Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **13** (2018), No. 3, pp. 351-363
DOI: 10.21915/BIMAS.2018304

CANONICAL LEFT CELLS AND THE LOWEST TWO-SIDED CELL IN AN AFFINE WEYL GROUP

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Dedicated to Professor George Lusztig on the occasion of his 70th birthday

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Abstract

We give some discussions to the relations between canonical left cells and the lowest two-sided cell of an affine Weyl group. In particular, we use the relations to construct irreducible modules attached to the lowest two-sided cell and some one dimensional representations of an affine Hecke algebra.

Canonical left cells of an affine Weyl group are interesting in understanding cells in affine Weyl groups and have nice relations with the structure and representations of algebraic groups. However, it is not easy to describe canonical left cells. In this paper we give some discussions to the relations between canonical left cells and the lowest two-sided cell of an affine Weyl group. In particular, we use the relations to construct irreducible modules attached to the lowest two-sided cell (see Theorem4.1) and some one dimensional representations of an affine Hecke algebra (see Theorem 3.5). For convenience we work with an extend affine Weyl group. This work was partially motivated by [1].

Received August 1, 2016 and in revised form December 19, 2016.
AMS Subject Classification: Primary 20C08.
Key words and phrases: Affine Weyl group, canonical left cell, lowest two-sided cell.
This work is partially supported by Natural Sciences Foundation of China (No. 11321101).

1. Canonical Left Cells

1.1. Let R be an irreducible root system and P the corresponding weight lattice. The Weyl group W_0 acts on P naturally and the semi-direct product $W = W_0 \ltimes P$ is an extended affine Weyl group, which contains the affine Weyl group $W_a = W_0 \ltimes \mathbb{Z}R$. Let S be the set of simple reflections of W_a. The partial order $\leq$ and the length function l on W are well defined. *The operation on P will be written in multiplication.*

For $w \in W$, set $L(w) = \{s \in S \,|\, sw \leq w\}$ and $R(w) = \{s \in S \,|\, ws \leq w\}$. Let s_0 be the unique simple reflection of W_a out of W_0. Define $Y_0 = \{w \in W \,|\, R(w) \subseteq \{s_0\}\}$. Then $Y_0 \cap \Omega$ is a left cell for any two-sided cell Ω of W, called a canonical left cell.

In general it is not easy to describe a canonical left cell. However, it is easy to describe the set Y_0. Let w_0 be the longest element of W_0. The set of anti-dominant weights in P is defined to be $P^- = \{x \in P \,|\, l(xw_0) = l(w_0) + l(x)\}$ and the set of dominant weights is $P^+ = \{x \in P \,|\, l(w_0x) = l(w_0) + l(x)\}$.

Proposition 1.2. $Y_0 = \{wx \,|\, w \in W_0,\ x \in P^- \text{ and } R(w) \subseteq L(x)\}$.

Proof. Let $u \in W$. Then there exist unique $w, v \in W_0$ and $x \in P^-$ such that $R(w) \subseteq L(x)$ and $u = wxv$. Moreover, we have $l(u) = l(x)+l(v)-l(w)$. The proposition follows.

1.3. It would be interesting to see when two elements in Y_0 are in a left cell. Let ρ be the product of all fundamental dominant weights. Then the set $\{wx\rho^{-1} \,|\, w \in W_0,\ x \in P^- \text{ and } R(w) \subseteq L(x)\}$ is the canonical left cell in the lowest two-sided cell c_0 of W. In general, for any $x \in P^-$ there exists a positive integer a (depending on x) such that x^b and x^a are in a left cell if $b \geq a$ (see [8, Lemma 3.2]). It seems that the number a is small, in many cases, it is among 1,2,3.

Let $S_0 = S \cap W_0$ and denote by Γ_0 the left cell $\{w \in W \,|\, R(w) = S_0\}$, which is in the lowest two-sided cell c_0 of W. For $x \in P$, denote by n_x (resp. m_x) the unique shortest element in the coset xW_0 (resp. the double coset W_0xW_0). The map $x \to n_x$ defines a one-to-one correspondence from P to Y_0, and the map $n_x \to n_xw_0$ defines a one-to-one correspondence from Y_0 to Γ_0. Also the map $x \to m_x$ defines a one-to-one correspondence between

P^+ and $Y_0 \cap Y_0^{-1}$. The sets Y_0 and Γ_0 produce naturally two modules of an affine Hecke algebra of (W, S). In next section we will see that the two modules are essentially the same.

