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EXACT LAWS OF LARGE NUMBERS FOR INDEPENDENT PARETO RANDOM VARIABLES

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Abstract

This article explores weighted laws of large numbers, namely *exact laws*, for independent *Pareto* distributions with infinite mean. It contains not only exact weak laws but also exact strong laws. Moreover, we give a simple example of the exact strong law applying the algorithm of Adler and Wittmann (1994).

1. Introduction

1.1. Notation

For positive sequences $\{a_n\}$ and $\{b_n\} \subset \mathbb{R}$ symbols $a_n \sim b_n$ and $a_n = o(b_n)$ stand for $\lim a_n/b_n = 1$, $\lim a_n/b_n = 0$, respectively. The indicator random variable is defined by $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and 0 if $\omega \notin A$ for each event A. We redefine the natural logarithm as the meaning of $\log x := \max\{\ln(x), 1\}$, where $\ln x$ denotes the ordinary natural logarithm. Moreover, the symbol $\lfloor x \rfloor$ for $x \in \mathbb{R}$ denote the integer part $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

1.2. Exact weak law of large numbers

Letting $\{X_n\}$ be independent and identically distributed (i.i.d.) random variables with common distribution $\mathrm{P}(X_1 = 2^k) = 2^{-k}$ for $k = 1, 2, \ldots$, we

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see $\mathrm{E}(X_1) = \infty$, which is known as the *St. Petersburg game* (see Chapter X.4 in [8]). In his textbook, Feller wrote

$$\lim_{n\to\infty} \sum_{k=1}^{n} X_k/(\log n/\log 2) \overset{P}{=} 1,$$

where 'P' denotes the convergence in probability. In general, for independent random variables $\{X_k\}$ when there exist constant sequences $\{a_k\}$ and $\{b_n\}$ satisfying that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k X_k/b_n = 1$, it is said to be an *exact law* of large numbers. More precisely, we call an *exact weak law* when the convergence is in probability, and an *exact strong law* when it is almost sure (see page 142 of Adler [1]). We discuss the exact weak law in this subsection, and the exact strong law in the next subsection.

Giving some examples, Adler and his collaborators investigated exact weak laws for i.i.d. random variables (see [1] and references therein). These studies are based on the efficient application of the *degenerate convergence criterion* (see Theorem 10.1.1 of [7]). When they are no longer identically distributed, the calculation requires careful handling. Hence Adler [2] investigated the exact weak law for independent random variables $\{X_n\}$ with $\mathrm{P}(X_n \le x) = 1 - (x+n)^{-1}$. For convenience, we use the following terminology which is not so ordinary.

Definition 1.1. For $h \ge 1$, a distribution of a nonnegative random variable X is said to be *Pareto with parameter* h, if the law is characterized by

$$\mathrm{P}(X \le x) = 1 - \frac{1}{x+h} \quad \text{for } x > 0 \quad \text{and} \quad \mathrm{P}(X = 0) = \frac{h}{1+h}. \tag{1}$$

Let us note that the probability density function of Equation (1) is $1/(x+h)^2$ for $x > 0$.

Theorem 1.1 (Theorem 1 of [2])**.** *Suppose that* $\{X_j\}$ *are independent and Pareto with parameter* j*, respectively. Then we have*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} (1/j) X_j}{\log n \log\log n} \overset{P}{=} 1. \tag{2}$$

This result is naturally extended as follows.

Theorem 1.2 (Theorem 3.1 of [10]). *Suppose that $\{X_j\}$ are independent and Pareto with parameter h_j, respectively. If*

$$a_j := 1/h_j, \quad A_n := \sum_{j=1}^{n} a_j, \quad \lim_{n\to\infty} A_n = \infty, \tag{3}$$

then we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \log A_n} \stackrel{P}{=} 1. \tag{4}$$

Note that Adler [4] recently extends Theorem 1.1 by considering $\mathrm{P}(X_j > x) = \{\log(x+j)\}^{\alpha}/(x+j)$ for $\alpha > -1$.

