Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **13** (2018), No. 1, pp. 31-42
DOI: 10.21915/BIMAS.2018102

KOSTKA-SHOJI POLYNOMIALS AND LUSZTIG'S CONVOLUTION DIAGRAM

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To George Lusztig on his 70th birthday, with admiration

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Abstract

We propose an r-variable version of Kostka-Shoji polynomials $K^-_{\boldsymbol{\lambda\mu}}$ for r-multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$. Our version has positive integral coefficients and encodes the graded multiplicities in the space of global sections of a line bundle over Lusztig's iterated convolution diagram for the cyclic quiver $\tilde{A}_{r-1}$.

1. Introduction

Let G be a reductive complex algebraic group. According to G. Lusztig [9], the IC-stalks of $G[[z]]$-orbit closures in the affine Grassmannian Gr_G are encoded by the Kostka polynomials associated to the Langlands dual group $G^\vee$. According to R. Brylinski [4], the same Kostka polynomials encode the graded multiplicities in the global sections of line bundles on the cotangent bundle of the flag variety of $G^\vee$. According to [2], [5], the IC-stalks of $GL_N[[z]]$-orbit closures in the mirabolic affine Grassmannian of GL_N are encoded by the Kostka-Shoji polynomials [16]. We note that the same Kostka-Shoji polynomials encode the graded multiplicities in the global sections of

Received May 8, 2016 and in revised form September 12, 2016.
AMS Subject Classification: 05E15 (14M15, 13A35).
Key words and phrases: Kostka-Shoji polynomials, cyclic quiver, convolution diagram, Frobenius splitting, affine flag variety, Bott-Samelson-Demazure-Hansen resolution.

line bundles on a certain vector bundle over the square of the flag variety of GL_N. This vector bundle is nothing but Lusztig's iterated convolution diagram for the cyclic $\tilde{A}_1$ quiver [11]. The higher cohomology vanishing of the above line bundles follows from the Frobenius splitting of this convolution diagram which in turn follows from the fact that the convolution diagram is related to a Bott-Samelson-Demazure-Hansen (BSDH for short) variety of affine type A [10].

The dilation action of $\mathbb{G}_m$ on Lusztig's convolution diagram extends to an action of $\mathbb{G}_m \times \mathbb{G}_m$ which gives rise to a 2-variable version of Kostka-Shoji polynomials $K_{\boldsymbol{\lambda\mu}}(t_1, t_2)$ such that $K_{\boldsymbol{\lambda\mu}}(t, t) = K_{\boldsymbol{\lambda\mu}}(t)$ (the classical Kostka-Shoji polynomial). Note that the realization of Kostka-Shoji polynomials via the IC-stalks on mirabolic affine Grassmannian cannot give rise to a 2-variable version since these stalks are pure Tate [2], [18], [5].

The (multi)graded multiplicities in the global sections of line bundles on Lusztig's iterated convolution diagram for the cyclic $\tilde{A}_{r-1}$ quiver are encoded conjecturally by an r-variable version of Kostka-Shoji polynomials $K^-_{\boldsymbol{\lambda\mu}}(t)$ [16] for r-multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$. The higher cohomology vanishing is proved by the same argument as above. It would be interesting to find out if Lusztig's convolution diagrams for more general quivers are Frobenius split.

We are grateful to P. Achar, R. Bezrukavnikov, A. Braverman, B. Feigin, V. Ginzburg, S. Kato, A. Kuznetsov, L. Rybnikov, V. Serganova, T. Shoji, R. Travkin, M. Vikulina, L. Yanushevich for the helpful discussions. The work of M. F. was supported by the Russian Foundation for Sciences (project no. 14-50-00150) and carried out at the Institute for Information Transmission Problems of the Russian Academy of Sciences. The study of A. I. has been funded by the Russian Academic Excellence Project '5-100'. A. I. was supported in part by Dobrushin stipend and grant RFBR 15-01-09242.

2. Kostka-Shoji Polynomials

2.1. Dominance order on multipartitions

We denote by $\mathcal{P}^r_N \subset \mathbb{Z}^{rN}$ the set of generalized r-multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ such that for any $s = 1, \dots, r$ the corresponding $\lambda^{(s)} = (\lambda^{(s)}_1 \geq \lambda^{(s)}_2 \geq \dots \geq \lambda^{(s)}_N)$ is a weakly decreasing sequence of integers of length N.

We order the entries of $\boldsymbol{\lambda}$ lexicographically as follows:

$$\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(r)}, \lambda_2^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_2^{(r)}, \ldots, \lambda_N^{(r-1)}, \lambda_N^{(r)}.$$

For any $n = 1, \ldots, rN$ we denote by $\Sigma_n(\boldsymbol{\lambda})$ the sum of the first n entries in the above order. We say that $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ in the *dominance order* if $\Sigma_n(\boldsymbol{\lambda}) \geq \Sigma_n(\boldsymbol{\mu})$ for any $n = 1, \ldots, rN-1$, and $\Sigma_{rN}(\boldsymbol{\lambda}) = \Sigma_{rN}(\boldsymbol{\mu})$. If $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$, then $\alpha := \boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathbb{N}^{rN-1}$ is the vector with coordinates $(\Sigma_n(\boldsymbol{\lambda}) - \Sigma_n(\boldsymbol{\mu}))_{n=1,\ldots,rN-1}$.