2. Cell Modules of Affine Hecke Algebras

2.1. Let H be the Hecke algebra of (W, S) over a field k with parameter q. Assume that k contains square roots of q. Let $\{T_w\}_{w\in W}$ be its standard basis. For any w in W, let

$$C_w = q^{-\frac{l(w)}{2}} \sum_{y\leq w} P_{y,w}(q) T_y,$$

and

$$C'_w = q^{\frac{l(w)}{2}} \sum_{y\leq w} (-1)^{l(w)-l(y)} q^{-l(y)} P_{y,w}(q^{-1}) T_y,$$

where $P_{y,w}$ are the Kazhdan-Lusztig polynomials. Then the elements C_w, $w \in W$, form a basis of H, and the elements $C'_w, w \in W$, form a basis of H as well, see [2].

For any $x \in P$ there is a well defined element $\theta_x = q^{-\frac{l(y)}{2}} T_y q^{\frac{l(z)}{2}} T_z^{-1}$. where $y, z \in P^+$ such that $x = yz^{-1}$. Then $\theta_x \theta_y = \theta_y \theta_x$ for any $x, y \in P$ and the elements $T_w \theta_x$ (resp. $\theta_x T_w$), $w \in W_0$, $x \in P$, form a basis of H. See [4].

The group algebra $k[P]$ is isomorphic to the subalgebra Θ of H generated by all $\theta_x,\ x \in P$. Lusztig defined several H-module structures on $k[P]$, see [5, Section 7]. They are actually isomorphic to the modules provided by the left cell Γ_0. Let M (resp. M') be the subspace of H spanned by all $C_w,\ w \in \Gamma_0$ (resp. $C'_w,\ w \in \Gamma_0$). Then M and M' are left ideals of H and generated by $C = C_{w_0}$ and $C' = C'_{w_0}$ respectively. The elements $\theta_x C$, $x \in P$, form a basis of M and the elements $\theta_x C'$, $x \in P$, form a basis of M'.

Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the subspace of H spanned by all $C_w,\ w \in W - Y_0$ (resp. $C'_w,\ w \in W - Y_0$). Then $\mathcal{I}$ and $\mathcal{I}'$ are left ideals of H. Let $N = H/\mathcal{I}$ and $N' = H/\mathcal{I}'$. Essentially the following result is due to Arkhipov and Bezrukavnikov (see [1, 1.1.1]).

Lemma 2.2. *As H-modules N is isomorphic to M', and N' is isomorphic to M.*

Proof. Consider the surjective homomorphism $H \to M'$, $h \to hC'$. It is easy to check that the kernel is $\mathcal{I}$. So N is isomorphic to M'. Similarly the surjective homomorphism $H \to M$, $h \to hC$ induces an isomorphism $N' \to M$ of H-module. The lemma is proved. $\square$

2.3. The geometric explanation of the isomorphism in the above lemma is that Thom isomorphism for a certain equivariant K-group of the cotangent bundle of flag variety is compatible with certain actions of the affine Hecke algebra H, see [5, Section 7].

Lemma 2.2 seems helpful in understanding the structure of H-modules M and M', and may be useful to understand canonical left cells. A natural question is to consider the submodule of M' (resp. M) generated by all C_wC' (resp. C'_wC), $w \in c_0 \cap Y_0$. Modulo a central character of H, we can get a finite dimensional quotient algebra of H. In next two sections we will give some discussion to the images in such quotient algebras of the submodules. We will show that the images in such a quotient algebra is either irreducible H-module or zero when k is algebraically closed (Theorem 4.1).

3. A Realization of Some One Dimensional Representations

In this section we construct some one dimensional representations of the affine Hecke algebra H through certain quotient algebras of H (see Theorem 3.5).

3.1. From now on we assume that k is algebraically closed. Recall that Θ is the subalgebra of H generated by all θ_x, $x \in P$. Let the Weyl group W_0 act on Θ by $w(\theta_x) = \theta_{w(x)}$.

