I (Harada [1])**.** Let p be a prime divisor of $|G|$ and J be a subset of $\mathrm{Irr}(G)$. If the linear combination

$$\sum_{\chi \in J} \chi(1)\chi$$

vanishes on all p-singular elements, then the set J is a union of some of p-blocks (i.e. $J = \mathrm{Irr}(B) \cup \mathrm{Irr}(B') \cup \cdots$.) In particular, $\mathrm{Irr}(B_i)$'s are the minimal subset $X \subseteq \mathrm{Irr}(B_i)$ such that the given linear combination vanishes on all p-singular elements.

The conjecture proposes that a (degree weighted) partial sum of the regular character ρ_G vanishes on all the p-singular elements only when it involves all irreducible characters belonging to some union of blocks. In fact, the converse of Conjecture I holds true since for a block B of G, we have

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi = \sum_{\psi \in \mathrm{IBr}(B)} \psi(1)\eta_\psi,$$

where $\mathrm{IBr}(B)$ is the set of all irreducible Brauer characters belonging to the block B and η_ψ is the principal indecomposable character associated with

$\psi \in \mathrm{IBr}(B)$. It is known that every principal indecomposable character η_ψ vanishes on the p-singular elements.

The conjecture was shown to be true if the Sylow p-subgroups of G are cyclic [1]. It has also been affirmatively proved for p-solvable groups in Kiyota-Okuyama (1981) [5], and other related results have been obtained in Kiyota (1984) [4], Ikeda (1994) [3], and in Miyamoto (2010) [6].

In this article, we also propose the following conjecture:

Conjecture II. For every finite group G, the quotient

$$\frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(1)}$$

is a natural integer.

Examples. If $G \cong S_3$, then

$$\frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(1)} = \frac{1\cdot 3\cdot 2}{1\cdot 2\cdot 1} = 3,$$

and if $G \cong A_5$, then

$$\frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(1)} = \frac{1\cdot 15\cdot 20\cdot 12\cdot 12}{1\cdot 3\cdot 3\cdot 4\cdot 5} = 240.$$

Remark. Some supportive evidence by computer for the validity of Conjecture II has been communicated by Chigira, who has also reported the proposed integer given in Conjecture II appears to be an integer multiple of the order of the commutator subgroup G' of the group G.

2. Osima's result and Conjecture I

Theorem (block orthogonality relation). If B is a block, then

$$\sum_{\chi \in \mathrm{Irr}(B)} \overline{\chi(x)}\chi(y) = 0$$

for all $x \in G^0$ and $y \in G - G^0$.

An analogous result holds trivially if B is replaced by a union J of some blocks. Conversely:

Theorem (Osima). Let $J \subseteq \mathrm{Irr}(G)$ and suppose

$$\sum_{\chi \in J} \overline{\chi(x)}\chi(y) = 0$$

for all $x \in G^0$ and $y \in G - G^0$, then J is a union of blocks.

Remark. If we set $x = 1$ in Osima's result, we obtain the proposed Conjecture I.

3. Motivation of Conjecture II

Define two matrices X and W of size $s \times s$ as follows:

$$X = (\chi_i(x_j))_{1 \le i,j \le s},$$

and

$$W = (\omega_{ij})_{1 \le i,j \le s},$$

where

$$\omega_{ij} = \frac{|K_j|\chi_i(x_j)}{\chi_i(1)}.$$

The matrix X is called the character table of G. Every entry of each of X and of W is an algebraic integer.

Example. $G = A_5$.

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & \alpha & \beta \\ 3 & -1 & 0 & \beta & \alpha \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 1 & 15 & 20 & 12 & 12 \\ 1 & -5 & 0 & \alpha' & \beta' \\ 1 & -5 & 0 & \beta' & \alpha' \\ 1 & 0 & 5 & -3 & -3 \\ 1 & 3 & -4 & 0 & 0 \end{pmatrix}$$

$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$; $\alpha' = 2(1+\sqrt{5}), \beta' = 2(1-\sqrt{5})$,
$\det X = 60\sqrt{5}; \det W = 14400\sqrt{5} = 2^6 3^2 5^2 \sqrt{5} = 240 \det X$.

Obviously, the matrix W is written as

$$W = diag.mat.\left(\frac{1}{\chi_1(1)}, \ldots, \frac{1}{\chi_s(1)}\right) \cdot X \cdot diag.mat.\left(|K_1|, \ldots, |K_s|\right),$$

where *diag.mat.* denotes a diagonal maritx. Therefore, we have

$$\det W = \frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(s)} \cdot \det X,$$

and so Conjecture II proposes that $\det W$ is a multiple of $\det X$ in the ring $\mathfrak{O}$ of the algebraic integers.