2.2. Partition function

Let $\delta_n,\ 1 \leq n \leq rN-1$, be the base of $\mathbb{N}^{rN-1}$. For $1 \leq m < n \leq rN$ we set $\alpha_{mn} := \sum_{l=m}^{n-1} \delta_l$. We define a finite subset $R_r^+ \subset \mathbb{N}^{rN-1}$ of positive pseudoroots as follows: $R_r^+ := \{\alpha_{mn}\}_{n-m=1 \pmod r}$.

Given $\alpha \in \mathbb{N}^{rN-1}$ we define a polynomial $L_r^\alpha(t)$ (Lusztig's partition function) as follows: $L_r^\alpha(t) := \sum p_d t^d$ where p_d is the number of (unordered) partitions of α into a sum of d positive pseudoroots. We extend $L_r(t)$ from $\mathbb{N}^{rN-1}$ to $\mathbb{Z}^{rN-1}$ by zero.

We also introduce a multivariable version of $L_r^\alpha(t_1, \ldots, t_r)$ where the variables are numbered by $\mathbb{Z}/r\mathbb{Z} = \{1, \ldots, r\}$. Namely, $L_r^\alpha(t_1, \ldots, t_r) = \sum p_{\underline{d}} \prod_{s \in \mathbb{Z}/r\mathbb{Z}} t_s^{d_s}$ where $\underline{d} = (d_1, \ldots, d_r) \in \mathbb{N}^{\mathbb{Z}/r\mathbb{Z}}$, and $p_{\underline{d}}$ is the number of unorderd partitions of α into a sum of positive pseudoroots having d_s summands α_{mn} with $m = d_s \pmod r$ for any $s \in \mathbb{Z}/r\mathbb{Z}$. Clearly, the restriction of $L_r^\alpha(t_1, \ldots, t_r)$ to the diagonal $t_1 = \ldots = t_r = t$ coincides with $L_r^\alpha(t)$. We extend $L_r(t_1, \ldots, t_r)$ from $\mathbb{N}^{rN-1}$ to $\mathbb{Z}^{rN-1}$ by zero.

2.3. Lusztig-Kato formula

We set $\rho = (N, N-1, \ldots, 2, 1)$, and $\boldsymbol{\rho} = (\rho, \ldots, \rho) \in \mathcal{P}_N^r$. Given $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_N^r$ we define $K_{\boldsymbol{\lambda}\boldsymbol{\mu}}(t) := \sum_{\boldsymbol{\sigma} \in \mathfrak{S}_N^r} (-1)^{\boldsymbol{\sigma}} L_r^{\boldsymbol{\sigma}(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho}-\boldsymbol{\mu}}(t)$, the sum over the product of r copies of the symmetric group $\mathfrak{S}_N$ acting on $(\mathbb{Z}^N)^r$ by permutations of entries of each composition.

We also introduce a multivariable version

$$K_{\boldsymbol{\lambda}\boldsymbol{\mu}}(t_1, \ldots, t_r) := \sum_{\boldsymbol{\sigma} \in \mathfrak{S}_N^r} (-1)^{\boldsymbol{\sigma}} L_r^{\boldsymbol{\sigma}(\boldsymbol{\lambda}+\boldsymbol{\rho})-\boldsymbol{\rho}-\boldsymbol{\mu}}(t_1, \ldots, t_r).$$

Clearly, $K_{\boldsymbol{\lambda\mu}}(t,\dots,t) = K_{\boldsymbol{\lambda\mu}}(t)$.

Recall the Kostka-Shoji polynomials $K^{\pm}_{\boldsymbol{\lambda\mu}}(t)$ [16, 3.1]. In case $r = 1$, $K^{+}_{\lambda\mu}(t) = K^{-}_{\lambda\mu}(t)$ is the classical Kostka polynomial, and it was proved by I. G. Macdonald [12, page 243] that $K^{+}_{\lambda\mu}(t) = K^{-}_{\lambda\mu}(t) = K_{\lambda\mu}(t)$ for $\lambda \geq \mu$.[1] In case $r = 2$, the identity $K^{+}_{\boldsymbol{\lambda\mu}}(t) = K^{-}_{\boldsymbol{\lambda\mu}}(t) = K_{\boldsymbol{\lambda\mu}}(t)$ for $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ was proved by T. Shoji [16, Proof of Proposition 3.3]. The following generalization of these identities for arbitrary r is supported by the calculations by L. Yanushevich for multipartitions of total size ≤ 7, using P. Achar's code [1].

Conjecture 2.4. *For multipartitions $\boldsymbol{\lambda} \geq \boldsymbol{\mu} \in \mathcal{P}^r_N$ we have $K^{-}_{\boldsymbol{\lambda\mu}}(t) = K_{\boldsymbol{\lambda\mu}}(t)$.*