We shall need several formulas in H. Let $x \in P$, the Macdonald formula says (see [6, Theorem 2.22])

$$C\theta_xC = q^{-\frac{l(w_0)}{2}} \sum_{w\in W_0} w(\theta_x \prod_{\alpha\in R^+} \frac{1-q\theta_\alpha}{1-\theta_\alpha})C. \tag{1}$$

Let Δ be the set of simple roots of R and denote x_α the fundamental dominant weight corresponding to a simple root α. Recall that $\rho = x_\Delta$ is the product of all fundamental dominant weights. Using [4, Corollary 7.8, Lemma 7.4 (iii)] we get

$$C'\theta_{\rho^{-1}}C \;=\; q^{-\frac{\nu}{2}}C'\sum_{I\subseteq R^+}(-q)^{|I|}\theta_{\rho^{-1}}\theta_{\alpha_I}, \tag{2}$$

$$C'\theta_{\rho}C \;=\; q^{\frac{\nu}{2}}C'\sum_{I\subseteq R^+}(-q)^{-|I|}\theta_{\rho}\theta_{\alpha_I^{-1}}, \tag{3}$$

where $\nu = l(w_0) = |R^+|$, α_I is the sum of all roots in I and $|I|$ is the cardinality of I.

There is a unique involutive anti-automorphism $h \to \tilde{h}$ of the k-algebra H such that $\tilde{T_r} = T_r$ $(r \in S_0)$, $\tilde{\theta_x} = \theta_x$ $(x \in P)$ [3, 2.13(c)]. Note that $\tilde{C} = C$ and $\tilde{C'} = C'$. Applying this anti-automorphism to the formulas (2) and (3) we get

$$C\theta_{\rho^{-1}}C' \;=\; q^{-\frac{\nu}{2}}\sum_{I\subseteq R^+}(-q)^{|I|}\theta_{\rho^{-1}}\theta_{\alpha_I}C', \tag{4}$$

$$C\theta_{\rho}C' \;=\; q^{\frac{\nu}{2}}\sum_{I\subseteq R^+}(-q)^{-|I|}\theta_{\rho}\theta_{\alpha_I^{-1}}C'. \tag{5}$$

(For a K-theoretic understanding of formula (4) see [13, 2.6], note that the C' here is the C in loc.cit.)

3.2. The center of H

We have

(a) The center $Z(H)$ of H consists of W_0-invariant elements in Θ, i.e. $Z(H) = \Theta^{W_0}$ (see [4]).

Therefore the center $Z(H)$ of H is isomorphic to $k \otimes_{\mathbb{Z}} R_G$, where G is a simply connected simple algebraic group over k with root system R and R_G is the representation ring of G.

For $w \in W_0$ define

$$e_w = w\Big(\prod_{\substack{\alpha\in\Delta\\ w(\alpha)\in R^-}} x_\alpha\Big).$$

(Recall that here x_α is the fundamental dominant weight corresponding to $\alpha \in \Delta$.) Then (see [7])

(b) Θ is a free $Z(H)$-module with a basis $\{\theta_{e_w} \mid w \in W_0\}$.

Hence,

(c) for any $x \in P$, the elements $\theta_x \theta_{e_w},\ w \in W_0$, form a $Z(H)$-basis of Θ.

(d) For $A, B \in \Theta$, define

$$(A, B) = (-1)^\nu \theta_\rho \prod_{\alpha \in R^+} (1 - \theta_\alpha)^{-1} \sum_{w \in W_0} (-1)^{l(w)} w(AB\theta_\rho) \in Z(H).$$

By [3, p.163] there exist $\theta'_u \in \Theta$ ($u \in W_0$) such that $(\theta_{e_w}, \theta'_u) = \delta_{w,u}$ and the elements θ'_u form a $Z(H)$-basis of Θ.

3.3. Let T be a maximal torus of G and identify P with the character group $\mathrm{Hom}(T, k^*)$ of T. Then Θ is isomorphic to the group algebra $k[P]$ of P. For $t \in T$, we have a k-algebra homomorphism $\phi_t : \Theta \to k$ defined by $\theta_x \to x(t)$ for all $x \in P$. It is known that the map $t \to \varphi_t$ defines a bijection between T and the set of k-algebra homomorphisms from Θ to k.

