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SPACE CURVES WITH A PRESCRIBED INDEX OF REGULARITY

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Abstract

We prove the existence of many smooth space curves of degree d and genus g with prescribed index of regularity (we have $g \le Kd^{3/2}$ with $K = 0.02$ for large d).

1. Introduction

Let $X \subset \mathbb{P}^3$ be any locally Cohen-Macaulay curve. The *index of regularity* or *regularity index* $r(X)$ of X is the minimal integer t such that $h^1(\mathcal{I}_X(x)) = 0$ for all $x \ge t$, with the convention $r(X) = -\infty$ if X is arithmetically Cohen-Macaulay. The integer $r(X)$ is related to the Castelnuovo-Mumford regularity of X, but it is often easier to compute. The computation of integer $r(X)$ was classically done for special classes of curves , e.g. curves with a singular model with only a small number of singularities or smooth curves on a smooth quadric surface, but the case of space curves seems to be of a different order of difficulty. After [1] we came back to this topic, using the statements and ideas in [1], but not its proof. We state our main result in an axiomatic form (if somebody produce a very good curve $C \subset \mathbb{P}^3$, then in the same irreducible component of the Hilbert space of smooth curves there are curves $X \subset \mathbb{P}^r$ with certain prescribed $r(X)$). It is [1] which assures the existence of such curves C for many pairs of degree and genera.

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For any curve $C \subset \mathbb{P}^3$ let N_C denote the normal sheaf of C in $\mathbb{P}^3$ (it is a vector bundle if C is smooth or nodal). In this note we prove the following result.

Theorem 1.1. *Fix positive integer d, m, e, ρ, δ such that $m \geq 13$, $e < m$, $d < \frac{m^2+4m+6}{3}$, $\rho \geq m+6$, $\rho+1 \leq \delta \leq \rho+1+\lfloor(\rho-3)/2\rfloor\lceil m/6\rceil$ and assume the existence of a smooth and connected curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$, $h^1(\mathcal{O}_C(e)) = 0$, $h^1(N_C(-1)) = 0$ and $h^1(\mathcal{I}_C(t)) = 0$ for each $t \geq m$. Then there is a smooth and connected curve $X \subset \mathbb{P}^3$ with $\deg(X) = d+\delta$, $p_a(X) = p_a(C)$, $r(X) = \rho$ and $h^1(N_X(-1)) = h^1(\mathcal{O}_X(e)) = 0$.*

The condition "$d < \frac{m^2+4m+6}{3}$" implies that the pair (d, m) is in the range A in the sense of [4, 5] if we add the condition $h^0(\mathcal{I}_C(m-1)) = 0$, which we do not impose in Theorem 1.1. We do not assume that m is the minimal integer such that $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq m$. The proof works relaxing the assumption $d < \frac{m^2+4m+6}{3}$ at the expense of a stronger assumption on δ. Conversely, stronger upper bounds on d in terms of m (as in [1]) allow any interested reader to weaken the assumption on the upper bound of ρ in terms of δ and m. To get a smooth curve $C \subset \mathbb{P}^3$ of degree d and genus g with $h^1(N_C(-1)) = 0$ (or just with $h^1(N_C) = 0$) we have $d^{3/2}/g$ upper bounded for $g \gg 0$ ([3], [8, page 11]).

Take X as in Theorem 1.1. Since $h^1(N_X(-1)) = 0$, we have $h^1(N_X) = 0$ and hence the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^3)$ of $\mathbb{P}^3$ is smooth at $[X]$ and of dimension $4\deg(X)$. We will also show how to construct the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ containing X if we know the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ containing C. If $\mu > m$, then $d < \frac{\mu^2+4\mu+6}{3}$ and hence we may use Theorem 1.1 for C and μ. See Lemmas 3.7 and 3.8 for the use of curves C' of degree $d' > d$ and with genus $p_a(C)$ to which we may apply the statement of Theorem 1.1.

Remark 1.2. In Theorem 1.1 set $g := p_a(X)$. By [1, Theorem 1] we may take as (d, g, m) in Theorem 1.1 any triple (d, g, m) with $g \geq 17052$ and $g \leq Cm^3$ with $C = \sqrt{20}/600$. We cover in this way the pairs (d, g) for (degree,genus) with $g \leq 0.02d^{3/2}$ and all $d \gg 0$ ([1, Corollary 1]).

We also find irreducible components of the Hilbert scheme of space curves containing two or more curves with prescribed (and different) index of regularity (Remark 3.9).

We think that the approach used in this paper may be used to attach other problems concerning the Hilbert functions of space curves, but there is a big warning. Since it heavily uses vanishing for the normal bundle of the curve, it may only be used in a range of (degree,genera) = (d, g) with $g/d^{3/2}$ upper bounded when $d \gg 0$ ([3], [8, §6]). The general aim is the stratification by degree, genus and Hilbert function of the set of all smooth and connected space curves. In this generality it is an hopeless project for several reasons, but in the Range A something may be done.

To get that a space curve $X \subset \mathbb{P}^3$ for which we know that $h^1(\mathcal{I}_X(x)) = 0$ has $r(X) = x$, we use the following remark (in the set-up of Theorem 1.1 we have $e < m < \rho$ and we take $x = \rho$).

