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A SURVEY: BOB GRIESS'S WORK ON SIMPLE GROUPS AND THEIR CLASSIFICATION

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Abstract

This is a brief survey of the research accomplishments of Bob Griess, focusing on the work primarily related to simple groups and their classification. It does not attempt to also cover Bob's many contributions to the theory of vertex operator algebras. (This is only because I am unqualified to survey that VOA material.)

For background references on simple groups and their classification, I'll mainly use the "outline" book [1]

Over half of Bob Griess's 85 papers on MathSciNet are more or less directly concerned with simple groups. Obviously I can only briefly describe the contents of so much work. (And I have left his work on vertex algebras etc to articles in this volume by more expert authors.)

Background: Quasisimple components and the list of simple groups

We first review some standard material from the early part of [1, Sec 0.3]. (More experienced readers can skip ahead to the subsequent subsection on the list of simple groups.)

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Components and the generalized Fitting subgroup

The study of (nonabelian) simple groups leads naturally to consideration of groups L which are:

quasisimple: namely, $L/Z(L)$ is nonabelian simple; with $L = [L, L]$.

Such L can arise inside a group X in various ways; notably when L is a:

component of X: namely, L is quasisimple; and subnormal in X.

Components are in turn the (commuting) factors of $E(X)$, in the:

generalized Fitting subgroup of X: $F^*(X) := E(X)F(X)$,

where the classical Fitting subgroup $F(X)$ is the largest normal nilpotent subgroup (and so in particular is a product of p-groups, for various primes p). The crucial property of $F^*(X)$ is that it is self-centralizing in X [1, 0.3.1]:

$$C_X\big(F^*(X)\big) = Z\big(F^*(X)\big).$$

Notice first that this means we can expect $F^*(X)$ to be nontrivial: for if $F^*(X) = 1$, then $X = C_X(1) = Z(1) = 1$. Indeed it shows roughly that the rest of X can't just ignore $F^*(X)$:

$X/Z\big(F^*(X)\big)$ acts *faithfully* as automorphisms of $F^*(X)$.

In particular $X/F^*(X)$ embeds in the outer automorphism group $\mathrm{Out}\big(F^*(X)\big)$.

Next we consider some basic implications of the above material; particularly in the context of the classification of the finite simple groups (CFSG): We take a minimal counterexample G to the CFSG; so we can use induction on $|G|$. We often need to consider, for a suitable prime p, a *p-local subgroup* X of G: namely the normalizer in G of some nonidentity p-subgroup.

Of course as G is simple, we have $X < G$. Hence by induction, we at least know the possible simple composition factors of X. But as we will soon see, we can typically even expect that there will be one or more components L in $E(X)$. So by induction, we even know the possibilities for the simple quotient $L/Z(L)$.

This focuses attention on:

The list of simple groups

In the CFSG, we begin with a list of "known" simple groups; and we must end by showing that the list is complete. Here is a summarized form of that initial+final CFSG-List (e.g. [1, 0.1.1]:

Theorem 0.1 (Classification of Finite Simple Groups)**.** *A (nonabelian) finite simple group must be one of:*

(A) *An alternating group A_n $(n \geq 5)$.*
(L) *A Lie-type group (certain matrix groups (linear, symplectic ...) over finite fields).*
(S) *One of 26 "sporadic" groups (not fitting into any infinite family).*

At various later points, we'll refer back to this rough case-list (A,L,S).

Remark 0.2 (When that list was still "in progress"...)**.** Keep in mind that, when Bob was a graduate student around 1970, not all the sporadic groups in sublist (S) were even known yet. (And a proof of the completeness of the full (A,L,S)-list still seemed a fairly visionary dream.)

In fact Bob's Ph.D. thesis work made a significant contribution to the knowledge of one particular property of the simple groups in the list:

1. Determining Schur Multipliers of (known!) Simple Groups

Assume S is simple; and that L is a quasisimple group, with $L/Z(L) \cong S$. An obvious question to ask is: How much larger than S can L be? Or equivalently: how large can $Z(L)$ be?