Now the center $Z(H)$ of H is a free Θ-module of rank W_0. Therefore every k-algebra homomorphisms from $Z(H)$ to k is the restriction $\phi_t|_{Z(H)}$ of some algebra homomorphism $\phi_t : \Theta \to k$. Moreover, $\phi_t|_{Z(H)} = \phi_s|_{Z(H)}$ if and only if s and t are conjugate by some element of W_0, since $Z(H) = \Theta^{W_0}$. Thus the set of k-algebra homomorphisms from $Z(H)$ to k is in one-to-one correspondence with the set of semisimple classes of G.

For $t \in T$, let $\bar{t}$ be the semisimple class of G containing t and let $\phi_{\bar{t}} = \phi_t|_{Z(H)} : Z(H) \to k$ be the corresponding homomorphism. Then let $\mathcal{Z}_t$ be the two-sided ideal of H generated by all $z - \phi_t(z),\ z \in Z(H)$. Define $H_t = H/\mathcal{Z}_t$. We have $\dim H_t = |W_0|^2$.

For each simple H-module L, there exist some t in T such that $Z(H)$ acts on L through the homomorphism ϕ_t. So to study simple modules of H it is enough to study simple modules of the quotient algebras H_t for $t \in T$. We shall use the same notations $C'_w, D_w, C, C', \theta_x, \ldots$ for their images in H_t.

Theorem 3.4. *Let $t \in T$. The following statements are equivalent.*

(a) $CH_tC = 0$. (*Recall that $C = C_{w_0}$ and $C' = C'_{w_0}$.*)

(b) $CH_tC' = 0$.

(c) $C'H_tC = 0$.

(d) $C'H_{t^{-1}}C' = 0$.

(e) *For any simple H_t-module L we have $CL = 0$.*

(f) *For any simple $H_{t^{-1}}$-module L we have $C'L = 0$.*

Proof. There is a unique involutive automorphism $h \to h^*$ of the k-algebra H such that $T_r^* = -qT_r^{-1} = q - 1 - T_r$ $(r \in S_0)$, $\theta_x^* = \theta_{x^{-1}}$ $(x \in P)$ [3, 2.13(d)]. Noting that $C^* = (-1)^{l(w_0)}C'$, we see that (a) and (d) are equivalent, (e) and (f) are equivalent.

Using the involutive anti-automorphism $h \to \tilde{h}$ of the k-algebra H defined by $\tilde{T}_r = T_r$ $(r \in S_0)$, $\tilde{\theta_x} = \theta_x$ $(x \in P)$ [3, 2.13(c)] and noting that $\tilde{C} = C$ and $\tilde{C}' = C'$, we see that (b) and (c) are equivalent.

Since the two-sided ideal H_{c_0} of H spanned by all C_w, $w \in c_0$ is generated by C, using [11, 7.7] we know that (a) and (e) are equivalent.

Now we show that (a) and (b) are equivalent. Since $T_wC = CT_w = q^{l(w)}C$ if $w \in W_0$, we see that CHC is spanned by $C\theta_xC$. By formula (1) in 3.1, we have

$$C\theta_xC = q^{-\frac{l(w_0)}{2}} \sum_{w \in W_0} w(\theta_x \prod_{\alpha \in R^+} \frac{1 - q\theta_\alpha}{1 - \theta_\alpha})C.$$

So we have the following assertion.

(i) The condition $CH_tC = 0$ is equivalent to

$$\phi_t(\sum_{w \in W_0} w(\theta_x \prod_{\alpha \in R^+} \frac{1 - q\theta_\alpha}{1 - \theta_\alpha})) = 0, \quad \text{for all } x \in P.$$

Using 3.2 (c) we know that H is spanned by all $T_wz\theta_\rho\theta_{e_u}$, $w, u \in W_0$, $z \in Z(H)$. Therefore we have the claim below.

(ii) The condition $CH_tC = 0$ is equivalent to

$$\phi_t(\sum_{w \in W_0} w(\theta_\rho\theta_{e_u} \prod_{\alpha \in R^+} \frac{1 - q\theta_\alpha}{1 - \theta_\alpha})) = 0, \quad \text{for all } u \in W_0.$$

Similar to [10, Lemma 2.10], we see that HC' is spanned by all $T_w z\theta_I C'$, $w \in W_0$, $z \in Z(H)$, $I \subseteq \Delta$, where $\theta_I = \prod_{\alpha\in I}\theta_{x_\alpha}$. Since $T_wC = CT_w = q^{l(w)}C$ if $w \in W_0$ and $C\theta_I C' = 0$ if $I \neq \Delta$, as a $Z(H)$-module, CHC' is generated by $C\theta_\rho C'$.