Remark 1.3. Let $X \subset \mathbb{P}^3$ be a closed subscheme with $\dim X \leq 1$. Fix $x \in \mathbb{N}$ such that $h^1(\mathcal{I}_X(x)) = 0$ and $h^1(\mathcal{O}_X(x-1)) = 0$. Since $\dim X \leq 1$, we have $h^2(\mathcal{O}_X(x)) = 0$ for all $x \in \mathbb{Z}$. Thus $h^2(\mathcal{I}_X(x-1)) = 0$. The Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_X(t)) = 0$ for all $t > x$. Thus if $h^1(\mathcal{I}_X(x-1)) \neq 0$ the integer x is the index of regularity of X. To get $h^1(\mathcal{I}_X(x-1))$ we will construct X such that $h^2(\mathcal{I}_X(x-2)) = 0$ and there is a line $D \subset \mathbb{P}^3$ with $\deg(D \cap X) = x+1$. Since every $G \in |\mathcal{I}_X(x)|$ contains D and $D \nsubseteq X$, the homogeneous ideal of X is not generated by forms of degree at most x. Since $h^2(\mathcal{I}_X(x-2)) = 0$, the Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_X(x-1)) \neq 0$.

By Remark 1.3 the general strategy is to construct a pair (X, D), where X is a smooth curve of genus g and degree d with $h^1(\mathcal{I}_X(\rho)) = 0$ and D is a line with $\deg(X \cap D) = \rho + 1$. In the set-up of Theorem 1.1 we take as X a very special smoothing (we need a line D with $\deg(X \cap D) = \rho + 1$) of a union of C and $d - \delta$ lines.

2. The Components of the Hilbert Schemes in which We Land

Lemma 2.1. *Let $C \subset \mathbb{P}^3$ be a smooth curve with $h^1(N_C(-1)) = 0$. Let $D \subset \mathbb{P}^3$ be a smooth rational curve intersecting C at a unique point, q, and quasi-transversally. Then $C \cup D$ is smoothable and $h^1(N_{C\cup D}(-1)) = 0$. If e is a positive integer with $h^1(\mathcal{O}_C(e)) = 0$, then $h^1(\mathcal{O}_{C\cup D}(e)) = 0$.*

Proof. The curve $C \cup D$ is smoothable by [7, Theorem 4.1] or [9, Proposition 1.6].

Let N be the vector bundle on C obtained from N_C making a positive elementary transformation at q in the direction of the tangent line of D at q. Let M be the vector bundle on D obtained from N_D making a positive elementary transformation at q in the direction of the tangent line of C at q. We have $N = N_{C\cup D|C}$ and $M = N_{C\cup D|D}$ ([7, Corollary 3.2]). Thus we have the Mayer-Vietoris exact sequence

$$0 \to N_{C\cup D}(-1) \to N(-1) \oplus M(-1) \to M(-1)_{|\{q\}} \to 0 \qquad (1)$$

Since $h^1(N_C(-1)) = 0$ and $N_C(-1)$ is a subsheaf of $N(-1)$ such that $N(-1)/N_C(-1)$ has zero-dimensional support, we have $h^1(M(-1)) = 0$. If $M(-1)$ is spanned, then the restriction map $H^0(M(-1)) \to H^0(M(-1)_{|\{q\}})$ is surjective and therefore the map $H^0(N(-1) \oplus M(-1)) \to H^0(M(-1)_{|\{q\}})$ induced by (1) is surjective. By (1) to prove that $h^1(N_{C\cup D}(-1)) = 0$ it is sufficient to prove that $h^1(M(-1)) = 0$ and that $M(-1)$ is spanned. Since D is smooth, N_D is a quotient of $T\mathbb{P}^3_{|D}$, sequence of $T\mathbb{P}^3$ gives a surjection $\mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to T\mathbb{P}^3$. Restricting to D we get that $N_D(-1)$ is spanned. Thus by the classification of vector bundles on $\mathbb{P}^1$ the vector bundle $N_D(-1)$ is a direct sum of 2 line bundles of degree ≥ 0. Hence $M(-1)$ is a direct sum of line bundles of degree ≥ 0. Thus $h^1(M(-1)) = 0$ and $M(-1)$ is spanned.

Since $\mathcal{O}_D(e)$ is spanned, the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{C\cup D}(e) \to \mathcal{O}_C(e) \oplus \mathcal{O}_D(e) \to \mathcal{O}_q(e) \to 0$$

gives $h^1(N_{C\cup D}(e)) = 0$. $\square$

Notation 2.2. Let Γ be an irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ whose general element is a smooth and connected curve C with $h^1(N_C(-1)) = 0$. For each integer $t > 0$ let Γ_t be the only irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ containing $C \cup D$, where D is a smooth rational curve of degree t intersecting transversally C at a unique point (Lemma 2.1). We have $h^1(N_X(-1)) = 0$ for a general $X \in \Gamma_t$ by Lemma 2.1.

Lemma 2.3. *If* $t \geq 2$, *then* $\Gamma_t = (\Gamma_1)_{t-1}$.

Proof. Take $C \in \Gamma$ with $h^1(N_C(-1)) = 0$ and a smooth degree t rational curve D with $\sharp(D \cap C) = 1$, say $C \cap D = \{q\}$, and intersecting quasi-transversally C at q. We may degenerate D in a family of curves containing q to a curve $R \cup A$ with R a line, $q \in R$, R is not the tangent line of C at q, $R \cap C = \{q\}$, A is a degree $t-1$ smooth rational curve with $A \cap C = \emptyset$, A intersects R at a single point, a, and quasi-transversally. Since $h^1(N_{C \cup R \cup A}) = 0$ (easier that the case done in the proof of Lemma 2.1), there is a unique irreducible component Ψ of $\mathrm{Hilb}(\mathbb{P}^3)$ containing $C \cup R \cup A$. By the definition of Γ_t and the fact that $C \cup R \cup A$ is a degeneration of $C \cup D$, this component is just Γ_t. We have $C \cup R \in \Gamma_1$. Since $h^1(N_{C \cup R}(-1)) = 0$ (Lemma 2.1), we have $h^1(N_B(-1)) = 0$ for a general $B \in \Gamma_1$. Let $\pi : \mathcal{X} \to \Delta$ be a flat family with special fiber $\pi^{-1}(o)$ and $B = \pi^{-1}(b)$ as a general fiber. Since a is a smooth point of $C \cup R$, up to a finite covering of Δ we may find a section u of π with $u(o) = a$. Up to a quasi-finite covering of Δ with image containing both o and b we may find a family of smooth degree $t-1$ rational curves $\eta : \mathcal{D} \to \Delta$ and a section v of η with $\eta^{-1}(o) = A$, $\eta^{-1}(b) \cap B = v(b)$, $\eta^{-1}(b)$ and B are quasi-transversal and $v(o) = a$. The curve $B \cup \eta^{-1}(b)$ is in $(\Gamma_1)_{t-1}$. □