1.3. Exact strong law of large numbers

Let us consider nonnegative i.i.d. random variables $\{X_k\}$ with $\mathrm{E}(X_1) = \infty$. In this case, it is known that there does not exist b_n such that $\lim_{n\to\infty} \sum_{k=1}^{n} X_k/b_n \stackrel{a.s.}{=} 1$, where '*a.s.*' denotes the almost sure convergence. When choosing a suitable weight $\{a_k\}$ and $\{b_n\}$, we can obtain $\lim_{n\to\infty} \sum_{k=1}^{n} a_k X_k/b_n \stackrel{a.s.}{=} 1$, namely the exact strong law, under some conditions. For example, the exact law for the distribution of the St. Petersburg game was given in Example 7 of [1].

It is known that for i.i.d. cases if the common tail function $x \mapsto \mathrm{P}(X_1 > x)$ is regularly varying with any exponent except -1, then the exact strong law fails (see page 142 of [1]). Therefore the distributions which have the regularly varying tail with exponent -1 deserve more than a passing notice. The independent Pareto random variables with parameter n may be considered as one of the easiest distributions with this property which are not identically distributed.

Now, for Theorems 1.1 and 1.2 let us examine the almost sure convergence. Theorem 2 of [2] and Remark 3.1 of [10] tell us

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \log A_n} \stackrel{a.s.}{=} \infty, \tag{5}$$

when $a_j = j^{-1}$. Adler recently showed that if Equation (3) and

$$\sum_{n=1}^{\infty} \frac{a_n}{A_n \log A_n} = \infty \tag{6}$$

are satisfied, then Equation (5) follows (see Theorem 2 of [3]). To obtain Equation (5) we may use the following statement rather than Equation (6).

Proposition 1.1. *Let us suppose Equation* (3)*. If*

$$a_n \geq a_{n+1}, \tag{7}$$

then Equation (5) *follows.*

From this, the exact strong law naturally fails when assuming Equations (3) and (7). Although Theorem 1.2 may be considered as a natural extension of Theorem 1.1, Equation (3) is not nice for the exact strong law. Here, to obtain the exact strong law, we propose a wider condition including Equation (3).

1.4. Our contribution

Theorem 1.3 (exact weak law)**.** *Suppose that* $\{X_j\}$ *are independent and Pareto with parameter* h_j*, respectively. For positive sequences* $\{a_j\}$ *and* $\{b_n\}$ *with*

$$\lim_{n\to\infty} b_n^{-1} \sum_{j=1}^{n} a_j = 0, \tag{8}$$

if there exists $0 < A < \infty$ *which satisfies*

$$\lim_{n\to\infty} b_n^{-1} \sum_{j=1}^{n} a_j \log\left(1 + \frac{b_n}{a_j h_j}\right) = A, \tag{9}$$

then we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{b_n} \stackrel{P}{=} A. \tag{10}$$

Remark 1.1. Let us observe the case that $\{a_j\}$ and $\{b_n\}$ satisfy Equation (3) and $b_n = A_n \log A_n$. Then Equations (8) and (9) with $A = 1$ follow.

Therefore Theorem 1.3 may be considered as an extension of Theorem 1.2. We note that if Equation (3) is fulfilled, then the antilogarithm part in Equation (9) does not depend on j.

Corollary 1.1. *Let us suppose the assumptions of Theorem* 1.3.

(i) *If*

$$\begin{cases} h_j = j^{\alpha} \quad \text{for } 0 \le \alpha < 1, \\ a_j = (\log j)^{b-2}/j, \ b_n = (\log n)^b \ \text{for } b > 0, \end{cases} \tag{11}$$

then we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{b_n} \stackrel{P}{=} \frac{1-\alpha}{b}. \tag{12}$$

(ii) *If*

$$h_j = j, \quad a_j = (\log j)^{b-1}/j, \ b_n = (\log n)^b (\log\log n) \ \text{for } b > 0, \tag{13}$$

then we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{b_n} \stackrel{P}{=} \frac{1}{b}. \tag{14}$$

Theorem 1.4 (exact strong law)**.** *Suppose that $\{X_k\}$ are independent and Pareto with parameter h_k, respectively. For positive sequences $\{a_n\}$ and $\{b_n\}$ with*