3. Lusztig's Convolution Diagram

3.1. A vector bundle over a flag variety

We consider the following ordered base of an rN-dimensional vector space $\mathbb{C}^{rN}$: $v^{(1)}_1,\dots,v^{(r)}_1,v^{(1)}_2,\dots,v^{(r)}_2,\dots,v^{(1)}_N,\dots,v^{(r)}_N$. Sometimes, for $1 \leq s \leq r$, $1 \leq j \leq N$, we denote $v^{(s)}_j$ by $v_{r(j-1)+s}$. It gives rise to an embedding $GL^r_N \hookrightarrow GL_{rN}$ (s-th copy of GL_N acts in the summand spanned by $v^{(s)}_1,\dots,v^{(s)}_N$), and also to an embedding of the Borel upper triangular subgroups $B^r_N \hookrightarrow B_{rN}$. In the adjoint representation of GL_{rN} restricted to B^r_N we consider a subrepresentation $\mathfrak{n}_r$ (of B^r_N) spanned by the elementary matrices E_{mn}, $1 \leq m < n \leq rN$ such that $n - m = 1 \pmod r$. It gives rise to a GL^r_N-equivariant vector bundle $\mathcal{T}^*_r\mathcal{B}^r_N = GL^r_N \times^{B^r_N} \mathfrak{n}_r$ over the flag variety $\mathcal{B}^r_N$ of GL^r_N. Note that when $r = 1$, the vector bundle $\mathcal{T}^*_1\mathcal{B}_N$ over the flag variety $\mathcal{B}_N$ is nothing but the cotangent bundle.

Let $x_1,\dots,x_{rN}$ stand for the characters of the diagonal Cartan torus T_{rN} of GL_{rN} corresponding to the diagonal matrix entries. Sometimes, for $1 \leq s \leq r$, $1 \leq j \leq N$, we denote $x_{r(j-1)+s}$ by $x^{(s)}_j$. For $1 \leq m < n \leq rN$ we set $x^{\alpha_{mn}} = x^{-1}_m x_n$. This is the weight of the elementary matrix E_{nm}. This rule extends to a homomorphism $\mathbb{N}^{rN-1} \to X^*(T_{rN})$, $\alpha \to x^\alpha$. The symmetric algebra $\mathrm{Sym}^\bullet\mathfrak{n}^\vee_r$ is graded, and its character is a formal series in $x_1,\dots,x_{rN},t$. In fact, $\mathrm{Sym}^\bullet\mathfrak{n}^\vee_r$ has a finer grading by $\mathbb{N}^{\mathbb{Z}/r\mathbb{Z}}$ arising

[1] A similar identity for arbitrary finite root systems was conjectured by G. Lusztig [9, (9.4)] and proved by S.-I. Kato [7, Theorem 1.3].

from a $\mathbb{Z}/r\mathbb{Z}$-grading of $\mathfrak{n}_r^\vee$: $\deg E_{nm} := m \pmod r$. Hence the character of $\mathrm{Sym}^\bullet \mathfrak{n}_r^\vee$ is a formal series χ in $x_1, \ldots, x_{rN}, t_1, \ldots, t_r$.

Lemma 3.2. $\chi = \sum_{\alpha \in \mathbb{N}^{rN-1}} L^\alpha(t_1, \ldots, t_r) x^\alpha$.

Proof. Clear. □

Given a multipartition $\boldsymbol{\mu} \in \mathcal{P}_N^r$, we consider the corresponding GL_N^r-equivariant line bundle $\mathcal{O}(\boldsymbol{\mu})$ on $\mathcal{B}_N^r$: the action of B_N^r on its fiber at the point $B_N^r \in \mathcal{B}_N^r$ is via the character $\prod (x_j^{(s)})^{-\mu_j^{(s)}}$. Its global sections $\Gamma(\mathcal{B}_N^r, \mathcal{O}(\boldsymbol{\mu}))$ is an irreducible GL_N^r-module $V^{\boldsymbol{\mu}}$ with lowest weight $-\boldsymbol{\mu}$. The character of $V^{\boldsymbol{\mu}}$ will be denoted $\chi^{\boldsymbol{\mu}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{rN}^{\pm 1}]$. The pullback of $\mathcal{O}(\boldsymbol{\mu})$ to $\mathcal{T}_r^* \mathcal{B}_N^r$ will be also denoted $\mathcal{O}(\boldsymbol{\mu})$ when no confusion is likely. We consider the equivariant Euler characteristic $\chi(\mathcal{T}_r^* \mathcal{B}_N^r, \mathcal{O}(\boldsymbol{\mu})) = \chi(\mathcal{B}_N^r, \mathrm{Sym}^\bullet \mathcal{T}_r \mathcal{B}_N^r \otimes \mathcal{O}(\boldsymbol{\mu}))$ where $\mathcal{T}_r \mathcal{B}_N^r = GL_N^r \times^{B_N^r} \mathfrak{n}_r^\vee$ stands for the vector bundle over $\mathcal{B}_N^r$ dual to $\mathcal{T}_r^* \mathcal{B}_N^r$. The $\mathbb{N}^{\mathbb{Z}/r\mathbb{Z}}$-grading of $\mathrm{Sym}^\bullet \mathfrak{n}_r^\vee$ gives rise to a $\mathbb{N}^{\mathbb{Z}/r\mathbb{Z}}$-grading of $\mathrm{Sym}^\bullet \mathcal{T}_r \mathcal{B}_N^r \otimes \mathcal{O}(\boldsymbol{\mu})$, and hence $\chi(\mathcal{B}_N^r, \mathrm{Sym}^\bullet \mathcal{T}_r \mathcal{B}_N^r \otimes \mathcal{O}(\boldsymbol{\mu}))$ is a formal series in $x_1, \ldots, x_{rN}, t_1, \ldots, t_r$.

Corollary 3.3. $\chi(\mathcal{B}_N^r, \mathrm{Sym}^\bullet \mathcal{T}_r \mathcal{B}_N^r \otimes \mathcal{O}(\boldsymbol{\mu})) = \sum_{\boldsymbol{\lambda} \geq \boldsymbol{\mu}} K_{\boldsymbol{\lambda}\boldsymbol{\mu}}(t_1, \ldots, t_r) \chi^{\boldsymbol{\lambda}}$.