Remark 2.4. Take a smooth and connected curve $C \subset \mathbb{P}^3$, an integer $t \geq 2$, and lines $L_i \subset \mathbb{P}^3$, $1 \leq i \leq t$, such that L_1 intersects C quasi-transversally and at a unique point, while for all $i = 2, \dots, t$ the line L_i meets $C \cup L_1 \cup \cdots \cup L_{i-1}$ quasi-transversally and at a unique point. As in the proof of Lemma 2.3 we see that $C \cup L_1 \cup \cdots \cup L_t \in \Gamma_t$.

3. The index of regularity

For any closed subscheme $Z \subset \mathbb{P}^3$ and any plane $H \subset \mathbb{P}^3$ the residual scheme of Z with respect to H is the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. We have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z \cap H, H}(t) \to 0$$

For any $o \in \mathbb{P}^3$ let $2o$ denote the first infinitesimal neighborhood of o in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_o)^2$ as its ideal sheaf (it is a zero-dimensional scheme with degree 3 and $2o_{\mathrm{red}} = \{o\}$). If $H \subset \mathbb{P}^3$ is a plane with $o \in H$, then $\mathrm{Res}_H(2o) = \{o\}$ and $2o \cap H$ is the first infinitesimal neighborhood of o in H. Hence for any scheme $T \subset H$, then $\mathrm{Res}_H(T \cup 2o) = \{o\}$. If T is a curve, which is singular at $\{o\}$, then $(T \cup 2o) \cap H = T$.

Lemma 3.1. *Let $C \subset \mathbb{P}^3$ be a smooth and connected curve with $h^1(N_C(-1)) = 0$. Let $H \subset \mathbb{P}^3$ be a plane intersecting transversally C. Fix $S \subsetneq C \cap H$, $o \in H \cap C \setminus S$ and a general $q \in H$. Let $D \subset H$ be the line spanned by o and q. Then $C \cup D$ is smoothable, $h^1(N_{C\cup D}(-1)) = 0$ and there are a flat family $\{X_t\}_{t\in\Delta}$ of space curves and $a \in \Delta$ with Δ an irreducible curve, $X_a = C \cup D$, X_t smooth if $t \neq a$ and $S \cup \{q\} \subset X_t$ for all $t \in \Delta$.*

Proof. Concerning the point q we only need $q \neq o$ (to define D) and that the line D meets C only at o. Set $w := \sharp(S)$. By Lemma 2.1 $Y := C \cup D$ is smoothable and $h^1(N_Y(-1)) = 0$. We need to prove that Y is smoothable in a family of space curves all of them containing $S \cup \{q\}$. Since $S \cup \{o\} \subset H$, we have $h^1(N_C(S - \{o\}) \leq h^1(N_C(-1)) = 0$. Since $N_D \cong \mathcal{O}_D(1)^{\oplus 2}$, we have $h^1(N_D(-o-q)) = 0$ and $N_D(-q) \cong \mathcal{O}_D^{\oplus 2}$. Since $Y \cup D$ is nodal the Mayer-Vietoris exact sequence of $Y = C \cup D$ gives $h^1(N_Y(-(S \cup \{q\})) = 0$. Let $\pi : W \to \mathbb{P}^3$ be the blowing up of $\mathbb{P}^3$ along $S \cup \{q\}$. For any curve $E \subset W$ let $N_{E,W}$ denote the normal sheaf of E in W. Let Y', C' and D' be the strict transforms of Y, C and D in W. Since Y is smooth at all points of $S \cup \{q\}$, π induces an isomorphism between Y', C', D' and Y, C, D, respectively and this isomorphism maps $N_Y(-(S \cup \{q\}))$, $N_C(-S)$, $N_D(-q)$ isomorphically onto $N_{Y',W}$, $N_{C',W}$ and $N_{D',W}$. Hence $H^0(N_Y(-S - \{q\}))$ is the tangent space at $C \cup D$ of the deformation functor of the curves containing $S \cup \{q\}$, while $H^1(N_Y(-S - \{q\}))$ is an obstruction space for the same functor (see [8] for several uses of this set-up). At this point we have basically won, but we show how to adapt the proof in [7, Theorem 4.1] to conclude the proof.

Claim 1: Y' is smoothable inside W.