Recall the formula (5) in 3.1:

$$C\theta_\rho C' = q^{\frac{\nu}{2}} \sum_{I\subseteq R^+} (-q)^{-|I|}\theta_\rho\theta_{\alpha_I^{-1}}C',$$

where $\nu = l(w_0) = |R^+|$, α_I is the sum of all roots in I and $|I|$ is the cardinality of I.

Let $A = q^{\frac{\nu}{2}}\sum_{I\subseteq R^+}(-q)^{-|I|}\theta_\rho\theta_{\alpha_I^{-1}}$. Note that

$$A = (-1)^\nu q^{-\frac{\nu}{2}}\theta_\rho^{-1}\prod_{\alpha\in R^+}(1-q\theta_\alpha).$$

Thus

$$(A,\theta_{e_u}) = (-1)^\nu q^{-\frac{\nu}{2}}\sum_{w\in W_0} w(\theta_{\rho e_u}\prod_{\alpha\in R^+}\frac{1-q\theta_\alpha}{1-\theta_\alpha}). \tag{6}$$

Since $A = \sum_{u\in W_0}(A,\theta_{e_u})\theta'_u$ in H and θ'_uC', $u \in W_0$, are linearly independent in H_tC', we obtain the following equivalence condition.

(iii) The condition $CH_tC' = 0$ is equivalent to

$$\phi_t(\sum_{w\in W_0} w(\theta_{\rho e_u}\prod_{\alpha\in R^+}\frac{1-q\theta_\alpha}{1-\theta_\alpha}) = 0 \quad \text{for all } u \in W_0.$$

Using (ii) and (iii) we see that (a) and (b) are equivalent. The theorem is proved.

Theorem 3.5. *Let $t \in T$ be such that $\alpha(t) = q$ for all simple roots α of R. Then*

(a) *CH_tC' (resp. $C'H_tC$) is a two-sided ideal of H_t with dimension 1 if $\sum_{w\in W_0} q^{l(w)} \neq 0$.*
(b) *$CH_tC' = 0$ if $\sum_{w\in W_0} q^{l(w)} = 0$.*

Proof. We have seen that CH_tC' is spanned by the image in H_t of $C\theta_\rho C'$. To see it is a two-sided ideal of H_t it suffices to prove that the images in H_t

of $\theta_x C\theta_\rho C'$ and $C\theta_\rho C'\theta_x$ for all $x \in \Theta$ are scalar multiples of the image of $C\theta_\rho C'$ in H_t.

(i) If w is not the neutral element of W_0, then there exists a positive root β such that $w(\beta) = \alpha^{-1}$ for some simple root α. Thus $w(1 - q\beta)(t) = 0$.

Let A be as in the proof of Theorem 3.4. Then $\theta_x C\theta_\rho C' = A\theta_x C'$. Since

$$(A\theta_x, \theta_{e_u}) = (-1)^\nu q^{-\frac{\nu}{2}} \sum_{w\in W_0} w(\theta_{\rho x e_u} \prod_{\alpha\in R^+} \frac{1 - q\theta_\alpha}{1 - \theta_\alpha}),$$

using (i) we get

$$(A\theta_x, \theta_{e_u})(t) = \rho(t)x(t)e_u(t) \prod_{a\in R^+} \frac{1 - q^{1+\langle\rho,\alpha^\vee\rangle}}{1 - q^{\langle\rho,\alpha^\vee\rangle}},$$

if $1 - q^{\langle\rho,\alpha^\vee\rangle} \neq 0$ for all positive roots α. We have (see for example [6, Corollary 2.17])

$$\prod_{a\in R^+} \frac{1 - q^{1+\langle\rho,\alpha^\vee\rangle}}{1 - q^{\langle\rho,\alpha^\vee\rangle}} = \sum_{w\in W_0} q^{l(w)},$$