Schur multipliers and quasisimple covers

The largest-possible $Z(L)$ is given by the classical *Schur multiplier* of S. See for example [6, Sec 6.1] for fuller details—and for the now-known list of the maximal $Z(L)$, for each simple S. (We mention that these groups are quite small, relative to the size of the corresponding S.)

To give a rough initial summary of the history of that multiplier-list: Various authors treated various cases, from the list of (then-known) S in the simple-groups list (A,L,S); for example, typically the discoverer of some sporadic group would at the same time determine its multiplier. Finally, Bob cleaned up all remaining details—and his list became the standard reference in the literature.

So let's look at that history, in just a little more detail:

Case (A)—alternating groups A_n : Schur in 1911 (cf. [27]) showed that A_n has multiplier $\mathbb{Z}_2$—except that $\mathbb{Z}_6$ arises for $n = 6, 7$.

Case (S)—sporadic groups: As mentioned above, the corresponding multipliers were often determined by the discoverer(s) of the group. (And recall that Bob was the co-discoverer, with Fischer, of the Monster M.) Bob made a number of additional contributions here: e.g. [14] on several sporadics, and [21] on the McLaughlin group McL.

Case (L)—Lie type groups: Steinberg did much of the work here, focusing on methods from the Lie theory underlying the construction of the groups. Bob's thesis [10] continued the work, including more general approaches, followed by [13].

Bob also corrected some erroneous calculations (e.g. M_{22}) which had been previously accepted in the literature.

The completed treatment appeared in the papers [11, 17, 19]. These were subsequently used as a standard reference for Schur multipliers, in much of the original literature on the CFSG. (For example, at least two dozen uses are visible in the index of [1].)

Unfortunately the methods and results are too technical for the expository level of this short survey; the reader can consult e.g. [6, Sec 6.1] for a more detailed overview.

Determining the multipliers for simple S also allows determination of the possible quasisimple covers L.

Digression: the cohomology viewpoint

And for the moment (I'll return to this aspect later, in Section 4) I'll just quickly add:

The multiplier of S can also be described topologically: as the homology group $H_2(S,\mathbb{Z})$, or dually as the cohomology group $H^2(S,\mathbb{C}^\times)$. Similarly, *other* kinds of nonsplit extensions for S (e.g. over nontrivial modules) also correspond to suitable nonzero cohomology groups.

And, at least among simple-group theorists, Bob rapidly became a world expert on such matters—especially, for the relation of these questions to the more exotic structure of p-local subgroups in sporadic groups.

(Also for cohomology of Lie-type groups: the CFSG often referred to e.g. Cline-Parshall-Scott [4] and Jones-Parshall [26]. The reader is directed to Scott's article in this volume, for recent results in this area.)

Let's now broaden our focus to the CFSG, and Bob's contributions in that vast project:

2. Works Classifying Groups of Standard form/type in the CFSG

For context, we'll first give a few pages of discussion on the main case division (Odd versus Even) in the CFSG. (The more experienced reader may wish to skip over this subsection, and proceed to the two following subsections on the actual treatment of the Odd and Even Cases.)

The Odd/Even Dichotomy Theorem

This is basically a summary of material in [1, Ch 0].

How might we get started on the classification of finite simple groups? Here is a fairly simple-minded approach:

By the celebrated Feit-Thompson Odd Order Theorem (e.g. [1, 1.2.1], a simple group G has must contain an *involution* x (that is, an element of order 2). Then the involution *centralizer* $C_G(x)$ has x in its center (and $C_G(x)$ is in particular a 2-local subgroup of G).