***Proof of Claim* 1.** Y' has a unique singular point, the only point $o' \in W$ with $\pi(o') = o$, and the curves C' and D' meets transversally at o'. Since $D \cap (S \cup \{q\}) = q$ and $N_D \cong \mathcal{O}_D^{\oplus 2}(1)$, $N_{D',W} \cong \mathcal{O}_{D'}^{\oplus 2}$. Hence $h^1(D', \mathcal{W}) = 0$ for any sheaf $\mathcal{W}$ obtained from $N_{D',W}$ making one negative elementary transformation at o'. Since $h^1(N_C(-1)) = 0$ and $S \subseteq C \cap H$, we have $h^1(N_C(-S)) = 0$, i.e. $h^1(N_{C',W}) = 0$. Apply [7, Theorem 4.1]. $\square$

Since $h^1(N_{Y'}) = 0$, the Hilbert scheme of W is smooth at Y'. Call X' any smooth curve belonging to the irreducible component of the Hilbert scheme of W containing Y' (it exists by Claim 1). W has $w+1$ exceptional divisors, i.e. smooth rational curves with self-intersection -1 mapped by π to different points of $S\cup\{q\}$. we have $\mathrm{Pic}(W) \cong \mathbb{Z}^{w+2}$ with as free generators $\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and these $w+1$ exceptional divisors. Let J be one these exceptional divisors, say $J = \pi^{-1}(a)$ with $a \in S \cup \{q\}$. Since a is a smooth point of Y, we have $\deg(Y' \cap J) = 1$ and hence $Y' \cdot J = 1$ in the intersection ring of J. Hence $Y' \cap J = 1$. Thus $a \in \pi(X')$. Thus $\pi(X') \supset S \cup \{q\}$. Since $Y' \cdot J = 1$, Y' intersects transversally J at a unique point. Since Y' is smooth, we get that $\pi(X')$ is smooth at a for all $a \in S \cup \{q\}$, concluding the proof of the lemma. $\square$

Remark 3.2. For any non-degenerate, reduced and connected $C \subset \mathbb{P}^3$ and any plane $H \subset \mathbb{P}^3$ the scheme $C \cap H$ spans H (use the the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_C(1) \to \mathcal{I}_{C\cap H,H}(1) \to 0$$

and that $h^1(\mathcal{I}_C) = 0$, because $h^0(\mathcal{O}_C) = 1$). Hence if H intersects transversally C and $S \subsetneq C\cap H$ is formed by collinear points, then we have $S \neq C\cap H$ and hence we may find $o \in C \cap H \setminus H$ to which we apply the statement of Lemma 2.1.

Lemma 3.3. *Let $C \subset \mathbb{P}^3$ be a smooth and connected curve with $h^1(N_C(-1)) = 0$. Let $H \subset \mathbb{P}^3$ be a plane intersecting transversally C. Fix an integer $t > 0$, t distinct points $p_1, \dots, p_t \in H \cap C$ and t distinct points $o_1, \dots, o_t \in H \setminus H \cap C$. Let L_i be the line spanned by $\{o_i, p_i\}$. Assume that all lines L_i are distinct, no point of H is contained in 3 of them and that $\{p_1, \dots, p_t\} = C \cap T$, where $T := \cup_{i=1}^t L_i$. Set $E := \mathrm{Sing}(T)$ and $\chi := \cup_{o\in E} 2o$. Fix a set $S \subseteq C \cap H \setminus T \cap C$. Then $C \cup T \cup \chi$ is smoothable in a family of curves all containing $\{o_1, \dots, o_t\} \cup S$.*

Proof. The case $t = 1$ is true by Lemma 3.1. Now assume $t > 1$ and that the lemma is true for the integer $t-1$. Set $T' := L_1 \cup \cdots \cup L_{t-1}$ and $\chi' := \mathrm{Sing}(T')$. By the inductive assumption $C \cup T' \cup \chi'$ is smoothable in a family of curves containing $A \cup \{o_1, \dots, o_t\}$ and whose general member Y satisfies $h^1(N_Y(-1)) = 0$. There are an integral affine curve, a family of lines $\{R_b\}_{b\in\Delta}$ of $\mathbb{P}^3$ and $a \in \Delta$ such that $R_a = L_t$, R_b meets C at a unique point, u_b, and quasi-transversally and $u_b \notin H$ for all $b \neq a$. Obviously $C \cup T \cup \chi$ is a degeneration of the family $\{C \cup T' \cup R_b\}_{b\in\Delta\setminus\{a\}}$. Hence $C \cup T \cup \chi$ is smoothable. But we also need to check that it is smoothable preserving

$A \cup \{o_1, \dots, o_t\}$ and that a general smoothing Y has $h^1(N_Y(-1)) = 0$. The scheme $C \cup T \cup \chi$ is a flat limit of a family of nodal curves $\{W_a\}_{a\in\Delta}$ with $W_b = C \cup T \cup \chi$ for some $b \in \Delta$ and $W_a = C \cup D_1(a) \cup \dots \cup D_t(a)$ a connected nodal curve, each $D_i(a)$ a line intersecting quasi-transversally C and at a unique point $p_i(a) \in C \setminus C \cap H$, each $D_i(a)$ containing o_i and $D_i(a) \cap D_j(a) = \emptyset$ for all $i \neq j$ and all $a \in \Delta \setminus \{a\}$ (use that through two different points of $\mathbb{P}^3$ there is a unique line and that this line depends regularly on the points). Fix a general $a \in \Delta$ and set $F := D_1(a) \cup \dots \cup D_t(a)$, $q_i := p_i(a)$ and $W := W_a$. Note that $W = C \cup F$ and $C \cap F\{q_1, \dots, q_t\} = \mathrm{Sing}(W)$. Since W is nodal, $N_W(-1)$ is locally free. Thus we have the Mayer-Vietoris exact sequence

$$0 \to N_W(-1) \to N_W(-1)_{|C} \oplus N_W(-1)_{|F} \to N_W(-1)_{\{q_1,\dots,q_t\}} \to 0 \qquad (1)$$