Proof. Same as the proof of [4, Lemma 6.1]. □

3.4. Convolution diagram

Recall the notations of [11, Section 1]. We consider the type $\tilde{A}_{r-1}$ cyclic quiver Q with the set $\mathbb{Z}/r\mathbb{Z}$ of vertices, and with arrows $s \to s-1$, $s \in \mathbb{Z}/r\mathbb{Z}$. Let $\mathbf{V}$ be a $\mathbb{Z}/r\mathbb{Z}$-graded vector space such that $\dim \mathbf{V}_s = N$ for any $s \in \mathbb{Z}/r\mathbb{Z}$. Let $\mathbf{i}$ be a length rN periodic sequence $(r, r-1, \ldots, 2, 1, r, r-1, \ldots, 2, 1, \ldots, r, \ldots, 1)$ of vertices, and let $\mathbf{a}$ be a length rN sequence $(1, 1, \ldots, 1)$ of positive integers. Then the variety $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ of all flags of type $(\mathbf{i}, \mathbf{a})$ in $\mathbf{V}$ is nothing but $\mathcal{B}_N^r$. Moreover, the iterated convolution diagram $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ of [11, 1.5] is nothing but $\mathcal{T}_r^* \mathcal{B}_N^r$. In effect, we identify $\mathbf{V}_s$ with a vector subspace of $\mathbb{C}^{rN}$ spanned by $v_1^{(s)}, \ldots, v_N^{(s)}$ (notations of 3.1). Then the fiber of the natural $GL(\mathbf{V}) = GL_N^r$-equivariant projection $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \to \mathcal{F}_{\mathbf{i},\mathbf{a}}$ over the flag $\mathbf{V} = \mathbb{C}v_1^{(1)} \oplus \ldots \oplus \mathbb{C}v_N^{(r)} \supset \mathbb{C}v_1^{(1)} \oplus \ldots \oplus v_N^{(r-1)} \supset \cdots \supset \mathbb{C}v_1^{(1)} \oplus \ldots \oplus \mathbb{C}v_1^{(r)} \oplus \mathbb{C}v_2^{(1)} \supset \mathbb{C}v_1^{(1)} \oplus \ldots \oplus \mathbb{C}v_1^{(r)} \supset \cdots \supset \mathbb{C}v_1^{(1)} \oplus \mathbb{C}v_1^{(2)} \supset \mathbb{C}v_1^{(1)} \supset 0$ is nothing but $\mathfrak{n}_r$.

4. Frobenius Splitting of $\mathcal{T}^*_r\mathcal{B}^r_N$

In this section we replace $\mathbb{C}$ by an algebraic closure k of the finite field $\mathbb{F}_p$ of characteristic p. The present section is devoted to the proof of the following

Theorem 4.1. *$\mathcal{T}^*_r\mathcal{B}^r_N$ is Frobenius split.*

Our proof is a variation of the one in [14].

4.2. The canonical bundle of $\mathcal{T}^*_r\mathcal{B}^r_N$

We have a subgroup $SL^r_N \subset GL^r_N$ (product of r copies of SL_N).

Lemma 4.3. *The canonical line bundle ω of $\mathcal{T}^*_r\mathcal{B}^r_N$ is SL^r_N-equivariantly trivial.*

Proof. The product w_1 of T_{rN}-weights in $\mathfrak{n}_r$ is

$$\prod_{s=2}^{r}\prod_{1\le k\le l\le N} x_{r(k-1)+s-1}x^{-1}_{r(l-1)+s}\cdot\prod_{1\le k<l\le N} x_{rk}x^{-1}_{r(l-1)+1}$$
$$=\prod_{s=1}^{r}\prod_{k=1}^{N} x^{N+1-2k}_{r(k-1)+s}\cdot\prod_{k=1}^{N} x_{rk}x^{-1}_{r(k-1)+1}.$$

The product w_2 of T_{rN}-weights in the tangent space of $\mathcal{B}^r_N$ at $B^r_N \in \mathcal{B}^r_N$ is

$$\prod_{s=1}^{r}\prod_{N\ge k>l\ge 1} x_{r(k-1)+s}x^{-1}_{r(l-1)+s}=\prod_{s=1}^{r}\prod_{k=1}^{N} x^{2k-N-1}_{r(k-1)+s}.$$

The T_{rN}-weight in the fiber of the canonical bundle at the point $B^r_N \in \mathcal{B}^r_N \subset \mathcal{T}^*_r\mathcal{B}^r_N$ is $w = w_1^{-1}w_2^{-1} = \prod_{k=1}^N x^{-1}_{rk}x_{r(k-1)+1}$. When we restrict w to the maximal torus of SL^r_N we get the trivial weight, hence the canonical line bundle ω is SL^r_N-equivariantly trivial. □

4.4. A splitting section

According to [3, Theorem 1.3.8], in order to prove the Frobenius splitting of $\mathcal{T}^*_r\mathcal{B}^r_N$ it suffices to construct a section $\phi \in \Gamma(\mathcal{T}^*_r\mathcal{B}^r_N, \omega^{1-p})$ in whose expansion with respect to some local coordinates $z_1, \dots, z_d$ the monomial

$z_1^{p-1} \cdots z_d^{p-1}$ occurs with coefficient 1. Since ω is SL_N^r-equivariantly trivial, it has an SL_N^r-invariant nowhere vanishing section ϖ, and we will look for the desired section ϕ in the form $\phi = f\varpi^{1-p}$ for some $f \in \mathsf{k}[\mathcal{T}_r^*\mathcal{B}_N^r]$.