By the orthogonality relations of the irreducible characters of G, we have

$$\bar{X}^T X = diag.mat.(|C_G(x_1)|, |C_G(x_2)|, \ldots, |C_G(x_s)|) \tag{1}$$

Here the superscript T denotes the transpose. Therefore,

$$||\det X|| = \pm\sqrt{|C_G(x_1)||C_G(x_2)|\cdots|C_G(x_s)|}.$$

Put $\det X = a + bi = a + b\sqrt{-1}, a, b \in \mathbb{R}$. Then, by taking the complex conjugate of each side and counting the number of pairs of complex characters, we obtain

$$\det \overline{X} = a - bi = a + bi \quad or \quad \det \overline{X} = a - bi = -a - bi.$$

In particular, we have $b = 0$ or $a = 0$. In other words, $\det X$ is either real or pure imaginary. Hence,

$$\det X \;=\; \pm\sqrt{|C_G(x_1)||C_G(x_2)|\cdots|C_G(x_s)|}$$

or

$$\det X \;=\; \pm i\sqrt{|C_G(x_1)||C_G(x_2)|\cdots|C_G(x_s)|}.$$

Considering the order of the centralizer of a nonidentity element in the center of each Sylow subgroup of G and $|C_G(x_1)| = |G|$, we see

$$(*) \quad |G| \mid \det X.$$

In the next section, we show

$$(**) \quad |G| \mid \det W.$$

4. $|G|$ divides $\det W$

Whether or not the quotient $|K_1||K_2|\cdots|K_s|/\chi_1(1)\chi_2(1)\cdots\chi_s(1)$ is a rational integer is unknown, and therefore it is not obvious that $\det W$ is divisible by the group order $|G|$ in the ring $\mathfrak{O}$ of algebraic integers. The fact that it is true can be shown as follows.

Theorem 1. *The group order $|G|$ divides $\det W$ in the ring of algebraic integers $\mathfrak{O}$.*

Proof. We note first that every eigenvalue of the matrix W is an algebraic integer, since the characteristic polynomial of W is monic (by definition), all of its coefficients are algebraic integers, and the ring $\mathfrak{O}$ is integrally closed.

Let

$$\mathbf{v}^{\pm} = \begin{pmatrix} 1 \pm \sqrt{|G|} \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

be s-dimensional (column) vectors. Then, using the orthogonality relations of the characters of G, it is straight forward to see that

$$W\mathbf{v}^{+} = \sqrt{|G|}\mathbf{v}^{+} \quad \textit{and} \quad W\mathbf{v}^{-} = -\sqrt{|G|}\mathbf{v}^{-}$$

and so $\mathbf{v}^{\pm}$ are eigenvectors of W with respect to eigenvalues $\pm\sqrt{|G|}$. In particular, characteristic polynomial of W has a factor $x^2 - |G|$ and $\det W$ is divisible by $|G|$ in $\mathfrak{O}$.

Remark. We note that the characteristic polynomial of a matrix depends on ordering of columns and rows. Theorem 1 above shows that if we choose $K_1 = \{1\}$ and $\chi_1 = 1$, then the matrix W always possesses two eigenvalues $\pm\sqrt{|G|}$.

5. Additional eigenvalues of W

As for additional eigenvalues of the matrix W, we have the following:

Theorem 2. *Suppose G possesses an irreducible character χ and a pair of (distinct) conjugacy classes $x \in K$ and $x' \in K'$ satisfying*

(1) $\langle x \rangle = \langle x' \rangle$ *and* $\chi(x) \neq \chi(x')$,

(2) *The field* $\mathbb{Q}(\chi(x))$ *is a quadratic extension of* $\mathbb{Q}$*; and,*

(3) $\eta(x) \in \mathbb{Q}$ *for all irreducible characters η different from χ, χ' where χ' is an algebraic conjugate of χ induced by the Galois automorphism of* $\mathbb{Q}(\chi(x))$.

Then, with some suitable ordering of $\{K_2, \ldots, K_s\}$ and of $\{\chi_2, \ldots, \chi_s\}$, W possesses an eigenvalue $\frac{|K|(\chi(x)-\chi(x'))}{\chi(1)}$.