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} < \infty, \tag{15}$$

if there exists $0 < B < \infty$ which satisfies

$$\lim_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} a_k \log\left(1 + \frac{b_k}{a_k h_k}\right) = B, \tag{16}$$

then we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} \stackrel{a.s.}{=} B. \tag{17}$$

Corollary 1.2. *Let us suppose the assumptions of Theorem* 1.4. *If Equation* (11) *holds, then we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} \stackrel{a.s.}{=} \frac{1-\alpha}{b}. \tag{18}$$

Unfortunately, Theorem 1.4 cannot apply to the case of Equation (13), since $\sum_{n=1}^{\infty} a_n/b_n = \infty$. When $h_j = j$, we can also obtain the exact strong law making use of the algorithm of Adler and Wittmann [5]. However, parameters $\{a_k\}$ and $\{b_n\}$ for this law are so complicated, because the construction procedure of them is not so straightforward. Actually, no examples were given in [5]. Here, we have the following statement.

Proposition 1.2. *Suppose that $\{X_k\}$ are independent and Pareto with parameter k, respectively. Let us put*

$$a_n = k^{-5} e^{-100k^4} \quad \textit{for } n_{k-1} < n \le n_k,\ k \ge 1, \tag{19}$$

$$b_n = \sum_{k=1}^{n} a_k (r_k - r_{k-1}) \quad \textit{for } n \ge 1, \tag{20}$$

where

$$n_k = \begin{cases} \left\lfloor e^{e^{100k^4}} \right\rfloor + 1 & \textit{if } k \ge 1, \\ 0 & \textit{if } k = 0, \end{cases} \tag{21}$$

$$r_n = \begin{cases} \log n \log\log n & \textit{if } n \ge 1, \\ 0 & \textit{if } n = 0. \end{cases} \tag{22}$$

Then we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k X_k / b_n \overset{a.s.}{=} 1. \tag{23}$$

The plan of the article is as follows. In Section 2, we calculate the first and the second moments of the truncated Pareto random variable. In Section 3, we give all proofs of Propositions 1.1, 1.2, Theorems 1.3, 1.4 and Corollaries 1.1, 1.2, respectively.

2. Preliminary

Lemma 2.1. *Suppose that X is Pareto with parameter $h \ge 1$. For $a > 0$ we have*

$$\mathrm{E}(X\mathbf{1}\{X \le a\}) = \log\left(1 + \frac{a}{h}\right) - \frac{a}{h+a} \tag{24}$$

and

$$\mathrm{E}(X^2\mathbf{1}\{X \le a\}) = a - 2h\log\left(1 + \frac{a}{h}\right) + \frac{ah}{h+a} < a. \tag{25}$$

Proof. It is easy to see that

$$\mathrm{E}(X\mathbf{1}\{X \le a\}) = \int_0^a \frac{tdt}{(t+h)^2} = \log\left(1+\frac{a}{h}\right) - \frac{a}{h+a}$$

and

$$\mathrm{E}(X^2\mathbf{1}\{X \le a\}) = \int_0^a \frac{t^2dt}{(t+h)^2} = a - 2h\log\left(1+\frac{a}{h}\right) + \frac{ah}{h+a} < a.$$

The fact that $\log(1+x) > x/(x+1)$ for $x > 0$ and some estimates yield the last inequality. $\square$

3. Proofs

3.1. Proof of Proposition 1.1

The proof is based on Lemma 6.18 of [11]. The mean value theorem implies

$$\int_{A_{n-1}}^{A_n} \frac{dx}{x\log x} = (A_n - A_{n-1})c_n = a_n c_n \quad \text{for } n \ge 2, \tag{26}$$

where

$$\frac{1}{A_n \log A_n} \le c_n \le \frac{1}{A_{n-1}\log A_{n-1}}. \tag{27}$$

It follows that $\int_{A_1}^{\infty} dx/(x\log x) = \infty$ by simple calculation. Therefore, Equation (26) and $\lim_{n\to\infty} A_n = \infty$ yield $\sum_{n=1}^{\infty} a_n c_n = \infty$. Consequently, Equation (6) holds because

$$\infty \overset{(27)}{=} \sum_{n=2}^{\infty} \frac{a_n}{A_{n-1}\log A_{n-1}} = \sum_{n=1}^{\infty} \frac{a_{n+1}}{A_n \log A_n} \overset{(7)}{\le} \sum_{n=1}^{\infty} \frac{a_n}{A_n\log A_n}.$$

The rest proof from Equation (6) to Equation (5) is the same one of Theorem 2 of [3].