By simplicity we have $C_G(x) < G$. So we may apply induction to $C_G(x)$. Thus for example, we know that the simple sections of $C_G(x)$ must lie among the "known" groups in the CFSG-list 0.1, which we are abbreviating by (A,L,S). (Indeed we'll even know the possible quasisimple components of $F^*\big(C_G(x)\big)$, when they arise very soon below...) This suggests a possible:

Approach to the CFSG:

— First, describe all groups H with an involution x in the center.

— Then for each H, find all simple G with $C_G(x) \cong H$.

And in rough outline, with various modifications, this was the approach actually used in the CFSG.

A little closer analysis of the Approach soon leads to a natural case subdivision. This will perhaps be clearer, if we first examine centralizers in some suitable explicit simple groups:

Examples motivating Odd and Even Cases

Now in effect, "most" simple G have Lie type. So those groups provide a model-case (and the usual "target") in the CFSG.

Indeed, many basic features can be viewed in what is perhaps the most natural and accessible matrix group, namely $G := GL_n(\mathbb{F}_{p^a})$ for some prime p. (Admittedly this group is not usually simple—but it is "close enough" for most expository purposes, since typically it is easy to transfer features to its determinant-1 subgroup, and then to the simple quotient modulo its center.) A major advantage of GL_n is that we can just draw pictures of square matrices.

Example 2.1 (Odd Case Example)**.** Assume first that p is *odd*. An obvious involution t is provided by a diagonal matrix, with say k nontrivial entries of -1.

$$t := \left(\begin{array}{c|c} -I_k & 0 \\ \hline 0 & I_{n-k} \end{array} \right)$$

By linear algebra, any matrix commuting with t preserves its (± 1)-eigenspaces; so the centralizer is expressed via block-matrices, with picture:

$$C_G(t) = \left(\begin{array}{c|c} GL_k(q) & 0 \\ \hline 0 & GL_{n-k}(q) \end{array} \right)$$

Now the subgroup SL_r in each factor GL_r is usually quasisimple; so since these subgroups are normal, they are in fact components of $C_G(t)$.

How can we abstract from this behavior?

Definition 2.2 (component type)**.** We say a group X is of *component type*: if for some involution $x \in X$, $C_X(x)/O_{2'}\big(C_X(x)\big)$ has a component.

(The reason for working modulo the "core"—the largest normal $2'$-subgroup $O_{2'}\big(C_X(x)\big)$—is embedded in the proof of the Dichotomy Theorem 2.5 below.)

The concept of component type in fact captures most of the odd-characteristic groups in Lie type (L); as well as most alternating groups (A), and some sporadic groups (S). So we turn to:

Example 2.3 (Even Case Example)**.** Now instead take $p = 2$. In characteristic 2, the field element -1 is the same as 1; so the matrix t in the earlier Odd Case Example 2.1 here becomes just the identity—of order 1, rather than an involution.

So now we instead take t given by a minimal Jordan-form matrix of order 2: consisting of Jordan blocks for the eigenvalue 1, all of dimension 1—except for just one of dimension 2. Here the sub-diagonal entry 1 in the 2×2 block can by conjugation in GL_n be taken to the lower-left position of the larger matrix t; which then takes the form:

$$t := \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & I_{n-2} & 0 \\ \hline 1 & 0 & 1 \end{array}\right)$$

With a little bit more linear-algebra, we can compute that:

$$C_t := C_G(t) = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & GL_{n-2}(2^a) & 0 \\ \hline * & * & * \end{array}\right)$$

We can in fact describe C_t as a semidirect product UL, where:

$$U := \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline * & I_{n-2} & 0 \\ \hline * & * & 1 \end{array}\right), \quad L := \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & GL_{n-2}(2^a) & 0 \\ \hline 0 & 0 & * \end{array}\right)$$

Here U is a 2-group, which is normal in C_t. And L contains a subgroup $GL_{n-2}(\mathbb{F}_{p^a})$—which (in contrast to the situation in the Odd Case Example 2.1) is *not* normal in C_t. (Check that conjugating L by U produces nonzero elements in the off-diagonal blocks; so the result is not in L.)