The restriction of $N_W(-1)$ to each connected component of F is the direct sum of a line bundle of degree 1 and a line bundle of degree 0. Thus the restriction map $H^0(N_W(-1)_{|F}) \to H^0(N_W(-1)_{\{q_1,\dots,q_t\}})$ is surjective. Therefore (1) gives $h^1(N_W(-1)) = 0$. By the semicontinuity theorem for cohomology we get the lemma. $\square$

Lemma 3.4. *Let $T = L_1 \cup \dots \cup L_b \subset \mathbb{P}^2$, $b \geq 2$, be a nodal union of b distinct lines. Let $E \subset \mathbb{P}^2$ be the union of all points $L_i \cap L_j$ with $i \geq j+2$. Then $h^1(\mathbb{P}^2, \mathcal{I}_E(b-3)) = 0$.*

Proof. If $b = 2$, then $E = \emptyset$ and hence the lemma is true in this case. Now assume $b > 2$ and that lemma is true for a smaller number of lines. We have $\sharp(L_b \cap E) = b-2$ and hence $h^1(L_b, \mathcal{I}_{E\cap L_b, L_b}(b-3)) = 0$. Use the residual exact sequence of sheaves on $\mathbb{P}^2$:

$$0 \to \mathcal{I}_{E \setminus E\cap L_b}(b-4) \to \mathcal{I}_E(b-3) \to \mathcal{I}_{E\cap L_b, L_b}(b-3) \to 0$$

and the inductive assumption. $\square$

Lemma 3.5. *Let $C \subset \mathbb{P}^3$ be a smooth curve with $h^1(N_C(-1)) = 0$. Let $L \subset \mathbb{P}^3$ be a line with $\sharp(L \cap C) = 1$ and L intersecting quasi-transversally C. Set $\{q\} := L \cap C$. Fix a line D intersecting quasi-transversally C, with $q \notin D$ and with $D \cap L \neq \emptyset$. Then there is a smoothing of $Y := C \cup L$ in a family of curves all containing the set $S := (C \cup L) \cap D$.*

Proof. Set $\{o\} := L \cap D$ and $S' := S \setminus \{o\}$. Let $\pi : W \to \mathbb{P}^3$ be the blowing up of S. Let C', L' and Y' be the strict transform in W of C, L and Y. Since C, L and Y are smooth at the points of S, π induces an isomorphism between C' (resp. L', resp. Y') and C (resp. L, resp. Y) and this isomorphism induces an isomorphism between $N_{C'}$ (resp. $N_{L'}$, resp. $N_{Y'}$) and $N_C(-S')$ (resp. $N_L(-o)$, resp. $N_Y(-S)$). To prove the lemma it is sufficient to smooth Y' in W, because in the smoothing family each element would intersect each exceptional divisor of π. Since $S' \subset D$, D is contained in a plane and $h^1(N_C(-1)) = 0$, we have $h^1(N_C(-S')) = 0$, i.e. $h^1(N_{C'}) = 0$. We have $N_L(-o) \cong \mathcal{O}_L^{\oplus 2}$, i.e. $N_{L'} \cong \mathcal{O}_{L'}^{\oplus 2}$. Hence the vector bundle M on L' obtained from $N_{L'}$ making a negative elementary transformation is a direct sum of a line bundle of degree -1 and a line bundle of degree 0. Thus $h^1(M) = 0$. The smoothing of Y' is obtained as in the proof of [7, Theorem 4.1]. $\square$

Lemma 3.6. *Fix an integer $b > 0$. Let $C \subset \mathbb{P}^3$ be a smooth curve with $h^1(N_C(-1)) = 0$. Let $D, L_1 \subset \mathbb{P}^3$ be lines with $\sharp(L_1 \cap C) = \sharp(D \cap C) = \sharp(D \cap L_1) = 1$, L_1 and D intersecting quasi-transversally C and $D \cap C \notin L_1$. If $b \geq 2$ define recursively the lines L_i, $2 \leq i \leq b$, in the following way. Let L_i, $2 \leq i \leq b$, be a general line intersecting both L_{i-1} and D. The curve $Y := C \cup L_1 \cup \cdots \cup L_b$ is nodal and connected and it may be smoothed in a family of curves all containing the set $D \cap (C \cup L_1 \cup \cdots \cup L_b)$.*

Proof. The case $b = 1$ is true by Lemma 3.5. Now assume $b > 1$ and that the lemma is true for a smaller number of lines. By Lemma 3.5 there is a smoothing $\{X_a\}_{a \in \Delta}$ of $C \cup L_1$ (say $X_o = C \cup L_1$ with $o \in \Delta$) in a family of curves containing $D \cap (C \cup L_1)$. By assumption the point $b := L_1 \cap L_2$ is a smooth point of $C \cup L_1$ and hence, taking if necessary a finite covering of Δ, there is a section u of $\{X_a\}_{a \in \Delta}$ with $u(o) = b$. Taking a smaller Δ if necessary we may assume that for every $a \in \Delta$ we have $u(a) \notin D \cup C$ and the line $L_2(a)$ spanned by $u(a)$ and $D \cap L_2$ does not intersects C. We take a general $a \in \Delta$ and apply Lemma 3.5 to $X_a \cup L_2(a)$ to get the case $b = 2$; call $\{T_c\}_{c \in \Theta}$ this smoothing and $o' \in \Theta$ with $T_{o'}$ smooth. Now assume $b > 2$. Since any 2 points of $\mathbb{P}^3$ span a line, we may find an equisingular deformation $\{L_3(a) \cup \cdots \cup L_b(a)\}_{a \in \Delta}$ of $L_3 \cup \cdots \cup L_b$ with $L_3(o) \cup \cdots \cup L_b(o) = L_3 \cup \cdots \cup L_b$ and $L_3(a)$ meeting $L_2(a)$ at a unique point and then (for a fixed general $a \in \Delta$) an equisingular deformation $\{R_3(c) \cup \cdots \cup R_b(c)\}_{c \in \Theta}$, $o' \in \Theta$, with