3.2. Proof of Theorem 1.3

The proof is based on Theorem 2.1 of [10]. Equation (8) implies that $\{a_j\}$ and $\{b_n\}$ satisfy Equation (4) with $\alpha = 1$ in Lemma 2.2 of [10]. Moreover, since it follows that

$$\mathrm{E}X_j\mathbf{1}\left\{|X_j|\le b_n/a_j\right\}\overset{(24)}{=}\log\left(1+\frac{b_n}{a_jh_j}\right)-\frac{b_n}{a_jh_j+b_n}\quad\text{for } 1\le j\le n,$$

we have

$$b_n^{-1}\sum_{j=1}^n a_j\mathrm{E}X\mathbf{1}\left\{|X|\le\frac{b_n}{a_j}\right\}=b_n^{-1}\sum_{j=1}^n a_j\log\left(1+\frac{b_n}{a_jh_j}\right)-\sum_{j=1}^n\frac{a_j}{a_jh_j+b_n}.$$

The first term converges to A as $n\to\infty$ because of Equation (9). The second term converges to 0 since Equation (8) yields

$$0\le\sum_{j=1}^n\frac{a_j}{a_jh_j+b_n}\le b_n^{-1}\sum_{j=1}^n a_j\to 0\quad\text{as } n\to\infty.$$

Hence, applying Theorem 2.1 of [10], we have Equation (10).

3.3. Proof of Corollary 1.1

1. Equation (8) follows because $\sum_{n=1}^\infty a_n/b_n=\sum_{n=1}^\infty 1/\{n(\log n)^2\}<\infty$ and the Kronecker lemma (Lemma A.6.2 of [9]). Since

$$\begin{aligned}&b_n^{-1}\sum_{j=1}^n a_j\log\left(1+\frac{b_n}{a_jh_j}\right)\\=\;&(\log n)^{-b}\sum_{j=1}^n(\log j)^{b-2}j^{-1}\log\left(1+(\log n)^bj^{1-\alpha}(\log j)^{2-b}\right)\\\sim\;&(1-\alpha)(\log n)^{-b}\sum_{j=1}^n j^{-1}(\log j)^{b-1}\sim\frac{1-\alpha}{b},\end{aligned}$$

   we have Equation (9) with $A=(1-\alpha)/b$. Therefore, Theorem 1.3 implies Equation (12).
2. Applying Theorem 3.2 of [10] with $\delta=0$ and $\gamma=b-1$, we can prove the desired result.

3.4. Proof of Theorem 1.4

The proof is based on Theorem 1 of [1]. For convenience, let us put

$c_k = b_k/a_k$. Then it follows that

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} = b_n^{-1} \sum_{k=1}^{n} a_k \left\{X_k \mathbf{1}(X_k \le c_k) - \mathrm{E}X_k \mathbf{1}(X_k \le c_k)\right\} \quad (28)$$

$$+b_n^{-1} \sum_{k=1}^{n} a_k X_k \mathbf{1}(X_k > c_k) \quad (29)$$

$$+b_n^{-1} \sum_{k=1}^{n} a_k \mathrm{E}X_k \mathbf{1}(X_k \le c_k). \quad (30)$$

Equation (28) converges to 0 almost surely. In fact, since it turns out that

$$\sum_{k=1}^{\infty} \frac{a_k^2}{b_k^2} \mathrm{E}X_k^2 \mathbf{1}(X_k \le c_k) \overset{(25)}{\le} \sum_{k=1}^{\infty} \frac{1}{c_k} \overset{(15)}{<} \infty,$$

this conclusion holds by using the Khinchine-Kolmogorov convergence theorem (Theorem 5.1.1 of [7]) and the Kronecker lemma.