To this end recall the decomposition $\mathsf{k}^{rN} = \bigoplus_{1\le s\le r} \mathbf{V}_s$ of Section 3.4. Accordingly, we will write down the matrices $A \in \mathfrak{gl}_{rN}$ in the block form

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

(for $1 \le s, u \le r$ the corresponding block A_{su} is an $N \times N$-matrix). The subgroup $B_N^r \subset GL_N^r \subset GL_{rN}$ is formed by all the matrices with upper-triangular diagonal blocks and vanishing non-diagonal blocks. The subspace $\mathfrak{n}_r \subset \mathfrak{gl}_{rN}$ is formed by all the matrices with strictly upper triangular block A_{r1}, nonstrictly upper triangular blocks $A_{s,s+1}$, $1 \le s \le r-1$, and all the other blocks vanishing. Hence $\mathcal{T}_r^*\mathcal{B}_N^r = GL_N^r \times^{B_N^r} \mathfrak{n}_r$ is the quotient of $GL_N^r \times \mathfrak{n}_r = \{(g_1, \dots, g_r; A_{12}, A_{23}, \dots, A_{r-1,r}, A_{r1})\}$ modulo the action of $B_N^r = \{(b_1, \dots, b_r)\}$ given by $(b_1, \dots, b_r) \cdot (g_1, \dots, g_r; A_{12}, A_{23}, \dots, A_{r-1,r}, A_{r1}) = (g_1 b_1^{-1}, \dots, g_r b_r^{-1}; b_1 A_{12} b_2^{-1} \dots, b_{r-1} A_{r-1,r} b_r^{-1}, b_r A_{r1} b_1^{-1})$. We define

$$\begin{aligned} & f(g_1, \dots, g_r; A_{12}, A_{23}, \dots, A_{r-1,r}, A_{r1}) \\ & := \prod_{s=1}^{r-1} \prod_{j=1}^{N} \Delta_j(g_s A_{s,s+1} g_{s+1}^{-1}) \cdot \prod_{j=1}^{N-1} \Delta_j(g_r A_{r1} g_1^{-1}) \end{aligned} \tag{4.1}$$

where Δ_j stands for the principal $j \times j$-minor in the upper left corner.

Proposition 4.5. *The section* $\phi = f^{p-1}\varpi^{1-p} \in \Gamma(\mathcal{T}_r^*\mathcal{B}_N^r, \omega^{1-p})$ *splits* $\mathcal{T}_r^*\mathcal{B}_N^r$.

The proof is given in Section 4.8 after a preparation in Section 4.6.

4.6. Residues

We recall the following construction [14, 3.5]. Given a smooth divisor Z in a smooth variety Y and a global section of the anticanonical class $\eta \in \Gamma(Y, \omega_Y^{-1})$ we construct the residue $\mathrm{res}\eta \in \Gamma(Z, \omega_Z^{-1})$ as follows. We start with an open subvariety $U \subset Z$ such that the normal bundle $\mathcal{N}_{Z/Y}|_U$ restricted to

U is trivial. We choose a nowhere vanishing section $\sigma \in \Gamma(U, \mathcal{N}_{Z/Y}|_U)$. Then $\mathrm{res}\eta|_U$ is defined by the requirement $\langle \mathrm{res}\eta, \zeta\rangle = \langle \eta, \zeta\frac{d\sigma}{\sigma}\rangle$ for any $\zeta \in \Gamma(U, \omega_U)$ where $\langle\ ,\ \rangle$ is the pairing between the anticanonical and canonical bundles. One can check that $\mathrm{res}\eta|_U$ is independent of the choice of σ, and for $U' \subset U$ we have $(\mathrm{res}\eta|_U)|_{U'} = \mathrm{res}\eta|_{U'}$, so the local sections $\mathrm{res}\eta|_U$ glue to the desired $\mathrm{res}\eta$. If we have a chain of smooth divisors $Y \supset Z_1 \supset \cdots \supset Z_n$ we can iterate the above construction to obtain $\mathrm{res}\colon \Gamma(Y, \omega_Y^{-1}) \to \Gamma(Z_n, \omega_{Z_n}^{-1})$.

Lemma 4.7. *There is a chain of smooth divisors $\mathcal{T}_r^*\mathcal{B}_N^r = Y \supset Z_1 \supset \cdots \supset Z_n = \mathcal{B}_N^r$ such that $(\mathrm{res} f\varpi^{-1})^{p-1} \in \Gamma(\mathcal{B}_N^r, \omega_{\mathcal{B}_N^r}^{-1})$ gives rise to a Frobenius splitting of $\mathcal{B}_N^r$ compatible with the splitting $\varphi\colon \mathrm{Fr}_* \mathcal{O}_{\mathcal{B}_N^r} \to \mathcal{O}_{\mathcal{B}_N^r}$ arising from $f^{p-1}\varpi^{1-p}$.*