This situation is usually described in the language of the generalized Fitting subgroup, namely $F^*(C_t) = O_2(C_t)$: for there are no components, and no (sub)normal q-groups for odd q. (For the final statement, work in the quotient modulo any scalar matrices.)

How can we abstract from this new behavior?

Definition 2.4 (characteristic 2 type)**.** We say a group X is of *characteristic 2 type*, if: For each involution $x \in X$, $F^*\big(C_X(x)\big) = O_2\big(C_X(x)\big)$.

(It turns out [1, B.1.6] that this condition on centralizers is equivalent to the ostensibly-stronger condition of having $F^*(N) = O_2(N)$ for all 2-local subgroups N.)

The concept of characteristic 2 type in fact captures the characteristic-2 groups of Lie type; as well as various sporadics (S).

Thus these two abstract concepts cover the behavior of the explicit simple groups in the *known*-list (A,L,S). But for our Approach to the CFSG, can we establish that they cover all "unknown" simple G?

Proving the Dichotomy Theorem

Earlier work of various authors had classified simple groups which are "small" in the sense of having low 2-rank: that is, groups with $m_2(G) \le 2$. Having $m_2(G) \ge 3$ allowed the use of *signalizer functors*—from ideas of Thompson, further refined by Gorenstein and Walter. These led around 1971 to the main case division (e.g. [1, 0.3.10]:

Theorem 2.5 ((Odd/Even) Dichotomy Theorem)**.** *If G is simple, with $m_2(G) \geq 3$,then G is either of component type, or of characteristic 2 type.*

For this survey, it probably suffices to know just the statement of this result. But certain ideas inside the proof may be instructive:

Digression 2.6 (A sketch of the proof of the Dichotomy Theorem)**.** The proof is comparatively elementary, and is developed in the later part of [1, Sec 0.3]; here we provide only a very rapid overview:

We begin by assuming G is *not* of component type. This says that for each involution t, with centralizer C_t, the quotient $\bar{C}_t := C_t/O_{2'}(C_t)$ has no components—so that $F^*(\bar{C}_t) = O_2(\bar{C}_t)$. And we must establish characteristic 2 type—which will follow, if we can show that $\theta(C_t) := O_{2'}(C_t) = 1$ for all t.

The approach is to extend θ to larger elementary abelian 2-subgroups: e.g. for $B = \langle t, u \rangle$, take $\theta(B)$ to be the span $\langle \theta(v) : v \in B^{\#} \rangle$; etc. And then try to show that this larger θ can have only trivial values.

Notice that it is not clear that $\theta(B)$ should be a $2'$-group. Here we quote the early, and fairly elementary, result called the Signalizer Functor Theorem (using balance and the hypothesis $m_2(G) \geq 3$). Hence these odd-order value-groups for θ are 2-*signalizers* in Thompson's terminology; and θ is then a *signalizer functor*.

The Theorem also states that θ must be *complete*: namely values are "globally" determined, by elementary subgroups A of rank at least 3; and in particular, $\theta(B) = \theta(A)$ for any B of index 2 in such A.

And completeness in turn suggests that we consider the graph on the various A of rank 3; where A and A' define an edge, if they share some B of rank 2.

If this graph is disconnected, then G has a strongly embedded subgroup; and we can quote the early and very-standard Bender-Suzuki Theorem—to conclude that G is a Lie-type group of rank 1, over a field of characteristic 2. In particular, G is of characteristic 2 type—so that we are done in this case.

Otherwise, the graph must be connected. But the completeness property above shows that the value of θ must be constant on any connected component—which in this case, contains the full G-conjugacy class of A. Then the constant value of $\theta(A)$ on this class is preserved by G, and so gives a normal subgroup. Thus by simplicity of G, we get $\theta(A) = 1$; and in particular $\theta(t) = 1$ for all t—which is what we had set out to prove.