if $1 - q^{\langle\rho,\alpha^\vee\rangle} \neq 0$ for all positive roots α. Now $(A\theta_x, \theta_{e_u})$ is in $Z(H)$, so $(A\theta_x, \theta_{e_w})(t)$ is a regular function in $q \in k^*$. Thus we have

$$(A\theta_x, \theta_{e_u})(t) = (-1)^\nu q^{-\frac{\nu}{2}} \rho(t)x(t)e_u(t) \sum_{w\in W_0} q^{l(w)} \tag{7}$$

for all $q \in k^*$. So the images in H_t of $\theta_x C\theta_\rho C'$ for all $x \in \Theta$ are scalar multiples of the image in H_t of $C\theta_\rho C'$, and CH_tC' is a left ideal of H_t. Using the involutions $h \to h^*$ and $h \to \tilde{h}$ of H several times we see that CH_tC' is a left ideal of H_t implies that it is also a right ideal of H_t.

The formula (7) also indicates that $CH_tC' = 0$ if and only if $\sum_{w\in W_0} q^{l(w)} = 0$. The theorem is proved.

It is easy to check that $T_sCH_tC' = qH_tC'$ and $CH_tC'T_s = -CH_tC'$ for all simple reflections s if $\alpha(t) = q$ for all simple roots α. So the ideals CH_tC' and $C'H_tC$ give natural realizations of some one dimensional representations of H_q.

4. Irreducible Modules Attached to the Lowest Two-Sided Cell

The main result of this section is the following.

Theorem 4.1. *Let $t \in T$, then*

(a) *The element $C\theta_\rho C'$ in H_t generates an irreducible module L_t of H if it is nonzero. Moreover, $CL_t \neq 0$ in this case.*

(b) *The element $C'\theta_\rho C$ in H_t generates an irreducible module L'_t of H if it is nonzero. Moreover, $C'L_t \neq 0$ in this case.*

Proof. Let J_{c_0} be the based ring of c_0. According to [9], J_{c_0} is isomorphic to a $|W_0| \times |W_0|$ matrix ring over R_G. Let $\mathbf{J}_{c_0} = \mathbb{C} \otimes J_{c_0}$. Then up to isomorphism, irreducible $\mathbf{J}_{c_0}$-modules are naturally in one-to-one correspondence with the semisimple classes of G. For the semisimple class containing t, let E_t be a corresponidng simple $\mathbf{J}_{c_0}$-module.

Let $\varphi_0 : H \to \mathbf{J}_{c_0}$ be Lusztig's homomorphism defined through the basis $C_w, w \in W$. Then E_t is endowed with an H-module structure through the homomorphism. Denote the H-module structure on E_t by E_{t,φ_0}. We have (see for example the proof of Theorem 3.5 in [13]) the following assertion.

(i) E_{t,φ_0} is isomorphic to H_tC.

According to [11, 7.7] and [12, Lemma 2.5], H_tC has a simple constituent L such that $CL \neq 0$ if and only if $CH_tC \neq 0$. In this case, L is the unique simple constituent of H_tC such that $CL \neq 0$ and L is also the unique simple quotient module of H_tC, i.e., the head of H_tC.

By Theorem 3.4, $CH_tC \neq 0$ is equivalent to $CH_tC' \neq 0$. Now assume that $CH_tC' \neq 0$. By the proof of [12, Lemma 2.5], the set

$$M_{t,0} = \{h \in H_tC \mid C_wh = 0, \quad \forall w \in c_0\}$$

is the unique maximal submodule of H_tC.

We have a natural H-module homomorphism: $H_tC \to H_tC\theta_\rho C'$, $h \to h\theta_\rho C'$. Therefore, to prove that $C\theta_\rho C'$ in H_t generates an irreducible module L_t of H it suffices to prove that

$$h \in M_{t,0} \Longleftrightarrow h\theta_\rho C' = 0.$$

Let Θ_t be the image of Θ in H_t. Then H_tC consists of $\theta C,\ \theta \in \Theta_t$. Since Θ is a free $Z(H)$-module with a basis $\{e_w \,|\, w \in W_0\}$, and for any $w \in c_0$ there exists $\xi,\ \eta \in \Theta$ such that $C_w = \xi C \eta$, we see that $\theta C \in H_tC$ is in $M_{t,0}$ if and only if $C\theta_{e_w}\theta C = 0$ in H_t for all $w \in W_0$. By formula (1) in 3.1, this is equivalent to

$$\phi_t\Big(\sum_{w\in W_0} w(\theta_{e_u}\theta \prod_{\alpha\in R^+}\frac{1-q\theta_\alpha}{1-\theta_\alpha})\Big) = 0, \quad \text{for all } u \in W_0. \tag{8}$$