Proof. It suffices to argue generically on $\mathcal{B}_N^r$. Let $X_N \subset \mathcal{B}_N$ be an open Bruhat cell: the open orbit of the strictly lower triangular subgroup $U_N^- \subset GL_N$. We consider an open cell $X := X_N^r \times \mathfrak{n}_r \subset \mathcal{T}_r^*\mathcal{B}_N^r$; we have $X \cap \mathcal{B}_N^r = X_N^r$. We will calculate residues on X, so in the definition (4.1) of the function f we will assume that g_s are strictly lower triangular for any $1 \le s \le r$. But for $g \in U_N^-$ and $A \in \mathfrak{gl}_N$ we have $\Delta_j(gA) = \Delta_j(A)$ for any $1 \le j \le N$. Hence $\Delta_j(g_s A_{s,s+1} g_{s+1}^{-1}) = \Delta_j(A_{s,s+1} g_{s+1}^{-1}) = \Delta_j(g_{s+1} A_{s,s+1} g_{s+1}^{-1})$. Therefore, we can identify $X \subset \mathcal{T}_r^*\mathcal{B}_N^r$ with $\hat{X} := (U_N^- \times \mathfrak{b}_N)^{r-1} \times (U_N^- \times \mathfrak{u}_N) \subset (GL_N \times^{B_N} \mathfrak{b}_N)^{r-1} \times (GL_N \times^{B_N} \mathfrak{u}_N)$ (where $\mathfrak{b}_N \supset \mathfrak{u}_N$ is the Lie algebra of B_N and its nilpotent radical) so that $f\varpi^{-1}|_X = \hat{f}\hat{\varpi}^{-1}|_{\hat{X}}$. Here $\hat{\varpi}$ is an SL_N^r-invariant nowhere vanishing volume form on $\hat{X}$, and

$$\hat{f}(g_1, A_1, \ldots, g_r, A_r) := \prod_{s=1}^{r}\prod_{j=1}^{N} \Delta_j(g_s A_s g_s^{-1}).$$

According to [14, Theorem 3.8], the chains of divisors required in the lemma exist for each factor $\hat{X}_s$ of $\hat{X}$ (that is, $GL_N \times^{B_N} \mathfrak{b}_N \supset \mathcal{B}_N$ or $GL_N \times^{B_N} \mathfrak{u}_N \supset \mathcal{B}_N$) equipped with the function $\hat{f}_s := \prod_{j=1}^{N} \Delta_j(g_s A_s g_s^{-1})$ and section $\hat{\varpi}_s$, and the required compatibilities hold. Hence the desired compatibility holds for their external product. ☐

4.8. Proof of Proposition 4.5

The section $f^{p-1}\varpi^{1-p} \in \Gamma(\mathcal{T}_r^*\mathcal{B}_N^r, \omega^{1-p})$ gives rise to a morphism $\varphi\colon \mathrm{Fr}_* \mathcal{O}_{\mathcal{T}_r^*\mathcal{B}_N^r} \to \mathcal{O}_{\mathcal{T}_r^*\mathcal{B}_N^r}$, and we have to check that the composition $\mathcal{O}_{\mathcal{T}_r^*\mathcal{B}_N^r} \to$

$\mathrm{Fr}_* \mathcal{O}_{\mathcal{T}_r^* \mathcal{B}_N^r} \to \mathcal{O}_{\mathcal{T}_r^* \mathcal{B}_N^r}$ is the identity morphism. It suffices to check $\varphi(1) = 1$. We know from Lemma 4.7 that $\varphi(1)|_{\mathcal{B}_N^r} = 1$.

We consider a one-parametric central subgroup $\mathbb{G}_m \hookrightarrow GL_N^r$ whose s-th component is $t \mapsto t^s$ for $1 \le s \le r$. Then $f\varpi^{-1}$ is a $\mathbb{G}_m$-eigensection with a nontrivial character. Hence the zero divisor of $\varphi(1)$ is $\mathbb{G}_m$-invariant. Since $(\mathcal{T}_r^* \mathcal{B}_N^r)^{\mathbb{G}_m} = \mathcal{B}_N^r$ and $\varphi(1)|_{\mathcal{B}_N^r} = 1$, the function $\varphi(1)$ has an empty zero divisor. Since the fibers of the projection $\mathcal{T}_r^* \mathcal{B}_N^r \to \mathcal{B}_N^r$ are vector spaces, and a nowhere vanishing function on a vector space is constant, we conclude $\varphi(1) = 1$. □

5. Frobenius Splitting of Lusztig's Convolution Diagrams

The aim of this Section is a proof of the following generalization of Theorem 4.1:

Theorem 5.1. *Let* $\mathbf{i}$ *be an arbitrary length* ℓ *sequence of vertices of the cyclic quiver* Q*. Let* $\mathbf{a}$ *be an arbitrary length* ℓ *sequence of positive integers. Then the iterated convolution diagram* $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ *is Frobenius split.*

Our proof follows [10, Section 11] covering $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ by an open subvariety of an affine type A BSDH resolution, and then applying Frobenius splitting for BSDH resolutions [13, Lemme 52].