Equation (29) also converges to 0 almost surely. In fact, since

$$\sum_{k=1}^{\infty} \mathrm{P}(X_k > c_k) = \sum_{k=1}^{\infty} \frac{1}{c_k + h_k} \le \sum_{k=1}^{\infty} \frac{1}{c_k} \overset{(15)}{<} \infty,$$

we have $\mathrm{P}(X_k > c_k, \text{infinitely often}) = 0$ by the first Borel-Cantelli lemma.

Equation (30) converges to B by the following reasons. It is computed as

$$b_n^{-1} \sum_{k=1}^{n} a_k \mathrm{E}X_k \mathbf{1}(X_k \le c_k) \overset{(24)}{=} b_n^{-1} \sum_{k=1}^{n} a_k \log\left(1 + \frac{b_k}{a_k h_k}\right) - b_n^{-1} \sum_{k=1}^{n} \frac{b_k}{h_k + c_k}. \quad (31)$$

Then the first term converges to B by Equation (16), and the second term converges to 0 by also applying the Kronecker lemma to $\sum_{n=1}^{\infty} b_n/\{b_n(c_n + h_n)\} \le \sum_{n=1}^{\infty} 1/c_n < \infty$.

3.5. Proof of Corollary 1.2

Equation (15) holds, because of $\sum_{n=1}^{\infty} a_n/b_n = \sum_{n=1}^{\infty} 1/\{n(\log n)^2\} <$

∞. Since it follows that

$$
\begin{aligned}
& b_n^{-1}\sum_{k=1}^{n} a_k \log\left(1+\frac{b_k}{a_k h_k}\right) \\
= \ & (\log n)^{-b}\sum_{k=1}^{n}(\log k)^{b-2}k^{-1}\log\left(1+k^{1-\alpha}(\log k)^2\right) \\
\sim \ & (1-\alpha)(\log n)^{-b}\sum_{k=1}^{n}k^{-1}(\log k)^{b-1}\sim\frac{1-\alpha}{b},
\end{aligned}
$$

we have Equation (16) with $B=(1-\alpha)/b$. Therefore, Theorem 1.4 implies Equation (18).

3.6. Proof of Proposition 1.2

Firstly, let us quote the following theorem.

Theorem 3.1 (Adler and Wittmann [5]). *Let $\{Y_k\}$ be independent random variables. If $\lim_{n\to\infty}\sum_{k=1}^{n} Y_k/r_n \overset{P}{=} 1$ for some constants $\{r_n\}$, then there exist $\{a_k\}$ and $\{b_n\}$ which satisfy $\lim_{n\to\infty}\sum_{k=1}^{n} a_kY_k/b_n \overset{a.s.}{=} 1$.*

The proof tells us an algorithm for the construction of $\{a_k\}$ and $\{b_n\}$ from $\{Y_k\}$ and $\{r_n\}$. We apply it to Theorem 1.1. For convenience, let us assume the following.

$$
\begin{cases}
\text{Let } \{X_k\} \text{ be independent and Pareto with parameter } k\text{, respectively,} \\
Y_k := X_k/k, \quad S_n := \sum_{k=1}^{n} Y_k.
\end{cases}
\tag{32}
$$

Note that Theorem 1.1 implies $\lim_{n\to\infty} S_n/r_n \overset{P}{=} 1$ for r_n defined by Equation (22). We use the following three lemmas to construct $\{a_k\}$ and $\{b_n\}$.

Lemma 3.1. *Under Equation* (32) *we have*

$$
\mathrm{P}\left(\left|\frac{S_n}{\log n}-\log\log n\right|>x\right)\le\frac{10}{x}\quad \textit{for } x\ge 25 \textit{ and } n\ge 25. \tag{33}
$$

Proof. This proof is based on Lemma 1 of [6]. First of all, we show

$$
\left(\frac{\sum_{k=1}^{n}\mu_{kn}-r_n}{\log n}\right)^2\le 2x\quad \text{for } x\ge 25 \text{ and } n\ge 25, \tag{34}
$$

where

$$\mu_{kn} = \mu_{kn}(x) := \mathrm{E}(Z_{kn}) = \mathrm{E}(Y_k \mathbf{1}\{Y_k \le x \log n\}),$$
$$Z_{kn} := Y_k \mathbf{1}\{Y_k \le x \log n\}, \quad W_{kn} := Y_k \mathbf{1}\{Y_k > x \log n\} \quad \text{for} \quad 1 \le k \le n$$

and r_n is defined by Equation (22). It is easy to see that

$$1 \le \frac{H_n}{\log n} \le 1 + \frac{1}{\log n}, \tag{35}$$

where $H_n = \sum_{k=1}^n 1/k$. Using this, we calculate the left hand side of Equation (34) as follows.