Now we return to our main theme. As a result of the Dichotomy Theorem 2.5: the Odd Case of component type, and the Even Case of characteristic 2 type, became the two "halves" of the rest of the CFSG.

And this provides the context for our indication of Bob Griess's contributions to the CFSG.

Treating the Odd Case—via standard components

The Odd half of the CFSG was handled first; for fuller details see [1, Ch 1].

Here we will over-simplify that material: The case of component-type G was reduced (with much work: the Unbalanced Group Theorem, the proof of the B-Conjecture, etc) to Aschbacher's situation of:

Definition 2.7 (standard form)**.** We say the component L of $C_G(x)$ is *standard* in G: if whenever $L \neq L^g$, we have: $[L, L^g] \neq 1$; and $|C_G(L, L^g)|$ odd.

The condition is obtained by taking $C_G(x)$ maximal in a suitable sense.

To see this in the earlier Odd Case Example 2.1: Take $k = 2$, to maximize the component SL_{n-2}; and take L^g to have its entries of -1 in the final two diagonal positions. (Then $L \cap L^g$ is large, giving the two desired conditions.)

So at that point, it remained to solve the "standard form problems"; namely:

— take each simple S from the known-list (A,L,S),

— form all quasisimple covers L of S (using the earlier multiplier-list)

— for each such L, find all simple G in which L is standard.

(Of course Bob's work is used for the list of Schur multipliers!)

Various authors then handled various cases of S. For the more particular purposes of this survey, I single out some of Bob's contributions:

The case of S a Bender group ($L_2(2^a)$, $U_3(2^a)$, $Sz(2^a)$): see Griess-Mason-Seitz [7].

The case $S \cong L_3(4)$ or the Held sporadic group He: see Griess-Solomon [25].

The case of L the quasisimple cover $4M_{22}$ of the Mathieu group M_{22}: Bob treated this in an unpublished work; a proof was later published by Harada-Solomon.

(For uses of some of these papers *beyond* standard form, see e.g. [1, Ch 1].)

Treating the Even Case—including the standard-type subcase

The Even half of the CFSG took longer to finish; for fuller details see [1, Ch 2]. Again over-simplifying:

Here we start with G of characteristic 2 type; so the expected answers in the conclusion should mainly be Lie-type groups in characteristic 2.

And the main idea is to mimic the uses of signalizer-functor methods—as used above to separate off the Odd Case. That is: inside an example-group in characteristic 2, follow the earlier Odd Case Example 2.1; but now taking t to be an element of *odd* prime order p—that is, having some p-th roots of unity ω on the diagonal. Again we get block-matrices giving components in $C_G(t)$. So go on to adapt the later notions, getting p-component type, characteristic p type, etc.

Since characteristic 2 type focuses on 2-local subgroups, the place of 2-rank $m_2(G)$ is taken by:

$e(G) :=$ the maximal p-rank, taken over all odd p, in all 2-locals of G.

The case $e(G) \leq 2$ ("quasithin") was eventually handled by Aschbacher-Smith [1, 3.0.1]. So we may assume that $e(G) \geq 3$, in analogy with earlier $m_2(G) \geq 3$.

Now the stage is set, to re-use the comparatively elementary argument 2.6 proving the Dichotomy Theorem 2.5—for x of *odd* prime order. This time, the outcome is a *tri*chotomy of cases:

— p-component type

— a disconnected case (since no odd-p analogue of Bender-Suzuki was available)

— characteristic $\{2,p\}$ type (since we started off already having characteristic 2 type)

This is called the Weak Trichotomy Theorem at [1, 2.2.1].