Example. Let $G = A_5$ and observe the character table of A_5 given previously. Now choose a new ordering of irreducible characters as given below.

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & \alpha & \beta \\ 3 & -1 & 0 & \beta & \alpha \end{pmatrix}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

The corresponding new W is:

$$W = \begin{pmatrix} 1 & 15 & 20 & 12 & 12 \\ 1 & 0 & 5 & -3 & -3 \\ 1 & 3 & -4 & 0 & 0 \\ 1 & -5 & 0 & \alpha' & \beta' \\ 1 & -5 & 0 & \beta' & \alpha' \end{pmatrix}$$

$\alpha' = 2(1+\sqrt{5}), \beta' = 2(1-\sqrt{5})$.

The (new) K_4 is a conjugacy class of an element x_4 of order 5 and K_5 is the conjugacy class of $x_5 = x_4^2$. An irreducible character χ_4 of degree 3 satisfies the properties given above in Theorem 2, since $\chi_4(x_4) = \frac{1+\sqrt{5}}{2}$ and $\chi_4(x_5) = \frac{1-\sqrt{5}}{2}$. Obviously, the 5-dim column vector

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

is an eigenvector of W with eigenvalue $4\sqrt{5}$. Therefore, $|G| \cdot 4\sqrt{5}$ divides $\det W$, and in particular $\det X = 60\sqrt{5}$ divides $\det W$ in $\mathfrak{O}$. Thus, the validity of the conjecture for $G = A_5$ is shown 'theoretically'. A similar argument applies to show the validity of the conjecture for $G = PSL_2(7)$.

***Proof of Theorem* 2.** We show that for some suitable orderings of $\mathrm{Irr}(G)$ and of $\mathrm{Conj}(G)$, the matrix W possesses an eigenvalue $\frac{|K|(\chi(x)-\chi(x'))}{\chi(1)}$. We note that the character χ mentioned in the theorem takes a quadratic irrational value and hence possesses the conjugate character χ' associated with the Galois automorphism of the quadratic field $\mathbb{Q}(\chi(x))$ over $\mathbb{Q}$. Our assumption implies that $\chi(x')$ is an algebraic conjugate of $\chi(x)$. Therefore, we conclude that $\mathbb{Q}(\chi(x')) = \mathbb{Q}(\chi(x))$. Now reorder the elements of the sets $\mathrm{Irr}(G)$ and $\mathrm{Conj}(G)$ so that $\chi = \chi_{s-1}$, $\chi' = \chi_s$, $K = K_{s-1}$, and $K' = K_s$. We then have $\chi_{s-1}(x_{s-1}) = \chi_s(x_s)$ and $\chi_{s-1}(x_s) = \chi_s(x_{s-1})$, the latter being an algebraic conjugate of the former.

Now let $\mathbf{u} = [0, 0, \ldots, 0, 1, -1]^T$ be an s-dimensional (column) vectors with only the last two entries of $\mathbf{u}$ being nonzero. Noting the condition (3) of the theorem, it is easy to compute that $\mathbf{u}$ is an eigenvector of W with eigenvalue $\frac{|K_{s-1}|(\chi_{s-1}(x_{s-1})-\chi_{s-1}(x_s))}{\chi_{s-1}(1)} = \frac{|K|(\chi(x)-\chi(x'))}{\chi(1)}$, which is nonzero by assumption.

6. Modular version of Conjecture II

As for Conjecture II, Kiyota has shown the following result.

Theorem 3 (Kiyota)**.** *Conjecture II holds true if all Sylow subgroups of G are abelian.*

The proof will be published by Kiyota elsewhere and here in these notes, we shall give only a sketch of his proof.

Choose an arbitrary prime divisor p of $|G|$. Let P be a Sylow p-subgroup of G, and $|P| = p^a$. Let $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_s\} = B_1 \cup B_1 \cup \cdots \cup B_t$ be the block decomposition of the irreducible characters of G with respect to the prime p. If d_i is the defect of the block B_i, then $|\deg \chi_{i'}|_p = p^{a-d_i+h_{i'}}$ for each irreducible character $\chi_{i'}$ belonging to B_i. Here $|\cdot|_p$ denotes the p-power of the natural integer $|\cdot|$ and $h_{i'} \geq 0$ is the height of $\chi_{i'}$.