$$\begin{aligned}
&\text{(LHS of Equation (34))} \\
&= \left\{ \frac{1}{\log n} \left(\sum_{k=1}^n \frac{1}{k} \mathrm{E}(X_k \mathbf{1}\{X_k \le kx \log n\}) \right) - \log\log n \right\}^2 \\
&\overset{(24)}{=} \left\{ \frac{H_n}{\log n} \left(\log(1 + x\log n) - \frac{x \log n}{1 + x \log n} \right) - \log\log n \right\}^2 \\
&= \left[\frac{H_n}{\log n} \left\{ \log x + \left(1 - \frac{\log n}{H_n}\right) \log\log n + \log\left(1 + \frac{1}{x\log n}\right) - \frac{x\log n}{1 + x\log n} \right\} \right]^2 \\
&\le \left[\frac{H_n}{\log n} \left\{ \log x + \left(1 - \frac{\log n}{H_n}\right) \log\log n + \log\left(1 + \frac{1}{x\log n}\right) \right\} \right]^2 \\
&\overset{(a)}{\le} \left\{ \frac{H_n}{\log n} \log x + \left(\frac{H_n}{\log n} - 1 \right) \log\log n + \frac{H_n}{x(\log n)^2} \right\}^2 \\
&\overset{(b)}{\le} \left\{ \left(1 + \frac{1}{\log n}\right) \log x + \frac{\log\log n}{\log n} + \frac{1}{x\log n}\left(1 + \frac{1}{\log n}\right) \right\}^2 \\
&\overset{(c)}{\le} \left\{ \left(1 + \frac{1}{\log n}\right) \sqrt{x} + \frac{\log\log n}{\log n} + \frac{1}{x\log n}\left(1 + \frac{1}{\log n}\right) \right\}^2 \\
&\overset{(d)}{\le} (1.32\sqrt{x} + 0.4)^2 \le (\sqrt{2x})^2
\end{aligned}$$

Inequality of (a) follows because $\log(1 + x) \le x$ for $x > 0$. Inequality of (b) follows from Equation (35). Inequality of (c) follows since $\log x \le \sqrt{x}$ for $x \ge 25$. Finally, inequality of (d) follows from $n \ge 25$ and $x \ge 25$.

On the other hand, we have

$$\begin{aligned}
\sigma_{kn}^2 &:= \mathrm{E}((Z_{kn} - \mu_{kn})^2) \le \mathrm{E}(Y_k^2 \mathbf{1}\{Y_k \le x \log n\}) \\
&= \frac{1}{k^2} \mathrm{E}(X_k^2 \mathbf{1}\{X_k \le kx \log n\}) \overset{(25)}{\le} \frac{x \log n}{k}.
\end{aligned}$$

Hence Equation (35) yields

$$\sum_{k=1}^{n} \sigma_{kn}^2 \le x(1+\log n)\log n \le 2x(\log n)^2. \tag{36}$$

Now, using the subadditivity and the Chebyshev inequality, we obtain

$$\begin{aligned}
& \mathrm{P}\left(\left|\frac{S_n}{\log n} - \log\log n\right| > x\right) \\
= \; & \mathrm{P}\left(\left|\sum_{k=1}^{n} Z_{kn} + \sum_{k=1}^{n} W_{kn} - r_n\right| > x\log n\right) \\
\le \; & \mathrm{P}\left(\left|\sum_{k=1}^{n} Z_{kn} - r_n\right| > x\log n\right) + \mathrm{P}\left(\sum_{k=1}^{n} W_{kn} > 0\right) \\
\le \; & \frac{\mathrm{E}\left(\left(\sum_{k=1}^{n} Z_{kn} - r_n\right)^2\right)}{x^2(\log n)^2} + \sum_{k=1}^{n} \mathrm{P}\left(W_{kn} > 0\right).
\end{aligned} \tag{37}$$