Let A be as in the proof of Theorem 3.4, then

$$\theta C\theta_\rho C' = \theta A C'.$$

Clearly, $\theta C\theta_\rho C' = 0$ in H_t if and only if $\theta_\rho^{-1}\theta C\theta_\rho C' = 0$ in H_t. Since

$$\theta_\rho^{-1}\theta C\theta_\rho C' = \sum_{u\in W_0}(\theta_\rho^{-1}\theta A, \theta_{e_u})\theta'_u C'$$

and $\theta'_u C',\ u \in W_0$, are linearly independent in H_tC', we see that $\theta C\theta_\rho C' = 0$ if and only if $\phi_t((\theta_\rho^{-1}\theta A, \theta_{e_u})) = 0$ for all $u \in W_0$. By the formula (6) established in the proof of Theorem 3.4, we have

$$(\theta_\rho^{-1}\theta A, \theta_{e_u}) = (-1)^\nu q^{-\frac{\nu}{2}} \sum_{w\in W_0} w(\theta_{e_u}\theta \prod_{\alpha\in R^+}\frac{1-q\theta_\alpha}{1-\theta_\alpha}).$$

Hence the condition $\phi_t((\theta_\rho^{-1}\theta A, \theta_{e_u})) = 0$ for all $u \in W_0$ is exactly the condition (8). We proved (a).

Since $C'\theta_{\rho^{-1}}C = (-1)^{l(w_0)}C'\theta_\rho C$, (b) follows from (a) by applying the involution $h \to h^*$. The theorem is proved.

4.2. The left ideal $\mathcal{I}$ defined in 2.1 is in the kernel of the H-module homomorphism $\psi : H \to H_tC', h \to hC'$. Thus ψ induces an H-module homomorphism $N \to H_tC'$, denoted again by ψ. Denote the image of C_w in $H/\mathcal{I}$ by the same notation C_w. Then $C_w,\ w \in Y_0$, form a basis of $N = H/\mathcal{I}$.

For each two-sided cell c, let $N^{\leq c}$ be the submodule of N spanned by all $C_w,\ w \in Y_0$ and $w \leq_L u$ for some $u \in c\cap Y_0$. Also let $N^{<c}$ be the submodule of N spanned by all $C_w,\ w \in Y_0 - Y_0 \cap c$ and $w \leq_L u$ for some $u \in c \cap Y_0$. Then $\psi(N^{<c})$ and $\psi(N^{\leq c})$ are submodules of H_tC'. This gives some natural

submodules of H_tC'. Theorem 4.1 asserts that $\psi(N^{\leq c_0}) = H_tC\theta_\rho C'$ is either 0 or an irreducible submodule of H_tC'. In general, $\psi(N^{\leq c})/\psi(N^{<c})$ maybe 0 or reducible, but it is not clear whether this module is semisimple if it is not 0. (We refer to [2] for the definition of preorder $\leq_L$.)

When k is the field of complex numbers, by [11, Theorem 7.8], $\psi(N^{\leq c_0})= H_tC\theta_\rho C'$ is 0 if and only if the set

$$\mathbf{g}_{t,q} = \{X \in \mathrm{Lie}(G) \mid \mathrm{Ad}(t)(X) = qX\}$$

contains nonzero semisimple elements.

For a simple H-module L, there exists a unique two-sided cell c of W such that $C_wL \neq 0$ for some $w \in c$ and $C_uL = 0$ for any $u \in W - c$ with $u \leq_{LR} w$. The two-sided cell c is denoted by c_L and is called the two-sided cell attached to L. (We refer to [2] for the definition of preorder $\leq_{LR}$.)

We have seen that for a simple H-module L, $c_L = c_0$ if and only if $CL \neq 0$. Theorem 4.1 gives a computable (in principle) realization of irreducible H-modules with attached two-sided cell c_0.

Acknowledgment

The author thanks the referee for helpful comments.

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