$R_3(o') \cup \cdots \cup R_b(o') = L_3(a) \cup \cdots \cup L_3(a)$ with $R_3(c)$ intersecting quasi-transversally T_c and at a unique point for all $c \in \Theta$. Thus we conclude by the inductive assumption. $\square$

Lemma 3.7. *Fix positive integers m, d and a smooth and connected curve $C \subset \mathbb{P}^3$ such that $h^1(N_C(-1)) = 0$, $h^1(\mathcal{O}_C(e)) = 0$, $h^1(\mathcal{I}_C(x)) = 0$ for all $x \geq m$ and $d := \deg(D) \leq (m^2+m+2)/2$. Let F be either a line or a smooth conic. Assume $\sharp(C \cap F) = 1$ and that F intersects quasi-transversally C. Set $Y := C \cup F$. Then Y is smoothable, $p_a(Y) = p_a(C)$, $h^1(N_Y(-1)) = 0$ and $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \geq m+1$.*

Proof. By Lemma 2.1 it is sufficient to prove the last assertion. Let H be a plane containing F. Since $h^1(N_C(-1)) = 0$, we may deform C in a family of curves containing the point $C \cap F$ so that (after this deformation), $C \cap (H \setminus F)$ is a general union of $d-1$ points of H. From the residual exact sequence of $\mathcal{I}_Y$ with respect to H we get the exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_Y(t) \to \mathcal{I}_{F\cup(C\cap(H\setminus F)),H}(t) \to 0 \tag{2}$$

Since $d-1 \leq (m+1)m/2$ and $C \cap (H \setminus F)$ is general in H, we have $h^1(H, \mathcal{I}_{D\cup(C\cap(H\setminus D)),H}(t)) = 0$ for all $t > m$. From (2) we get $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \geq m+1$. $\square$

Lemma 3.8. *Fix positive integers m, e, b, d with $b \geq 1$, $d-1 \leq \binom{m+4-b}{2}$ and $b < m$. Let C be a smooth and connected curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$, $h^1(N_C(-1)) = 0$, $h^1(\mathcal{O}_C(e)) = 0$, $h^1(\mathcal{I}_C(x)) = 0$ for all $x \geq m$. Let $Y = C \cup L_1 \cup \cdots \cup L_b$ be a general union of C, a general line L_1 intersecting C and, for $i = 2, \dots, b$, a general line L_i intersecting L_{i-1}. Then Y is smoothable $h^1(N_Y(-1)) = 0$ and $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \geq m+2$.*

Proof. As in Lemma 3.6 we see that Y is smoothable and that $h^1(N_Y(-1)) = 0$. We use a degeneration of Y and the semicontinuity theorem to prove that $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \geq m+2$. Set $R_1 := L_1$ and take a plane $H \supset R_1$. Since $h^1(N_C(-1)) = 0$, we may deform Y keeping fixed the point $\{q\} := R_1 \cap C$ so that the other $d-1$ points of $C \cap H$ are general in H. Fix general lines $R_2, \dots, R_b$ of H and let E be the union of the points $R_i \cap R_j$ with $i \geq j+2$. We have $\sharp(E) = (b-1)(b-2)/2$. Set $\chi := \cup_{o\in E} 2o$, $T := R_1 \cup \cdots \cup R_b$ and $M := C \cup T \cup \chi$. As in the proof of Lemma 3.3 we see by induction on b (using [2] or [6, Example 2.1.1] at each point of E) that

M is a flat limit of connected nodal curves with arithmetic genus $p_a(C)$ like $C \cup D_1 \cup \cdots \cup D_b$ with $D_1, \dots, D_b$ lines, $C \cap D_1 \neq \emptyset$ and $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We have the residual exact sequences of $\mathcal{I}_{C\cup E}$ and of $\mathcal{I}_M$ with respect to H:

$$0 \to \mathcal{I}_C(x-2) \to \mathcal{I}_{C\cup E}(x-1) \to \mathcal{I}_{E\cup(C\cap E),H}(x-1) \to 0,$$
$$0 \to \mathcal{I}_{C\cup E}(x-1) \to \mathcal{I}_M(x) \to \mathcal{I}_{T\cup(C\cap(H\setminus\{q\},H}(x) \to 0.$$

For each $x \geq m+2$ we have $h^1(H, \mathcal{I}_{T\cup(C\cap(H\setminus\{q\},H}(x)) = h^1(H, \mathcal{I}_{(C\cap H)\setminus\{q\}}(x+2-b)) = 0$ (by the assumption $d-1 \leq \binom{m+4-b}{2}$) and $h^1(H, \mathcal{I}_{E\cup(C\cap E),H}(x-1)) = 0$ (because $h^1(H, \mathcal{I}_E(b-3)) = 0$, $b \leq m$, $C \cap H$ is general in H and $d + (b-1)(b-2)/2 \leq \binom{m+3}{2}$. □

For any positive integer $x \geq 1$ a chain of x lines is a reduced, connected and nodal curve $F \subset \mathbb{P}^3$ such that $F = L_1 \cup \cdots \cup L_x$ with each L_i a line, $L_i \neq L_j$ for all $i \neq j$ and $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Since each irreducible component of F is a line, if $|i-j| = 1$, then L_i intersect quasi-transversally L_j and at a unique point. We have $p_a(F) = 0$ and F is smoothable to a smooth rational curve of degree x ([7]), but we need more if $x \geq 2$. Now assume $x \geq 2$ and fix $o \in L_1 \setminus L_1 \cap L_2$. Since F is nodal, N_F is a rank 2 vector bundle. As in [7] we see that $N_{F|L_i}$ is the direct sum of two line bundles of degree ≥ 1. As in the proof of Claim 1 in the proof of Lemma 3.1 we see that F may be smoothed in a family of space curves containing o.