In much lengthier work, Gorenstein and Lyons (with Aschbacher for $e(G)=3$) established a Strong Trichotomy Theorem [1, 2.3.9]; with corresponding cases much strengthened:

— standard type (a standard-component type for odd p)

— Uniqueness Type (the disconnected case—a strongly p-embedded 2-local be impossible)

— $GF(2)$ type (roughly, the large-extraspecial 2-subgroup problem—had already been handled)

Treatment of these three branches completed the logic of the CFSG.

For the the treatment of the latter two branches, see e.g. [1, Ch 7,8]. For the purposes of this survey, we emphasize the treatment of the first branch—that is, standard type.

Summary: all done—by Gilman-Griess [5] (Cf. [1, Ch 6].)

To re-emphasize: In striking contrast to the original standard form problems (for x an involution in the Odd Case)—where many different authors treated various components L—for standard type (where $|x|$ is odd in the Even Case), the single paper [5] of Gilman-Griess treated all possible components L.

Here is a very rough overview of the logic flow in that paper: The strong properties of a component L with standard type gives "neighbors" L, L^g; with considerable knowledge of their substantial overlap $L \cap L^g$. Typically then they can get p-locals J, K which resemble rank-2 parabolics, with $J \cap K$ resembling a rank-1 parabolic. In this situation, they can usually identify $\langle J, K\rangle$ as a known rank-3 Lie-type group in characteristic 2. This information, taken over all possibilities for pairs J, K, provides the Curtis-Tits relations—which can be used to identify G as higher-rank Lie type in characteristic 2.

(I mention one other work: for uses in the CFSG of Burgoyne-Griess-Lyons [2] on certain properties of Lie-type groups, see e.g. [1, index].)

3. Existence and Uniqueness of the Monster

In this section, I will only briefly indicate the Griess algebra of the Monster; and will leave vertex-operator algebra aspects to other contributors to this volume.

Existence of the Monster

Around 1973, Fischer and Griess independently gave evidence for a very large new sporadic simple group, the Monster M.

Bob summarized his approach in his 1975 Park City lecture [16]: based on two conjugacy classes z, x of involutions, with:

— $C_M(z) \cong 2^{1+24}Co_1$;

— $C_M(x) \cong 2B$ (the quasisimple double cover of the Baby Monster).

He especially focused on the (conjectured) smallest module—of dimension 196883.

There was at that time some (related) interest in *algebras* for groups; for example, Conway, Norton (algebras for 3-transposition groups),[1] Harada and his students ...

But Bob saw deeper than the rest of us: He had noted that the conjectured 196883-module is in $\mathrm{Sym}^2(196883)$—so that an M-algebra must exist there. He also wrote a preprint "Nonassociative commutative algebras whose automorphism groups are the symmetric groups"—studying the algebra for S_n on its permutation module, largely as a test case toward the Monster. (But apparently the referees didn't see so deeply? and the preprint remained unpublished. Who had the last laugh?)

Then (as all the world knows), in "The Friendly Giant" [18], Bob constructed M—as automorphisms of an algebra on a 196884-dimensional module B. This work was later (2010) awarded the Steele Prize.

[1] Cf. Jon Hall's article in this volume.

To give just a very quick overview of Bob's construction:

The multiplication is built from those on 3 components, of the restriction of the module B to the above subgroup $C := C_M(z) \cong 2^{1+24}Co_1$.

These component-products must then be stitched together very carefully: so that the result admits the action of a further element σ of order 3—designed to be in a subgroup $2^{2+11+22}(S_3 \times M_{24})$ of M, but not in C.

So: we now have $C < \langle C, \sigma \rangle$ in $\mathrm{Aut}_{\mathrm{alg}}(B)$. Bob works modulo various primes p, to show that this larger group is finite—and has the right order $|M|$. (Hence it gives a group "of type M" in the language of those days—since uniqueness of M had not yet been proved.)

From the later viewpoint of Borcherds and vertex operator algebras, B is the first term of a natural infinite sum of modules for M; but I am leaving these aspects to other articles in this volume.