By Brauer, there exists a block decomposition (though not necessarily unique) of the set of all conjugacy classes of G:

$$\mathrm{Conj}(G) = \{K_1, K_2, \ldots, K_s\} = \mathrm{Conj}(B_1) \cup \mathrm{Conj}(B_2) \cup \cdots \cup \mathrm{Conj}(B_t).$$

If $k(B_i)$ is the number of irreducible characters of G belonging to the block B_i, then $k(B_i) = |\mathrm{Conj}(B_i)|$ holds. For each conjugacy class $K_{i'}$ belonging to $\mathrm{Conj}(B_i)$, we have $|C_G(x_{i'})|_p = p^{d_i - e_{i'}}$ with some nonnegative integer $e_{i'}$. In particular, $|K_{i'}|_p = p^{a-d_i+e_{i'}} \geq p^{a-d_i}$.

Now decompose the quantity

$$\frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(1)}$$

into the product according to the block decomposition of irreducible characters and of the conjugacy classes.

Let

$$\frac{|K_{i_1}||K_{i_2}|\cdots|K_{i_{k_i}}|}{\chi_{i_1}(1)\chi_{i_2}(1)\cdots\chi_{i_{k_i}}(1)}$$

be the partial product corresponding to $\mathrm{Conj}(B_i)$ and $\mathrm{Irr}(B_i)$. Consider its p-power, which is easily seen to be

$$\begin{aligned}\left|\frac{|K_{i_1}||K_{i_2}|\cdots|K_{i_{k_i}}|}{\chi_{i_1}(1)\chi_{i_2}(1)\cdots\chi_{i_{k_i}}(1)}\right|_p &\geq p^{k_i(a-d_i)-k_i(a-d_i)+(e_{i_1}+e_{i_2}+\cdots+e_{i_{k_i}})-(h_{i_1}+h_{i_2}+\cdots+h_{i_{k_i}})}\\ &= p^{(e_{i_1}+e_{i_2}+\cdots+e_{i_{k_i}})-(h_{i_1}+h_{i_2}+\cdots+h_{i_{k_i}})}.\end{aligned}$$

Brauer's height conjecture. The height of an irreducible character of a group G belonging to some block B (with respect to a prime p) is always zero if and only if the defect group of B is abelian.

The Brauer's height conjecture has not been affirmatively confirmed fully, but the 'if'-part of it has been proved using the classification of all finite simple groups.

In our case, it is assumed that all Sylow subgroups are abelian, and so established part of the Brauer's conjecture implies that heights are all zero. In particular, $h_{i_1} = h_{i_2} = \cdots = h_{i_{k_i}} = 0$ which implies

$$\left| \frac{|K_{i_1}||K_{i_2}|\cdots|K_{i_{k_i}}|}{\chi_{i_1}(1)\chi_{i_2}(1)\cdots\chi_{i_{k_i}}(1)} \right|_p = p^{e_{i_1}+e_{i_2}+\cdots+e_{i_{k_i}}} \geq 1.$$

This equality holds for all prime divisors p of G and for all p-blocks, therefore we conclude that

$$\frac{|K_1||K_2|\cdots|K_s|}{\chi_1(1)\chi_2(1)\cdots\chi_s(1)}$$

is a natural integer, as desired. Hence the theorem of Kiyota.

7. Other Properties for the Product $|K_1||K_2|\cdots|K_s|$

We have not been able to obtain any general results for the product $\chi_1(1)\chi_2(1)\cdots\chi_s(1)$, but for the product $|K_1||K_2|\cdots|K_s|$ we have some results as given below. Our description here is somewhat sketchy since most arguments were already given in [2].

Theorem 4. *Under the same notation as above, the following conditions hold.*

(1) *The product $|K_1||K_2|\cdots|K_s|$ is an integer multiple of $|G/Z(G)|$.*

(2) *Let J be a subset of the index set $\{1, 2, \ldots, s\}$ satisfying that for every $\chi \in \mathrm{Irr}(G)$ with $\deg(\chi) \geq 2$, there exists at least one index $j \in J$ such that $\chi(x_j) = 0$. Then $\prod_{j\in J} |K_j|$ is an integer multiple of $|[G, G]|$.*

Here $Z(G)$ or $[G, G] = G'$ is the center or the commutator subgroup, of G respectively.

Corollary 5. *The product $|K_1||K_2|\cdots|K_s|$ is an integer multiple of $|[G, G]|$.*

***Proof of Corollary* 5.** If we take $J = \{1, 2, \ldots, s\}$, then by Burnside, the condition (2) of Theorem 4 is satisfied. Hence the corollary.