***Proof of Theorem* 1.1.** Call Γ the irreducible component of the Hilbert scheme of $\mathbb{P}^3$ containing C. We will get $X \in \Gamma_\delta$.

(a) In this step we assume $\rho - m$ even and $\delta := \rho + 1$. Set $c := (\rho - m)/2$.

Fix a line $D \subset \mathbb{P}^3$ such that $C \cap D = \emptyset$ and distinct planes H_i, $i \geq 1$, containing D. Note that $D = H_i \cap H_j$ for all $i \neq j$. Let a_1 be the maximal integer x such that $d - 1 \leq \binom{m+3-x}{2}$. We have $a_1 \geq 2$, because $d \leq m^2/2$. Set $b_1 := \min\{a_1, \delta\}$. Take an integer $i \geq 2$ and assume defined the integers a_j and b_j for all positive integers $j < i$. Let a_i be the maximal integer x such that $\binom{m+2i+1-x}{2} \geq d - 1 + b_1 + \cdots + b_{i-1}$. Set $b_i := \min\{a_i, m + 2i + 1 - b_1 - \cdots - b_{i-1}\}$. We obviously have $a_i \leq m + 2i - 1$ for all i. Note that $b_1 + \cdots + b_x = \rho + 1 = \delta$ for $x \gg 0$. If there is an integral curve $T \subset \mathbb{P}^3$ with $h^1(\mathcal{O}_T(\rho-1)) = 0$ and $\deg(T \cap D) \geq \rho + 1$, then $h^1(\mathcal{I}_T(\rho-1)) > 0$

by the Castelnuovo-Mumford lemma, because the homogeneous ideal of T is not generated by forms of degree $\le \rho$.

Claim 1: We have $a_1 \ge \lceil m/6 \rceil$.

Proof Claim 1. Since $d-1 < (m^2+4m)/3$, it is sufficient to use that

$$(5m/6+2)(5m/6+1)/2 = (5m+12)(5m+6)/72 \ge (m^2+4m)/3. \qquad \square$$

Claim 2: For all integers $i \ge 2$ we have $a_i \ge \lceil m/6 \rceil + i$.

Proof Claim 2. Since $b_1+\cdots+b_{i-1} \le m+2i-1$ and $d-1 < (m^2+4m)/3$, it is sufficient to use that

$$(5m/6+i+2)(5m/6+i+1)/2 \ge m+2i+(m^2+4m)/3. \qquad \square$$

By Claims 1 and 2 and the assumption $m \ge 13$, we have $a_i \ge 3$ for all i. Hence $b_i = 0$ for $i \gg 0$. Indeed we may take $b_i = 0$ if $i > (\rho - m)/2 = c$

For each $i = 1, \dots, c$ take $o_i \in C \cap H_i$. Since $D \cap C = \emptyset$, we have $o_i \notin H_j$ if $i \ne j$. Fix a general line L_{i1} of H_i containing o_i. If $b_i \le 1$ set $L_{ij} = \emptyset$, $E_{ij} = \emptyset$ for all $j > 1$ and $T_i := L_{i1}$. Now assume $b_i \ge 2$. Let L_{ij}, $2 \le j \le b_i$, be general lines of H_i. Set $T_i := L_{i1} \cup \cdots \cup L_{ib_i}$. Let $E_i \subset H_i$ be the union of all points $L_{ih} \cap L_{ij}$ with $j \ge h+2$. Since L_{ij} is a general line of H_i if $j \ge 2$, we have $E_i \cap H_t = \emptyset$ if $t \ne i$ and $E_i \cap D = \emptyset$.

Since $h^1(N_C(-1)) = 0$, as in [8, Théorème 1.5] (with H instead of a quadric surface) we see that we may deform C in such a way that all sets $C \cap H_i$ are general in H_i and in particular they have the Hilbert function of a general subset of H_i (if $c > 1$ we do not (and if $c > 2$ we cannot) assume that the set $C \cap (H_1 \cup \cdots \cup H_c)$ is a general subset of $H_1 \cup \cdots \cup H_c$ with the only restriction that its restriction to each H_i has the same cardinality). We may also assume that $C \cap D = \emptyset$. By Claim 1 and we have $d \ge a_i$ for all i. Note that $E_i \cap E_j = \emptyset$ for all $i \ne j$. Set $\chi_i := \cup_{o \in E_i} 2o_i$ and $J_i := T_i \cup \chi_i$, $Y_i := C \cup J_1 \cup \cdots \cup J_i$, $1 \le i \le c$, and $Y := Y_c$. Note that $T_i = J_i$ if $b_i \le 1$, $\mathrm{Res}_{H_1}(Y \cup E_1) = Y$, $\mathrm{Res}_{H_1}(Y_1) = Y \cup E_1$, $\mathrm{Res}_{H_i}(Y_i) = Y_{i-1} \cup E_i$ for all $i = 2, \dots, c$, and $\mathrm{Res}_{H_i}(Y_{i-1} \cup E_i) = Y_{i-1}$. By the proof of [6, Example 2.1.1] each J_i may be deformed to a chain of $\deg(T_i)$ lines in which the first line (in the ordering of the chain) is L_{i1} and all lines of the chain meets D. Hence we may find a deformed chain I_i meeting quasi-transversally C and at a unique point, o_i, and with $\deg(D \cap I_i) = \deg(T_i)$. Hence Y is deformable to a smooth curve X of degree d and genus g with $\deg(D \cap X) = d - \delta = \rho + 1$. Using Remark 2.4 and the semicontinuity theorem for cohomology we get $h^1(\mathcal{O}_X(e)) = 0$ and $h^1(N_X(-1)) = 0$. Since $\deg(D \cap X) = \rho + 1$, Remark 1.3 gives $h^1(\mathcal{I}_X(\rho - 1)) \neq 0$. By Lemmas 2.1 and 2.3 we have $X \in \Gamma_{d-\delta}$. Thus to prove Theorem 1.1 for the case $\rho = \delta - d + 1$, it is sufficient to prove that $h^1(\mathcal{I}_X(\rho)) = 0$. By semicontinuity it is sufficient to prove that $h^1(\mathcal{I}_Y(\rho)) = 0$.