Uniqueness of the Monster

Around 1980, Thompson and Norton (independently) gave "partial" uniqueness proofs for M—namely requiring certain additional hypotheses. An unrestricted uniqueness proof was not achieved until 1989, by Griess-Meierfrankenfeld-Segev [8].

In quick overview: For a group G, assumed to be of "type M" as above, the proof in fact shows uniqueness of a graph Γ, on the conjugacy class of the involution we had earlier denoted by x; where x, y is an edge in the graph, whenever xy is also in that conjugacy class.

The argument requires considerable detailed analysis, much of it related to suborbits, i.e. orbits of $C_G(x) \cong 2B$ on the graph Γ.

We mention that over the years, there has developed a considerable literature on uniqueness proofs for simple groups. And for example, the approach via graph-uniqueness is also prominent in a number of papers of Aschbacher and Segev.

4. "Sporadic" situations—nonsplit extensions, code loops, ...

Now, let's return to the cohomology-digression at the end of Section 1 of this survey: namely, Bob's interest in unusual cohomology and extensions—and especially their involvement in p-local subgroups of sporadic groups.

I count around a dozen papers by Bob on such themes; to sample a few:

Already by 1972 in [12]—Bob proceeds from extraspecial 2-groups to nonsplit extensions for their corresponding orthogonal automophism groups.

Dempwolff's famous nonsplit extension (occurring in a 2-local in the Thompson sporadic group F_3) is exhibited in a more general context in [15].

The Monster itself involves many other sporadic groups. (Bob dubbed these 20 groups the "Happy Family".)

There are some other contexts where Bob explored special configurations:

Codes and loops

Mathieu groups had long been studied via *codes* (namely the extended perfect ternary and binary Golay codes).

In an ostensibly-different direction: Parker in the early 1980s constructed certain Moufang *loops* (recall a loop satisfies the group laws, except possibly for associativity). These were used by Conway [3] for an alternative construction of the Monster.

Griess [20] combines the two above viewpoints: he obtained the Parker loops and others as "code loops" (based on doubly-even codes). He used code loops in later papers, to approach the O'Nan sporadic group $O'N$, the Rudvalis group Ru, and indeed various other groups.

Codes are also fundamental in Bob's sporadic-groups book [22].

Lattices

To the previous remark on codes should be added that *lattices* are also fundamental in Bob's book [22].

Indeed in [23], Bob extends the standard notion of selfdual lattices to *semiselfdual* lattices; and uses these to derive foundations for the Mathieu and Conway groups.

The theme of uniqueness for various special lattices is continued in a number of later papers, notably [24].

(I believe that other articles in this volume will indicate Bob's many papers on *lattice VOAs*.)

5. Embedding Simple Groups in Exceptional Lie Groups

There is a long history of the study of embeddings of finite groups (especially simple groups) in various (infinite) Lie groups over $\mathbb{R}$ or $\mathbb{C}$.

Of course an irreducible representation of a group gives an embedding at least in a linear group—and it is usually easy enough to check whether that embedding falls into one of the classical subgroups (unitary, orthogonal, symplectic).

But it is a subtler question, whether such an embedding falls into a Lie subgroup of *exceptional* type. (In particular, conjectures and questions of Kostant related to type E_8 motivated considerable research.)

I count about 15 papers of Bob in this area; 8 of them joint with Alex Ryba (and indeed 3 with Arjeh Cohen). So of course the reader is directed to the article of Alex Ryba in this volume, for the main treatment of that material.

(But I couldn't resist a quick preview of some salient features:)

Bob's interest in the area goes back (at least) to [15]—where the nonsplit Dempwolff extension $2^{5+10}L_5(2)$ is exhibited inside $E_8(\mathbb{C})$.

Various papers with Alex culminated in Griess-Ryba [9], giving the list of all quasisimple finite groups inside exceptional algebraic groups over $\mathbb{C}$. (Some other papers address questions of conjugacy among such subgroups.)

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