5.2. Recollections of [10], [11]

We fix $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{N}^{\mathbb{Z}/r\mathbb{Z}}$, and consider a $\mathbb{Z}/r\mathbb{Z}$-graded vector space $\mathbf{V} = \bigoplus_{s \in \mathbb{Z}/r\mathbb{Z}} \mathbf{V}_s$ such that $\dim \mathbf{V}_s = d_s$. Given a length ℓ sequence $\mathbf{i} = (s_1, \ldots, s_\ell) \in (\mathbb{Z}/r\mathbb{Z})^\ell$ and a length ℓ sequence $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{N}^\ell$ such that $\sum_{n\colon s_n = s} a_n = d_s$ for any $s \in \mathbb{Z}/r\mathbb{Z}$, we consider the iterated convolution diagram $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} = \{(\mathbf{V}^\bullet, f)\}$. Here $\mathbf{V}^\bullet$ is a $\mathbb{Z}/r\mathbb{Z}$-graded flag in $\mathbf{V}\colon \mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^\ell = 0$ such that $\mathbf{V}^{n-1}/\mathbf{V}^n$ is an a_n-dimensional vector space supported at the vertex s_n for any $n = 1, \ldots, \ell$, and $f = (f_s\colon \mathbf{V}_s \to \mathbf{V}_{s-1})_{s \in \mathbb{Z}/r\mathbb{Z}}$ is a Q-module structure on $\mathbf{V}$ such that $f\mathbf{V}^{n-1} \subset \mathbf{V}^n$ for any $1 \le n \le \ell$. The convolution diagram $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ is smooth, being a vector bundle over a flag variety of $GL(\mathbf{V}) = \prod_{s \in \mathbb{Z}/r\mathbb{Z}} GL(\mathbf{V}_s)$. We have a projection $\pi\colon \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \to \mathbf{E}_\mathbf{V}$, $(\mathbf{V}^\bullet, f) \mapsto f$ to the vector space $\mathbf{E}_\mathbf{V}$ of Q-modules with underlying space $\mathbf{V}$. The morphism π is proper, and its image is the closure $\overline{\mathbb{O}}_{\mathbf{i},\mathbf{a}}$ of a nilpotent $GL(\mathbf{V})$-orbit in $\mathbf{E}_\mathbf{V}$. The union of all nilpotent $GL(\mathbf{V})$-orbits in $\mathbf{E}_\mathbf{V}$ is a closed subvariety $\mathbf{E}_\mathbf{V}^{\mathrm{nil}} \subset \mathbf{E}_\mathbf{V}$ (possibly reducible).

Let $d := d_1 + \cdots + d_r$. Let F be a d-dimensional vector space over the Laurent series field $\mathsf{k}((\epsilon))$. We fix a flag of lattices $\cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$ in F such that $L_{s+r} = \epsilon L_s,\ L_s \supset L_{s+1}$ and $\dim(L_s/L_{s+1}) = d_{s\ (\mathrm{mod}\ r)}$ for any $s \in \mathbb{Z}$. We consider a type $\tilde{A}_{d-1}$ affine Schubert variety Z formed by all the flags of lattices $\cdots \supset M_{-1} \supset M_0 \supset M_1 \supset \cdots$ in F such that $M_{s+r} = \epsilon M_s,\ M_s \supset M_{s+1}$ and $M_s \subset L_s,\ \dim(L_s/M_s) = d_{s\ (\mathrm{mod}\ r)}$ for any $s \in \mathbb{Z}$. In [10, 11.4], G. Lusztig constructs an open dense embedding $\varphi\colon \mathbf{E}_\mathbf{V}^{\mathrm{nil}} \hookrightarrow Z$ (the image $\mathring{Z} = \varphi(\mathbf{E}_\mathbf{V}^{\mathrm{nil}}) \subset Z$ is specified by certain transversality conditions) such that for any nilpotent $GL(\mathbf{V})$-orbit closure $\overline{\mathbb{O}}_{\mathbf{i},\mathbf{a}} \subset \mathbf{E}_\mathbf{V}^{\mathrm{nil}}$ its image $\varphi(\overline{\mathbb{O}}_{\mathbf{i},\mathbf{a}})$ is the intersection of $\mathring{Z}$ with a Schubert subvariety $Z_{\mathbf{i},\mathbf{a}} \subset Z$.

The construction of [10, 11.4] yields an isomorphism $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \simeq \mathring{Z} \times_Z \widetilde{Z}_{\mathbf{i},\mathbf{a}}$ for a BSDH type resolution $\widetilde{Z}_{\mathbf{i},\mathbf{a}} \to Z_{\mathbf{i},\mathbf{a}}$ formed by all the collections $(M_s^n)_{s\in\mathbb{Z}}^{0\le n\le \ell}$ of double flags of lattices such that (a) $M_s^0 = L_s$; (b) $M_s^n \supset M_{s+1}^n$ and $M_{s+r}^n = \epsilon M_s^n$; (c) $M_s^{n-1} = M_s^n$ unless $s = s_n \pmod r$; (d) if $s = s_n \pmod r$, then $M_s^{n-1} \supset M_s^n$, and $\dim M_s^{n-1}/M_s^n = a_n$.

5.3. BSDH resolution

According to [3, Lemma 1.4.5], in order to construct a Frobenius splitting of $\widetilde{Z}_{\mathbf{i},\mathbf{a}}$ (and hence of its open subvariety $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$) it suffices to construct a proper dominant morphism $\varpi\colon \widehat{Z}_{\mathbf{i},\mathbf{a}} \to \widetilde{Z}_{\mathbf{i},\mathbf{a}}$ with connected fibers from a Frobenius split variety $\widehat{Z}_{\mathbf{i},\mathbf{a}}$. We will construct $\varpi\colon \widehat{Z}_{\mathbf{i},\mathbf{a}} \to \widetilde{Z}_{\mathbf{i},\mathbf{a}}$ in two steps.