Claim 3: We have $h^1(H_c, \mathcal{I}_{Y \cap H_c, H_c}(\rho)) = 0$.

Proof Claim 3. Since $E_i \cap H_c = \emptyset$ for all $i < c$, we have $Y \cap H_c = T_c \cup (T_1 \cup \cdots \cup T_{c-1} \cup Y) \cap H_c$. Since $(T_1 \cup \cdots \cup T_{c-1}) \cap H_c \subset D$ and $\deg((T_1 \cup \cdots \cup T_{c-1}) \cap H_c) \le \rho$, it is sufficient to prove that $h^1(H_c, \mathcal{I}_{C \cap (H_c \setminus T_c), H_c}(\rho - 1 - b_c)) = 0$. This is true by the definition of a_c and the assumption $b_c \le a_c$. □

Claim 4: We have $h^1(H_c, \mathcal{I}_{H_c \cap \mathrm{Res}_{H_c}(Y), H_c}(\rho - 1)) = 0$.

Proof Claim 4. We may apply Lemma 3.4, because $b_c \le \rho - 1$ and $H_c \cap C$ has the Hilbert function of a general subset of H_c with cardinality d. □

Claim 5: We have $h^1(\mathcal{I}_Y(\rho)) = 0$.

Proof Claim 5. By Claims 3 and 4 it is sufficient to prove that $h^1(\mathcal{I}_{Y_{c-1}}(\rho - b_c) = 0$. We first check as in the proof of Claim 3 we proved first $h^1(H_{c-1}, \mathcal{I}_{Y_{c-1} \cap H_{c-1}, H_{c-1}}(\rho - b_c)) = 0$. Then as in Claim 4 we proved that $h^1(H_{c-1}, \mathcal{I}_{(Y_{c-1} \cap H_{c-1}) \cup E_c, H_{c-1}}(\rho - b_c - 1)) = 0$ and then we continue using the planes $H_{c-2}, \ldots, H_1$. □

(b) In this step we assume $\delta = \rho + 1$ and $\rho - m$ odd, say $\rho = m + 2c + 1$ for some integer c. Take $H_1, \ldots, H_c$, $b_1, \ldots, b_c$ as in step (a), but with respect to the integer $\rho - 1$. We get a pair (Y, D) with Y a reducible curve $Y \in \Gamma_{\delta - 1)}$, $h^1(N_Y(-1)) = 0$, $h^1(\mathcal{I}_Y(x)) = 0$ for all $x \ge \rho - 1$, and D is a line with $\deg(D \cap Y) = \rho$. Take a general plane $H \supset D$ and let $D \subset H$ be a line containing exactly one point of $C \cap H$. Apply Lemma 3.7.

(c) Now we assume $\delta \ge \rho + 2$. Set $\delta' := \delta - \rho - 1$. Suppose we find an integer $\mu > m$ and $Y \in \Gamma(+\delta')$ with $h^1(\mathcal{I}_Y(x)) = 0$ for every $x \ge m'$ and

$d+\delta' \le \frac{\mu^2+4\mu+6}{3}$. We use Y, $d+\delta'$ and μ instead of C, d and m. We may use several times Lemma 3.8; if the integer $\mu - m$ is odd we also use once Lemma 3.7. Take for instance the first use of Lemma 3.8, from m and d to $m+2$ and $d+b$. Since $\frac{(m+2)^2+4(m+2)+6}{3} - \frac{m+2+4m+6}{3} = 8m/3 + 8/3 > b$, the only restriction is the upper bound for b assumed in Lemma 3.8. Since $d < \frac{m+2+4m+6}{3}$ we may take any $b \le \lceil m/6 \rceil$ and hence (taking $\mu = \rho - 3$) we may take as δ' any integer $\le \sum_{i=0}^{\lfloor(\rho-3)/2\rfloor-1} \lceil (m+2i)/6 \rceil$ and in particular any integer $\le \lfloor(\rho-3)/2\rfloor \lceil m/6 \rceil$. □

Remark 3.9. When $\delta \gg \rho + 1$ the construction easily gives irreducible components $\Psi = \Gamma_\delta$ with smooth curves $A, B \in \Psi$ with $h^1(N_A(-1)) = h^1(N_B(-1)) = 0$ and with $r(A) \ne r(B)$, e.g. $r(A) = \rho$ and $r(B) = \rho - 1$. In many cases we may find curves $A_1, \dots, A_k \in \Psi$ with $r(A_i) \ne r(A_j)$ for all $i \ne j$, but we do not have a quantitative version of this observation.

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