Since $(a+b)^2 \le 2(a^2+b^2)$, the first term of Equation (37) is bounded by

$$\begin{aligned}
\frac{\mathrm{E}\left(\left(\sum_{k=1}^{n} Z_{kn} - r_n\right)^2\right)}{x^2(\log n)^2} &= \frac{\mathrm{E}\left(\left\{\sum_{k=1}^{n}(Z_{kn} - \mu_{kn}) + \sum_{k=1}^{n} \mu_{kn} - r_n\right\}^2\right)}{x^2(\log n)^2} \\
&\le \frac{2\left\{\sum_{k=1}^{n} \sigma_{kn}^2 + \left(\sum_{k=1}^{n} \mu_{kn} - r_n\right)^2\right\}}{x^2(\log n)^2} \\
&\overset{(36)}{\le} \frac{2\left\{2x(\log n)^2 + \left(\sum_{k=1}^{n} \mu_{kn} - r_n\right)^2\right\}}{x^2(\log n)^2} \\
&\overset{(34)}{\le} 2\left(\frac{2}{x} + \frac{2}{x}\right) = \frac{8}{x}.
\end{aligned}$$

Since $n \ge 25$, the second term of Equation (37) is bounded by

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{P}\left(W_{kn} > 0\right) &= \sum_{k=1}^{n} \mathrm{P}\left(Y_k > x\log n\right) = \sum_{k=1}^{n} \mathrm{P}\left(X_k > kx\log n\right) \\
&= \sum_{k=1}^{n} \frac{1}{k(1+x\log n)} \overset{(35)}{\le} \frac{1+\log n}{1+x\log n} \\
&< \frac{1+\log n}{x\log n} = \frac{1}{x\log n} + \frac{1}{x} \le \frac{2}{x}.
\end{aligned}$$

Hence Equation (33) follows. □

Lemma 3.2. *Let us put* $\varepsilon_n = (\log\log n)^{-1/2}$. *Then* $\varepsilon_n r_n$ *is increasing to infinity, and* $\lim_{n\to\infty}(S_n - r_n)/(r_n\varepsilon_n) \overset{P}{=} 0$.

Proof. It is clear that $\varepsilon_n r_n = (\log n)\sqrt{\log\log n}$ is increasing to infinity. By Equation (33), it follows that

$$\mathrm{P}\left(\left|\frac{S_n - r_n}{r_n\varepsilon_n}\right| > \frac{x}{\sqrt{\log\log n}}\right) \le \frac{10}{x} \quad \text{for } x \ge 25 \text{ and } n \ge 25. \tag{38}$$

Let us fix $\delta > 0$. Then we see $x = \delta\sqrt{\log\log n} \ge 25$ for sufficiently large n. Inserting this to Equation (38), we have

$$0 \le \mathrm{P}\left(\left|\frac{S_n - r_n}{r_n\varepsilon_n}\right| > \delta\right) \le \frac{10}{\delta\sqrt{\log\log n}} \overset{n\to\infty}{\to} 0. \tag{39}$$

□

Recalling $\{n_k\}$ defined by Equation (21), we have the following lemma.

Lemma 3.3. *If* $m \ge n_k$, *then*

$$\mathrm{P}(|S_m - r_m| > \varepsilon_m r_m) < k^{-2} \tag{40}$$

and $r_{n_{k+1}} > 2r_{n_k}$ *for* $k \ge 1$.

Proof. Equation (21) yields $\log\log n_k \ge 100k^4$, whence $10/\sqrt{\log\log n_k} \le k^{-2}$. Applying Equation (39) with $\delta = 1$ to this, we have Equation (40). It is clear that $r_{n_{k+1}} > 2r_{n_k}$ because of $r_{n_k} \sim 100k^4 e^{100k^4}$. □

***Proof of Proposition* 1.2.** If $a_{n_k} = k^{-5}e^{-100k^4}$, then $a_{n_k}r_{n_k} \sim 100/k$. Therefore it is decreasing to 0 as $k \to \infty$ and $\sum_{k=1}^{\infty} a_{n_k}r_{n_k} = \infty$. Since the rest proof is followed by the argument of page 181 of [5], the proof is completed.

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