First we define $\widehat{Z}'_{\mathbf{i},\mathbf{a}} := \widetilde{Z}_{\mathbf{i},\mathbf{a}} \times_Z \mathcal{F}\ell_Z$ where $\mathcal{F}\ell_Z$ is formed by all the *complete* flags of lattices $\cdots \supset K_{-1} \supset K_0 \supset K_1 \supset \cdots$ (so that $K_{u+d} = \epsilon K_u,\ K_u \supset K_{u+1}$ and $\dim K_u/K_{u+1} = 1$ for any $u \in \mathbb{Z}$) such that for $s \ge 0,\ L_s \supset K_{d_0+d_1+\cdots+d_s}$ and $\dim L_s/K_{d_0+d_1+\cdots+d_s} = d_s$, while for $s < 0,\ L_s \supset K_{d_0-d_{-1}-\cdots-d_s}$ and $\dim L_s/K_{d_0-d_{-1}-\cdots-d_s} = d_s$. The evident projection $\mathcal{F}\ell_Z \to Z$ sends $K_\bullet$ to $M_\bullet$ where for $s \ge 0,\ M_s = K_{d_0+d_1+\cdots+d_s}$, while for $s < 0,\ M_s = K_{d_0-d_{-1}-\cdots-d_s}$. This projection is a fibration with a fiber isomorphic to a (finite) flag variety of type A.

Let us choose a base point $K^0_\bullet \in \mathcal{F}\ell_Z$ such that $K^0_0 = L_0$, and for $s > 0,\ K^0_{d_1+\cdots+d_s} = L_s$, and for $s < 0,\ K^0_{-d_{-1}-\cdots-d_s} = L_s$. Then the connected component $\mathcal{F}\ell$ of the ind-variety of complete flags of lattices containing $K^0_\bullet$ is identified with the affine flag variety of SL_d. The simple reflections of its affine Weyl group are numbered by $\mathbb{Z}/d\mathbb{Z}$, and any finite sequence $\underline{u} = (u(1), u(2), \ldots, u(k)),\ u(j) \in \mathbb{Z}/d\mathbb{Z}$, gives rise to a BSDH variety $D_{\underline{u}} \to \mathcal{F}\ell$ projecting to $\mathcal{F}\ell$ with connected fibers.

We consider a concatenated sequence $\underline{u} = (\underline{u}_\ell, \ldots, \underline{u}_1)$ where for $1 \leq n \leq \ell$ $\underline{u}_n$ is a sequence of integers in the interval

$$\left[d_1 + \cdots + d_{s_n} + 1 + \sum_{m<n:\ s_m=s_n} a_m,\ d_1 + \cdots + d_{s_n+1} - 1 + \sum_{m<n:\ s_m=s_n+1} a_m\right]$$

giving a reduced expression of the longest element of the (finite) parabolic Weyl subgroup generated by the simple reflections numbered by (the residues modulo d of) the integers in the above interval. Then there is a dominant projection $D_{\underline{u}} \to \widehat{Z}'_{\mathbf{i},\mathbf{a}}$ with connected fibers. Finally, $D_{\underline{u}}$ is Frobenius split according to [13, Lemme 52]. Theorem 5.1 is proved. □

6. Cohomology Vanishing

We say that a generalized multipartition $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)}) \in \mathcal{P}^r_N$ is *regular* if for any $s = 1, \ldots, r$ we have $\mu^{(s)}_1 > \mu^{(s)}_2 > \cdots > \mu^{(s)}_N$. In this case the line bundle $\mathcal{O}(\boldsymbol{\mu})$ on $\mathcal{T}^*_r\mathcal{B}^r_N$ is very ample, and we deduce from Theorem 4.1 and [3, Lemma 1.2.7(i)] the following

Corollary 6.1. *For a regular multipartition $\boldsymbol{\mu} \in \mathcal{P}^r_N$, we have the higher cohomology vanishing $H^{>0}(\mathcal{T}^*_r\mathcal{B}^r_N, \mathcal{O}(\boldsymbol{\mu})) = 0$.* □

Similarly to [3, Theorem 5.2.12] we put forth the following

Conjecture 6.2. *For a multipartition $\boldsymbol{\mu} \in \mathcal{P}^r_N$, we have the higher cohomology vanishing $H^{>0}(\mathcal{T}^*_r\mathcal{B}^r_N, \mathcal{O}(\boldsymbol{\mu})) = 0$.*

From Corollary 3.3 and Conjecture 6.2 we deduce

Corollary 6.3. *For any multipartition $\boldsymbol{\mu}$ we have $[\Gamma(\mathcal{B}^r_N, \mathrm{Sym}^\bullet\mathcal{T}_r\mathcal{B}^r_N \otimes \mathcal{O}(\boldsymbol{\mu}))] = \sum_{\boldsymbol{\lambda}\geq\boldsymbol{\mu}} K_{\boldsymbol{\lambda}\boldsymbol{\mu}}(t_1, \ldots, t_r)\chi^{\boldsymbol{\lambda}}$. Hence for any multipartitions $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ we have $K_{\boldsymbol{\lambda}\boldsymbol{\mu}}(t_1, \ldots, t_r) \in \mathbb{N}[t_1, \ldots, t_r]$.* □

7. Added in Proof

Conjecture 2.4 is proved in [17]. Conjecture 6.2 follows from [15] (see [6]). We are grateful to Wen-Wei Li and Yue Hu for bringing [15] to our attention.

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