***Proof of Theorem* 4.** As for (1), consider the action of G on the direct product set

$$\Omega = K_1 \times K_2 \times \cdots \times K_s$$

by conjugation on each component. If an element $g \in G$ fixes an element of Ω, then the centralizer $C_G(g)$ has a nontrivial intersection with every conjugacy class K_j of G. In particular, it holds that

$$G = \cup_{x \in G} C_G(g)^x.$$

This forces $g \in Z(G)$ as is well known. Therefore $G/Z(G)$ acts fixed-point-free on Ω. Hence (1).

(2) follows from an easy extension of the proof of the main theorem of [2]. For a subset S of G, define the sum of elements $\overline{S} = \sum_{x \in S} x \in \mathbb{C}[G]$ and its normalized sum $\hat{S} = \frac{\overline{S}}{|S|}$. Note that if H is a subgroup of G, then $\hat{H}$ is an idempotent in the group ring $\mathbb{Q}[G]$.

It was proved in [2] that

$$\hat{K}_1 \hat{K}_2 \cdots \hat{K}_s = x_1 x_2 \cdots x_s \hat{G'},$$

which is an extension of the following result of Brauer and Wielandt (1954, see [2] for details.)

Theorem (Brauer and Wielandt). The product $\prod_{j=1}^{s} \overline{K}_j$ is proportinal to $\overline{G}$ if and only if $G = [G, G]$.

Continuing the proof of the statement (2) of our theorem, choose a subset J subject to the condition stated in (2). Put

$$P_J = \prod_{j \in J} \hat{K}_j = \sum_{1 \le i \le s} a_i e_i, \quad a_i \in \mathbb{C}$$

where

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{x \in G} \chi_i(x^{-1}) x$$

is the central primitive idempotent corresponding to the irreducible character χ_i. It is well known that there exist exactly s algebra homomorphisms $\omega_i, 1 \leq i \leq s$, from $Z(\mathbb{C}G)$ to $\mathbb{C}$ corresponding to each χ_i. They satisfy

$$\omega_i(\hat{K}_j) = \frac{\chi_i(x_j)}{\chi_i(1)} \quad and \quad \omega_i(e_j) = \delta_{ij}.$$

Apply ω_i on P_J to obtain:

$$\omega_i(\prod_{j\in J} \hat{K}_j) = \prod_{j\in J} \frac{\chi_i(x_j)}{\chi_i(1)} = a_i.$$

Our condition on J forces that $a_i = 0$ if $\deg \chi_i \geq 2$. On the other hand, if $\deg \chi_i = 1$, then χ_i is a homomorphism from $\mathbb{C}G$ to $\mathbb{C}$ and so we obtain

$$a_i = \chi_i(\prod_{j\in J} x_j).$$

Therefore, it holds that

$$P_J = \sum_{\deg \chi_i=1} a_i e_i = \frac{1}{|G|} \sum{}' \sum_{x\in G} \chi_i(\alpha_J)\chi_i(x^{-1})x = \frac{1}{|G|} \sum_{x\in G} \sum{}' \chi_i(\alpha_J x^{-1})x,$$

where $\alpha_J = \prod_{j\in J} x_j$ and $\sum{}'$ runs over all irreducible characters of degree 1. Write $\alpha_J x^{-1} = y^{-1}$. Then $x = y\alpha_J$ and so

$$P_J = \frac{1}{|G|} \sum_{y\in G} \left(\sum{}' \chi_i(y^{-1})\right) y\alpha_J.$$

By the orthogonality relation of irreducible characters of the group $G/[G,G]$, $\sum{}'\chi_i(y^{-1})$ vanishes unless $y \in [G,G] = G'$. Thus,

$$P_J = \frac{1}{|G|} \sum_{y\in[G,G]} (|G/G'|)y\alpha_J = \alpha_J \hat{G}' = \prod_{j\in J} x_j \hat{G}',$$

and therefore, we have obtained:

$$\prod_{j\in J} \hat{K}_j = (\prod_{j\in J} x_j)\hat{G}'.$$

Write this equality using $\overline{K}_j$'s and $\overline{G}'$. Then, we have

$$\prod_{j\in J}\overline{K_j} = \frac{\prod_{j\in J}|K_J|}{|G'|}\left(\prod_{j\in J}x_j\right)\overline{G'}.$$

Now compare the number of group elements occurring on each side to obtain the statement (2).

Remark. Most results in this article have already been reported in Harada [7]. But due to its limited circulation, we reproduced them here with some deletions and additions.

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